ADJOINT VARIABLE METHODS FOR DESIGN SENSITIVITY ANALYSIS WITH THE METHOD OF MOMENTS

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Outline

Introduction and Objectives
design sensitivity analysis

The Adjoint Variable Method
the direct differentiation method
the adjoint variable method
computational efficiency, feasibility, accuracy

Applications with Frequency-Domain Solvers

Conclusions
Introduction and Objectives

Design sensitivity analysis

sensitivity of the state variables

sensitivity of the response (or objective) function

Objectives

obtain the response and its gradient in the design variable space through a single full-wave analysis

applications with frequency-domain solvers

feasibility of the approach
The Adjoint Variable Method

the linear EM problem

\[ Z(x)I = V \]

\[ x = [x_1 \cdots x_n]^T \quad - \text{design parameters} \]

\[ I = [I_1 \cdots I_m]^T \quad - \text{state variables} \]

define a scalar function (response function, objective function)

\[ f(x, \bar{I}(x)) \]

objective

\[ \nabla_x f \quad \text{subject to} \quad Z(x)I = V, \quad \nabla_x f = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right] \]
The Adjoint Variable Method

state variable sensitivity

(direct differentiation method, DDM)

J.W. Bandler, Optimization, vol. 1, Lecture Notes, 1994

\[ \nabla_x I = Z^{-1} \left[ \nabla_x V - \nabla_x (Z\bar{I}) \right] \]

\[
\frac{\partial I}{\partial x_i} = Z^{-1} \left[ \frac{\partial V}{\partial x_i} - \frac{\partial Z}{\partial x_i} \bar{I} \right], \quad i = 1, \ldots, n
\]

\[
\nabla_x f = \nabla^e_x f + \nabla_I f \cdot \nabla_x I
\]

\[
\nabla_I f = \begin{bmatrix}
\frac{\partial f}{\partial I_1} & \cdots & \frac{\partial f}{\partial I_m}
\end{bmatrix}; \quad \nabla_x I = \begin{bmatrix}
\frac{\partial I_1}{\partial x_1} & \cdots & \frac{\partial I_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial I_m}{\partial x_1} & \cdots & \frac{\partial I_m}{\partial x_n}
\end{bmatrix}
\]
The Adjoint Variable Method

response function sensitivity
(adjoint variable method, AVM)

\[ \nabla_x f = \nabla_x f + \nabla_I f \cdot Z^{-1} \left[ \nabla_x V - \nabla_x (Z\bar{I}) \right] \]

\[ \hat{I} = \left[ \nabla_I f \cdot Z^{-1} \right]^T = [Z^T]^{-1} \left[ \nabla_I f \right]^T \]

\[ Z^T \hat{I} = [\nabla_I f]^T \]

\[ \nabla_x f = \nabla_x f + \hat{I}^T \left[ \nabla_x V - \nabla_x (Z\bar{I}) \right] \]

\[ \frac{\partial f}{\partial x_i} = \frac{\partial e f}{\partial x_i} + \hat{I}^T \left[ \frac{\partial V}{\partial x_i} - \frac{\partial Z}{\partial x_i} \bar{I} \right], \quad i = 1, 2, \ldots, n \]
The Adjoint Variable Method

computational efficiency (single excitation mode)

<table>
<thead>
<tr>
<th></th>
<th>LU-decompositions</th>
<th>Back-substitutions</th>
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<tbody>
<tr>
<td>FDA</td>
<td>( n + 1 )</td>
<td>( n + 1 )</td>
</tr>
<tr>
<td>DDM</td>
<td>1</td>
<td>( n )</td>
</tr>
<tr>
<td>AVM</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
The Adjoint Variable Method

feasibility and accuracy of the AVM

finite-difference approximations within the AVM

the matrix sensitivity

\[
\frac{\partial f}{\partial x_i} = \frac{\partial e f}{\partial x_i} + \hat{I}^T \left[ \frac{\partial V}{\partial x_i} \right] \left( \frac{\partial Z}{\partial x_i} \right) \bar{I}, \quad i = 1, \ldots, n
\]

the adjoint excitation

\[
Z^T \hat{I} = [\nabla_{I f}]^T
\]

\[
\hat{V} = [\nabla_{I f}]^T
\]

\[
\hat{V} \approx \left[ \frac{\Delta f}{\Delta I_1}, \ldots, \frac{\Delta f}{\Delta I_m} \right]
\]
Applications

1. Input impedance of a dipole (Pocklington’s eqn., complex code)

sensitivity with respect to the normalized length

\[ L_n = \frac{L}{\lambda} \]

\[ \frac{\partial R_{in}}{\partial L_n} \quad \frac{\partial X_{in}}{\partial L_n} \quad \text{subject to } ZI = V \]

(1) finite-difference approach (FDA):

\[ \frac{\partial Z_{in}(L_n^{(k)})}{\partial L_n} \approx \frac{Z_{in}(L_n^{(k)} + \Delta L_n^{(k)}) - Z_{in}(L_n^{(k)})}{\Delta L_n^{(k)}} \]

\[ \Delta L_n^{(k)} = 0.01L_n^{(k)} \]
Applications

adjoint variable method (AVM):

the matrix sensitivity

\[
\frac{\Delta Z_{ij}}{\Delta L^{(k)}_n} \approx \frac{Z_{ij}(L^{(k)}_n + \Delta L^{(k)}_n) - Z_{ij}(L^{(k)}_n)}{\Delta L^{(k)}_n}
\]

\[\Delta L^{(k)}_n = 0.01L^{(k)}_n\]

(2) complete re-meshing: full \(\Delta Z\) matrix
(3,4) boundary layer: sparse \(\Delta Z\) matrix

Fig. 1. The dipole and the boundary layer concept (S. Amari, 2001).
Applications

the adjoint excitation (analytical)

\[
\hat{V}_b = \frac{\partial Z_{in}}{\partial I_b} = \frac{\partial (1/I_b)}{\partial I_b} = -\frac{1}{I_b^2}, \quad \hat{V}_j = 0 \text{ for } j \neq b
\]

Fig. 2. Derivative of the input resistance of the dipole with respect to \( L_n \).
Fig. 3. Derivative of the input reactance of the dipole with respect to $L_n$. 

- FDA ($\Delta L_n = 0.01L_n$)
- AVM, no BL ($\Delta L_n = 0.01L_n$)
- AVM, BL ($\Delta L_n = 0.1\delta_n$)
- AVM, BL ($\Delta L_n = 0.5\delta_n$)
Applications

2. Input impedance of a Yagi-Uda array

sensitivity with respect to the normalized separation distance driver-reflector and the normalized reflector length

\[ x = [l_{1n}, s_{1n}]^T \]

<table>
<thead>
<tr>
<th>( l_1 / \lambda )</th>
<th>( l_2 / \lambda )</th>
<th>( l_d / \lambda )</th>
<th>( s_1 / \lambda )</th>
<th>( s_d / \lambda )</th>
<th>( a / \lambda )</th>
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</thead>
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<td>0.45</td>
<td>0.406</td>
<td>0.2607</td>
<td>0.34</td>
<td>0.003</td>
</tr>
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</table>

\( l_3 = l_4 = l_5 = l_6 = l_d; \quad s_2 = s_3 = s_4 = s_5 = s_d \)

Fig. 4. The geometry of the Yagi-Uda array (initial design).
Fig. 5. Input resistance sensitivity with respect to $s_{1n}$. 
Fig. 6. Input reactance sensitivity with respect to $s_{1n}$. 

$\frac{\partial X_{in}}{\partial s_{1n}}$
Applications

3. Gain of a Yagi-Uda array (Pocklington’s eqn., real code)

sensitivity with respect to the normalized separation distances

\[ s_k = \frac{s}{\lambda}, \quad k = 1, \ldots, 5 \]

\[ \frac{\partial G}{\partial s_k} \quad \text{subject to} \quad \hat{Z}\hat{I} = \hat{V}, \quad \hat{I} = \begin{bmatrix} \Re\{\mathbf{I}\} \\ \Im\{\mathbf{I}\} \end{bmatrix} \]

the gain sensitivity depends on \( s_{ni} \) explicitly

\[ \frac{\partial G}{\partial s_{ni}} = \frac{\partial e G}{\partial s_{ni}} - \hat{I}^T \left( \frac{\partial Z}{\partial s_{ni}} \hat{I} \right), \quad i = 1, \ldots, 5 \]
Applications

the adjoint excitation $\mathbf{\hat{V}}$ is a full vector

\[
\begin{align*}
\mathbf{\hat{V}}_k &= \frac{\partial G}{\partial \text{Re}(I_k)} \\
\mathbf{\hat{V}}_{k+m} &= \frac{\partial G}{\partial \text{Im}(I_k)} \\
\end{align*}
\]

\[k = 1, \ldots, m\]

\[\text{analytical}\]

\[
\begin{align*}
\mathbf{\hat{V}}'_k &= \frac{\Delta G}{\Delta \text{Re}(I_k)} \\
\mathbf{\hat{V}}'_{k+m} &= \frac{\Delta G}{\Delta \text{Im}(I_k)} \\
\end{align*}
\]

\[\text{finite differences}\]
Fig. 7. Gain and gain sensitivity of the Yagi-Uda array with respect to $s_{4n}$. 

$\frac{\partial G}{\partial s_{3n}}$
Fig. 8. Gain sensitivity of the Yagi-Uda array with respect to \( \frac{s_{3n}}{\lambda} \), finite-difference approximation of \( \hat{\mathbf{V}} \).
Applications

4. Optimization of the Yagi-Uda array for maximum gain and an input impedance of 73 Ω

design parameters

\[ \mathbf{x} = \begin{bmatrix} s_{1n} & s_{2n} & s_{3n} & s_{4n} & s_{5n} \end{bmatrix}^T \]

objective function

\[ f(\mathbf{x}) = 0.5 \left[ (\text{Re}\{Z_{in}\} - 73)^2 + (\text{Im}\{Z_{in}\})^2 \right] - 0.5G^2 \]

we start from a design already optimized for gain only
Applications

Fig. 9. The progress of the objective function during the optimization of the input impedance and the gain of the Yagi-Uda array.
# Applications

## TABLE I

DESIGN PARAMETERS, INPUT IMPEDANCE AND GAIN OF THE YAGI-UDA ARRAY DESIGN

<table>
<thead>
<tr>
<th></th>
<th>$s_{1n}$</th>
<th>$s_{2n}$</th>
<th>$s_{3n}$</th>
<th>$s_{4n}$</th>
<th>$s_{5n}$</th>
<th>$R_{in}$</th>
<th>$X_{in}$</th>
<th>$G$</th>
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<td>0.3400</td>
<td>0.3735</td>
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<td>0.4519</td>
<td>0.4023</td>
<td>65.85</td>
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<td>0.3844</td>
<td>0.4822</td>
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<td>72.85</td>
<td>5.46</td>
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<tr>
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<tr>
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<td>71.80</td>
<td>0.38</td>
<td>13.75</td>
</tr>
</tbody>
</table>

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5. Optimization of a patch antenna for an input impedance of 50 Ω

**design parameters**

\[ x = [L \ W \ S]^T \]

**objective function**

\[ f(x) = (\text{Re}\{Z_{in}\} - 50)^2 + (\text{Im}\{Z_{in}\})^2 \]
Applications

$x^{(0)} = [50 \ 90 \ 14]^T \text{ (mm)} \quad \Rightarrow \quad x^{(4)} = [51.51 \ 96.39 \ 15.004]^T \text{ (mm)}$

**Fig. 10.** The progress of the objective function during the optimization of the input impedance of the patch antenna.
Conclusions

The AVM is implemented into a feasible technique for the frequency-domain DSA of HF structures

- reduction of the CPU time requirements for the DSA by a factor of $n$ to $(n+1)$
- improved accuracy and convergence
- feasibility: does not require significant modification of existing codes

Factors affecting the accuracy

- finite differences with the $\partial_{x_i} Z$ matrix: insignificant
- finite differences with $\partial_{I_k} f$: significant
Conclusions

Applications of the DSA based on the AVM

- optimization
- modeling
- statistical and yield analysis

Limitations

- linear frequency-domain analysis
- extension to nonlinear frequency-domain analysis is straightforward