The New Space Mapping Algorithms (since 2000)

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Objectives of Space Mapping

- Optimization of very expensive models
- Construct easy-to-calculate surrogate models

We assume two models of a physical object are available:

- an accurate fine model (expensive)
- a simpler coarse model (cheap)
Type of Problem Considered

Minimize w.r.t. \( x \in \mathbb{R}^n \) the absolute values of the deviations between response \( r(x; t_i) \) and specifications \( y_i \)

\[
f_i(x) = r(x; t_i) - y_i, \quad i = 1, \ldots, m
\]

Find \( x^* \in \arg\min_x \{ H(f(x)) \} \)
Original SM for Optimization

(Bandler et al., 1995)

Physical problem

\[ c \quad \quad \quad \quad \quad \quad f \]

coarse model

fine model

Connect similar responses

\[ f(x) \approx c(P(x)) \]
Original SM for Optimization

Problem: \( \text{minimize } H(f(x)) \)

SM strategy: \( \text{minimize } H(c(P(x))) \)

SM methodology:

For \( i = 0, 1, 2, \ldots \), find estimates \( P^{(i)} \) of \( P \) and
\( \text{minimize } H(c(P^{(i)}(x))) \)
Original SM: Basic Algorithm

Find initial estimate $x^{(0)}$ to $x^*$

for $i = 0,1,2, \ldots$ do

- calculate $f(x^{(i)})$
- find $P(x^{(i)})$
  
  based on previous points find an estimate $P^{(i)}$ of $P$
  
  minimize $H(c(P^{(i)}(x)))$ to find $x^{(i+1)}$

endo
Initialization

Find the coarse model solution $z^*$
Find the Space Mapping $P$

- by connecting similar responses

coarse model

\[ c \]

fine model

\[ f \]
Parameter Extraction: Find $x^{(0)}$

$$z^{(0)} = P(x^{(0)}) \equiv \arg \min_{z} \left\{ \left\| f(x^{(0)}) - c(z) \right\| \right\}$$
Find $x^{(1)}$

Intuition: $x^{(1)} = x^{(0)} + (z^* - z^{(0)})$
We assume \( f(x) \approx c(P(x)), \) i.e., \( P(x^*) = z^* \)

\[
P(x) \approx P(x^{(0)}) + J_P(x^{(0)})(x - x^{(0)})
\]

\[
P^{(0)}(x) \equiv P(x^{(0)}) + B^{(0)}(x - x^{(0)}) , \quad B^{(0)} = I
\]

\[
x^{(1)} = \arg\min \left\{ H(c(P^{(0)}(x))) \right\}
\]

\[
P^{(0)}(x) = z^* \implies P(x^{(0)}) + (x - x^{(0)}) = z^*
\]

\[
\implies z^{(0)} + (x - x^{(0)}) = z^*
\]

\[
\implies x^{(1)} = x = x^{(0)} + (z^* - z^{(0)})
\]
Original SM Algorithm

\[ x^{(0)} = z^* \]

for \( i = 0, 1, 2, \ldots \) (while not STOP) do

\[ f(x^{(i)}) \]

\[ z^{(i)} = P(x^{(i)}) \equiv \arg \min_z \left\{ \| f(x^{(i)}) - c(z) \| \right\} \]

compute \( P^{(i)} \) from \( P(x^{(i)}) \) and \( B^{(i)} \)

\[ x^{(i+1)} = \arg \min_x \left\{ H(c(P^{(i)}(x))) \right\} \]

enddo

Bandler, Biernacki, Chen, Hemmers, Madsen (1995)
i’th Iteration: Estimate P

Assume $P$ has been computed at $x^{(0)}, x^{(1)}, \ldots, x^{(i)}$

$$P(x) \approx P(x^{(i)}) + J_P(x^{(i)})(x - x^{(i)})$$

$$P^{(i)}(x) \equiv P(x_f^{(i)}) + B^{(i)}(x - x^{(i)})$$

where $B^{(i)} \approx J_P(x^{(i)})$ is, e.g., a Broyden update
Traditional Taylor-based Optimization

At the iterate $x^{(i)}$ minimize $H(s_T^{(i)}(x))$

where $s_T^{(i)}$ is a first order Taylor estimate of $f$ at $x^{(i)}$

Combined surrogate:
\[ s_{comb}^{(i)}(x) = \eta_i s_T^{(i)}(x) + (1 - \eta_i) s_{SM}^{(i)}(x), \quad 0 \leq \eta_i \leq 1 \]

Bandler, Bakr, Madsen, Søndergaard (2001)
Approximation Errors

Taylor error at \( x^{(i)} \)

\[
\left\| f(x) - f^{(i)}(x) \right\| \leq C_T \cdot \left\| x - x^{(i)} \right\|^2
\]

SM error at \( x^{(i)} \)

\[
\left\| f(x) - c(P^{(i)}(x)) \right\| \leq \varepsilon + \left\| J_c(P^{(i)}(x^{(i)})) \right\| \cdot C_{SM} \cdot \left\| x - x^{(i)} \right\|^2
\]
Approximation Errors

Space Mapping

Local (Taylor)
Convergence Theory

Convergence has been proved for the combination of the Space Mapping with a traditional algorithm. Vicente (2003), for the least squares objective. Madsen, Søndergaard (2004), for a general objective.
i’th Surrogate Model

Assume $P$ has been computed at $x^{(0)}, x^{(1)}, \ldots, x^{(i)}$

$$s_{SM}^{(i)}(x) \equiv c\left( B^{(i)}(x - x^{(i)}) + P(x^{(i)}) \right)$$
Input Surrogate Model

(Bandler et al., 1994)

Surrogate: \( s(x, p) \equiv c(Bx + d), \quad p = (B,d) \)

Assume \( f \) has been computed at \( x^{(0)}, x^{(1)}, \ldots, x^{(i)} \)

\[
p^{(i)} \in \arg \min_p \left\{ \sum_{k=0}^{i} w_k \| f(x^{(k)}) - s(x^{(k)}, p) \| \right\}
\]

\[
s^{(i)}(x) \equiv s(x, p^{(i)})
\]

\[
x^{(i+1)} = \arg \min_x H(s^{(i)}(x))
\]
Output Surrogate Model 
(*Bandler et al.*, 2003)

Surrogate: \[ s(x, p) \equiv Ac(x) + b, \quad p = (A, b) \]

Assume \( f \) has been computed at \( x^{(0)}, x^{(1)}, \ldots, x^{(i)} \)

\[ p^{(i)} \in \arg \min_p \left\{ \sum_{k=0}^i w_k \| f(x^{(k)}) - s(x^{(k)}, p) \| \right\} \]

\[ s^{(i)}(x) \equiv s(x, p^{(i)}) \]

\[ x^{(i+1)} = \arg \min_x H(s^{(i)}(x)) \]

(*John Dennis, private communication, 2002*)
Implicit Surrogate Model

*(Bandler et al., 2001)*

Surrogate: \( s(x, x_p) \equiv c(x, x_p) \)

Assume \( f \) has been computed at \( x^{(0)}, x^{(1)}, \ldots, x^{(i)} \)

\[
x^{(i)}_p \in \arg \min_{x_p} \left\{ \sum_{k=0}^{i} w_k \left\| f(x^{(k)}) - s(x^{(k)}, x_p) \right\| \right\}
\]

\[
s^{(i)}(x) \equiv s(x, x^{(i)}_p)
\]

\[
x^{(i+1)} = \arg \min_x H(s^{(i)}(x))
\]
Output/Implicit Surrogate Model

Output SM surrogate (additive): \[ s(x, d) \equiv c(x) + d \]

\[ d^{(i)} \in \arg \min_d \left\{ \sum_{k=0}^{i} w_k \| f(x^{(k)}) - (c(x^{(k)}) + d) \| \right\} \]

Implicit SM surrogate: Let \[ s(x, x_p) = c(x) + x_p \]

\[ x_p^{(i)} \in \arg \min_{x_p} \left\{ \sum_{k=0}^{i} w_k \| f(x^{(k)}) - (c(x^{(k)}) + x_p) \| \right\} \]

Thus: Additive Output SM is a special case of Implicit SM
Input / Implicit Surrogate Model

Input SM surrogate: \[ s(x, p) \equiv c(Bx + d), \quad p = (B, d) \]

\[ p^{(i)} \in \arg \min_{p} \left\{ \sum_{k=0}^{i} w_k \| f(x^{(k)}) - s(x^{(k)}, p) \| \right\} \]

Implicit SM surrogate:

\[ s(x, x_p) \equiv c(Bx + d), \quad x_p = (B, d) \]

\[ x_p^{(i)} \in \arg \min_{x_p} \left\{ \sum_{k=0}^{i} w_k \| f(x^{(k)}) - s(x^{(k)}, x_p) \| \right\} \]

Thus: Input SM can be considered a special case of Implicit SM
Space Mapping for Modelling

Star Distribution for SM-based Modelling

(Bandler et al., 2001)
Input/Output Surrogate Model
(Bandler et al., 2003)

Input surrogate: \( s(x, B, d) \equiv c(Bx + d) \)

Output surrogate: \( s(x, A, b) \equiv Ac(x) + b \)

Assume \( f \) has been computed at \( x^{(0)}, x^{(1)}, \ldots, x^{(N)} \)

\((A^{(i)}, B^{(i)}, c^{(i)}, d^{(i)})\)

\[ = \arg \min_{(A,B,c,d)} \left\{ \sum_{k=0}^{i} w_{k} \| f(x^{(k)}) - (A \cdot c(Bx^{(k)} + d) + b) \| \right\} \]

\[ s^{(i)}(x_f) \equiv A^{(i)}c(B^{(i)}x + c^{(i)}) + d^{(i)} \]
Space Mapping-based Modelling

\[ x \in X_R \equiv [x_{\text{mid}} - \delta, x_{\text{mid}} + \delta], \quad \delta = [\delta_1, \delta_2, \ldots, \delta_N] \]

\[(A^{(i)}, B^{(i)}, b^{(i)}, d^{(i)})\]

\[= \arg \min_{(A,B,b,d)} \left\{ \sum_{k=0}^{N} w_k \| f(x^{(k)}) - (A \cdot c(B x^{(k)} + d) + b) \| \right\} \]

\[s^{(i)}(x) \equiv A^{(i)} c(B^{(i)} x + d^{(i)}) + b^{(i)}\]

\[w_k \equiv w_k(x, C, \lambda) = \frac{\exp\left(-\frac{\| x - x^{(k)} \|^2}{C\lambda^2}\right)}{\sum_{j=1}^{N} \exp\left(-\frac{\| x - x^{(j)} \|^2}{C\lambda^2}\right)} \quad \lambda = \frac{2}{n \cdot N^{1/n}} \sum_{i=1}^{n} \delta_i\]
The SM based modelling technique is arbitrarily accurate:

**Theorem**

Let \( X_B \equiv \{ x^{(0)}, x^{(1)}, \ldots, x^{(N)} \} \).

Suppose certain regularity conditions are satisfied. Let \( \varepsilon > 0 \) be given. Then there exists \( \eta > 0 \) such that:

\[
\forall x \in X_R \quad \exists x^{(k)} \in X_B : \| x - x^{(k)} \| < \eta
\]

then \( \| f(x) - s(x) \| < \varepsilon \) for any \( x \in X_R \)

provided \( C > 0 \) is sufficiently small.

*Koziel, Bandler, Madsen (2006)*
Convergence Theory

Convergence of the different variations of the Space Mapping has been proved under certain regularity and Lipschitz conditions.

(Koziel, Bandler, Madsen (2005))

(Koziel’s talk at 16.45 tomorrow)
Conclusions

• Space Mapping provides powerful surrogate models (mapped coarse models) applicable in optimization as well as in modelling.

• Space Mapping has been successfully applied to numerous engineering problems.

• The Input SM, Output SM, and Implicit SM Space Mapping algorithms originate from engineering practice. They are similar in theory, however they perform differently in practice.

• Space Mapping is provably convergent.

• Space Mapping requires skilled engineers for designing the coarse models.