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J.W. Bandler, Editor

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IMPORTANT NOTE

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PART I: INTRODUCTION

AN INTRODUCTION TO SIMULATION AND OPTIMIZATION

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Invited Paper

Abstract

A review of recent work in simulation and optimization is made with the aim of introducing the designer to the benefits of automating optimal design procedures and to indicate limitations imposed by the current state of the art.

Introduction

This paper is directed to the engineer interested in using computer aids to modeling and design, and considering the application of optimization techniques. Limitations on the size and scope of problems which can be approached from the optimization point of view as imposed by the current state of the art are also indicated.

It seems that the gap between theoretical developments and their practical implementation is in danger of widening. With the plethora of literature in optimization methods and computer-aided circuit design, particularly articles laying claim to superiority of technique, a confused impression is created.

With these thoughts in mind, the author will attempt to direct the microwave reader to work which appears relevant, useful, instructive or stimulating within the domain of activity of the respective authors.

Review

It is felt that Calahan's book on computer-aided network design[1] is a good indicator of current trends and possible future developments in computer based circuit and system design techniques and philosophies. The collection of articles[2] considered by Director to be benchmarks in simulation and optimization is also recommended, again not so much for the details as for its point of view. Szentirmai's reprint volume[3] deals with various aspects of filter design, and appears representative of numerical advances in that area.

Complementary survey articles on optimization techniques are those by Bandler[4] and Charalambous[5]. Also appearing in the IEEE Transactions on Microwave Theory and Techniques and somewhat complementary in the areas of simulation and sensitivity analysis are papers by Bandler and Seviora[6] and Monaco and Tiberio[7]. A pragmatic article of particular interest to microwave designers is one by Perlman and Gelnovatch[8].

Analysis

Effort is being directed at solving larger systems more efficiently. See, for example, Wexler et al[9] and others[10,11]. As far as engineering design is concerned, it is important to stress that it is generally inefficient to put a conventional simulation program into an optimization loop without taking certain things into account. Assuming, for example, that the program exploits sparsity in the computations the question of efficient computation of the effects of parameter changes (indispensable to design) arises. In general, for economical and physical reasons, not all possible design variables or degrees of freedom are always utilized. Setting up the necessary equations and recomputing the entire response every time a relatively small number of parameters is changed will result in a much larger computing bill than is necessary. In considering the value of an analysis routine for design purposes, then, the manner in which the effects of component variation are handled is crucial.

Sensitivity

A much debated topic in the circuit literature, particularly in time-domain analysis, efficient sensitivity evaluation is a cornerstone to automatic design[2]. Branin[12] has dispelled some of the mystique shrouding the adjoint network method[2,6] by a compact, abstract presentation. The adjoint network approach whereby, for example, the first-order sensitivities of the output of a circuit may be efficiently evaluated with respect to all designable components using the results of only two circuit analyses has, however, been a powerful motivating force.

Extensions and applications of the adjoint network concept abound in the literature[7,13]. In the frequency domain for linear circuits, at least, it appears that, by suitable mathematical manipulations, higher-order sensitivities[14], large-change sensitivities[15], sensitivities with respect to frequency[16] etc., are available relatively efficiently by suitable programming. The most widely acclaimed optimization methods[5], however, require only first derivatives. Furthermore, the value of second- and higher-order sensitivities at points possibly far from the optimum has not been established.

Formulation

There appear to be two principal approaches to the formulation of design objectives. On the one hand, some designers attempt to approximate ideal performance specifications which, by definition, are unattainable. This approach requires the least preparation of the problem, but the results tend to be somewhat ambiguous in the context of meeting specifications and subsequent assignment of tolerances. On the other hand, more insight can be brought to bear if design problems are cast in the form of meeting or exceeding realistic performance specifications[4]. One can go a step further, exploiting more fully one's prior knowledge or insight into the problem at hand, by devising artificial specifications[17] in an attempt to anticipate more closely the actual optimum performance realizable by the configuration and thereby permit its more rapid evaluation. Optimal assignment of manufacturing tolerances appears to be more well-defined in the context of realistic specifications.

Objectives

The ubiquitous least squares objective[2,18], usually employed in conjunction with error-prone data or ideal

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specifications in the context, for example, of modeling or design, respectively, is probably the simplest to implement. Particularly in filter design, however, non-Euclidean measures of error have been widely applied historically. See, for example, Szentirmai[3]. Numerical approximation methods for minimax (Chebyshev) or near minimax solutions, contrary to prevailing assumptions, can, for all practical purposes be realized almost as easily as least squares solutions[19]. On paper, at least, they produce more impressive-looking responses. One reason is that one or more trial runs are usually required in practice to verify a solution. Once a run, for example, using a least squares objective has been performed, sufficient information about the properties of the problem is often available to allow one to subsequently force at least a near minimax solution with relatively little additional effort[19].

Algorithms

It is known that a well-conditioned problem in terms of selection of a well-behaved objective function and nonredundant variables which have been properly scaled allows the conventional steepest descent method to perform adequately. The Newton method, which may be viewed as steepest descent with respect to a different norm[20] is generally less sensitive to scaling but, unlike steepest descent, is affected by the properties of the second derivatives and convexity.

Modern gradient methods[21,22] attempt to overcome the limitations of the basic steepest descent and Newton methods, as do analogous methods in the minimax optimization of a set of functions[19,23,24]. In minimax problems, in particular, classical assumptions about the number or character of the equal (or active) extrema vis a vis the number of independent variables need not and, in general, do not hold.

Current efforts in optimization[20] are directed at developing robust algorithms, however, anticipation and alleviation of ill-conditioning, where possible, is desirable.

Centering

Centering a design usually implies the process of finding a nominal design somehow influenced by manufacturing tolerances and, possibly, post production tuning[25,26]. The procedure may involve optimal assignment of component tolerances, maximization of production yield, design subject to a specified yield, etc. The problem could be a worst-case one with design variables assumed independent; it might involve correlated elements, statistical distributions, and so on. A number of relevant works will provide the interested reader with further details[27]. It should be emphasized that, in general, all design parameters: nominal values, tolerances, tuning ranges and so on will interact in defining an optimal design[25,26]. A solution obtained from a least squares or minimax approximation in the usual sense does not necessarily provide the best nominal values. The centering problem is generally significantly more expensive to solve, requiring careful preparation.

Software

An excellent survey of both available and proprietary general purpose software for circuit designers has been made by Kaplan[18]. The article, however, appears limited to developments in the U.S. and probably places undue emphasis on least squares objectives. A number of optimization programs with documentation is available from the present author[28]. Two collections[29,30] of reprints, reports, notes and programs should also be mentioned. Documented listings of very useful optimization programs are also available from the U.K. Atomic Energy Research Establishment[31], and the Numerical Optimization Centre[32]. See also pp.242-243 of Gill and Murray[20].

Should one use a commercially available analysis and design package, for example, through a time-sharing facility? It is felt that current optimization features in these packages are generally weak, so that their use will probably be expensive in the long run.

New algorithms or packages should be tested on suitable examples and compared with respect to features, flexibility, ease of use, convergence to known solutions, memory required and running times. This is particularly appropriate in optimal design where, over an extended period of use, enormous numbers of simulations might be required.

Techniques which appear different may sometimes be alternative implementations of the same basic algorithm[5]. This is, understandably, often not realized at the time by the proponents of the techniques. As the state of the art advances, unification takes place and the techniques can be put into better perspective. See also Branin[12] and Bonfatti et al[11].

Conclusions

Having assimilated the essential past achievements (regrettably inadequately referenced because of limited space) where might one find indicators of possible new developments? Three additional recent works may be singled out: an advance in minimax algorithms where derivatives are not required[33], an advance in efficient design in the time domain employing sensitivity information[34], and an advance in centering which takes account of many uncertainties relevant to the microwave area[35]. A number of sessions at this year's IEEE International Symposium on Circuits and Systems (Munich, Germany, Apr. 1976) promise further achievements in simulation and optimization in all areas covered by this paper[36].

References

[1] D.A. Calahan, Computer-Aided Network Design (Revised Edition). New York: McGraw-Hill, 1972.

[2] S.W. Director, Ed., Computer-Aided Circuit Design: Simulation and Optimization. Stroudsburg, Penn.: Dowden, Hutchinson and Ross, 1973.

[3] G. Szentirmai, Ed., Computer-Aided Filter Design. New York: IEEE Press, 1973.

[4] J.W. Bandler, "Optimization methods for computer-aided design," IEEE Trans. Microwave Theory Tech., vol.MTT-17, Aug. 1969, pp.533-552.

[5] C. Charalambous, "A unified review of optimization," IEEE Trans. Microwave Theory Tech., vol. MTT-22, Mar. 1974, pp. 289-300.

[6] J.W. Bandler and R.E. Seviora, "Current trends in network optimization," IEEE Trans Microwave Theory Tech., vol. MTT-18, Dec. 1970, pp. 1159-1170.

[7] V.A. Monaco and P. Tiberio, "Computer-aided analysis of microwave circuits," IEEE Trans. Microwave Theory Tech., vol.MTT-22, Mar. 1974, pp. 249-263.

[8] B.S. Perlman and V.G. Gelnovatch, "Computer aided design, simulation and optimization," in Advances in

Microwaves vol.8, L. Young and H. Sobol, Eds. New York: Academic Press, 1974.

[9] A. Wexler et al, "Solution of large, sparse systems in design and analysis", IEEE Int. Microwave Symp. Digest (Palo Alto, Calif., May 1975).

[10] G.D. Hachtel, R.K. Brayton and F.G. Gustavson, "The sparse tableau approach to network analysis and design", IEEE Trans. Circuit Theory, vol. CT-18, Jan. 1971, pp.101-113.

[11] F. Bonfatti, V.A. Monaco and P. Tiberio, "Microwave circuit analysis by sparse-matrix techniques", IEEE Trans. Microwave Theory Tech., vol.MTT-22, Mar.1974, pp.264-269.

[12] F.H. Branin, Jr., "Network sensitivity and noise analysis simplified", IEEE Trans. Circuit Theory, vol. CT-20, May 1973, pp.285-288.

[13] G.C. Temes, R.M. Ebers and R.N. Gadenz, "Some applications of the adjoint network concept in frequency domain analysis and optimization", Computer Aided Design, vol. 4, Apr. 1972, pp. 129-134.

[14] A.K. Seth and P.H. Roe, "Hybrid formulation of explicit formulae for higher order network sensitivities", IEEE Trans. Circuits Syst., vol. CAS-22, May 1975, pp.475-478.

[15] R.N. Gadenz, M.G. Rezai-Fakhr and G.C. Temes, "A method for the computation of large tolerance effects", IEEE Trans. Circuit Theory, vol. CT-20, Nov.1973, pp.704-708.

[16] J.W. Bandler, M.R. M. Rizk and H. Tromp, "Efficient calculation of exact group delay sensitivities", IEEE Trans. Microwave Theory Tech., vol. MTT-24, April 1976.

[17] C. Charalambous and J.W. Bandler, "New algorithms for network optimization", IEEE Trans. Microwave Theory Tech., vol. MTT-21, Dec. 1973, pp.815-818.

[18] G. Kaplan, "Computer-aided design", IEEE Spectrum, vol.12, Oct.1975, pp.40-47.

[19] J.W. Bandler, C. Charalambous, J.H.K. Chen and W.Y. Chu, "New results in the least pth approach to minimax design", IEEE Trans. Microwave Theory Tech., vol. MTT-24, Feb.1976.

[20] P.E. Gill and W. Murray, Eds., Numerical Methods for Constrained Optimization. New York: Academic Press, 1974.

[21] R. Fletcher, "Fortran subroutines for minimization by quasi-Newton methods", Atomic Energy Research Establishment, Harwell, England, Report AERE-R7125,1972.

[22] P.E. Gill and W. Murray, "Quasi-Newton methods for unconstrained optimization", J.Inst.Math.Applics., vol.9, 1972, pp.91-108.

[23] K. Madsen, H. Schjaer-Jacobsen and J. Voldby, "Automated minimax design of networks", IEEE Trans. Circuits Syst., vol. CAS-22, Oct.1975, pp.791-796.

[24] C. Charalambous and A.R. Conn, "Optimization of microwave networks", IEEE Trans. Microwave Theory Tech., vol. MTT-23, Oct. 1975, pp.834-838.

[25] J.F. Pinel, K.A. Roberts and K. Singhal, "Tolerance assignment in network design", Proc. IEEE Int. Symp. Circuits Syst. (Newton, Mass., Apr. 1975),pp.317-320.

[26] J.W. Bandler, P.C. Liu and H. Tromp, "A nonlinear programming approach to optimal design centering, tolerancing and tuning", IEEE Trans. Circuits Syst., vol. CAS-23, Mar. 1976.

[27] "Tolerance assignment in network design", Special Session, Proc. IEEE Int. Symp. Circuits Syst. (Newton, Mass., Apr. 1975), pp.317-336.

[28] Group on Simulation, Optimization and Control, Faculty of Engineering, McMaster University, Hamilton, Canada L8S 4L7.

[29] W. Kinsner, Ed., "Notes on numerical methods in engineering analysis and design of fields, circuits and systems", Faculty of Engineering, McMaster Univ., Hamilton, Canada, Report SOC-93, June 1975.

[30] J.W. Bandler, Ed., "Notes on numerical methods of optimization with applications in optimal design", Faculty of Engineering, McMaster Univ., Hamilton, Canada, Report SOC-113, Nov.1975.

[31] Computer Sciences and Systems Division, Atomic Energy Research Establishment, Harwell, England OX11 ORA.

[32] Numerical Optimization Centre, Hatfield Polytechnic, Hatfield, Hertfordshire, England.

[33] K. Madsen, O. Nielsen, H. Schjaer-Jacobsen and L. Thrane, "Efficient minimax design of networks without using derivatives", IEEE Trans. Microwave Theory Tech., vol. MTT-23, Oct. 1975, pp.803-809.

[34] R.K. Brayton and S.W. Director, "Computation of delay time sensitivities for use in time domain optimization", IEEE Trans. Circuits Syst., vol. CAS-22, Dec. 1975, pp. 910-920.

[35] J.W. Bandler, P.C. Liu and H. Tromp, "Integrated approach to microwave design", IEEE Trans. Microwave Theory Tech., to appear.

[36] Regular and Special Sessions, IEEE International Symposium on Circuits and Systems, Munich, Germany, Apr. 1976.

THE TOLERANCE PROBLEM IN OPTIMAL DESIGN

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Invited Paper

This paper reviews recent results in the tolerance assignment problem. A mathematical statement of the problem is made and difficulties in solving it are pointed out. The approach is taken that component tolerance assignment is an integral part of computer-aided circuit and system design. It is shown that both the optimal nominal parameter values and tolerances should be determined simultaneously using optimization methods for the best results. A bibliography of recent or relevant work in the area of circuit design subject to tolerances is appended.

1. Introduction

A very important practical problem in design is the problem of optimal design subject to component tolerances. Computer-aided optimal design of microwave circuits subject to tolerances seems, however, to have been relatively unexplored from a systematic point of view. Basically, the problem is to ensure that a design when fabricated will meet performance or other specifications. Manufacturing tolerances, material uncertainties and parasitic effects, for example, will generally result in the constructed design not performing as well as an ideal theoretical design. Mass production of a design may be invisaged or only a few realizations may be produced. A yield of less than 100 percent may often be more economical than a 100 percent yield. In some instances, a 100 percent yield may be essential. Depending on such factors certain statistical or worst-case design approaches may be employed.

Most previous work has involved some kind of tolerance investigation after a nominal design has been specified. This work may be described as tolerance analysis. Other work has been done in which a function of first-order sensitivities has been minimized in order to improve the nominal design. Unsophisticated design "centering" techniques usually taking two design parameters at a time have also been proposed.

The purpose of this paper is to review some recent results in the tolerance assignment problem. The focus will be principally on worst-case design, but a bibliography of recent or relevant work in the area of circuit design subject to tolerances is appended to put the present discussion into perspective.

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2. The Tolerance Optimization Problem

The tolerance optimization problem consists of finding a nominal design point $\underset{\sim}{\phi}^o \triangleq [\phi_1^o \ \phi_2^o \ \ldots \ \phi_k^o]^T$ and a set of associated tolerances $\underset{\sim}{\varepsilon} \triangleq [\varepsilon_1 \ \varepsilon_2 \ \ldots \ \varepsilon_k]^T$, where k is the number of independent design parameters $\underset{\sim}{\phi} = [\phi_1 \ \phi_2 \ \ldots \ \phi_k]^T$, such that the tolerance region R_t, where $R_t \triangleq \{\underset{\sim}{\phi} \mid \phi_i^o - \varepsilon_i \leq \phi_i \leq \phi_i^o + \varepsilon_i, \ i=1,2,\ldots,k\}$, and R_c, the region of points $\underset{\sim}{\phi}$ such that all performance specifications and constraints are satisfied, intersect in such a way as to minimize the cost of production. For 100 percent yield $R_t \subset R_c$.

The conventional problem of finding a single point $\underset{\sim}{\phi}$ which best fits performance specifications and constraints is a difficult enough optimization problem. Moving an infinite number of possible designs around in a region is, of course, impossible in general. This has led, for example, to algorithms based on iterative use of the Monte Carlo approach, worst-case designs predicted by local linearization of the functions concerned, and so on.

As an example of the difficulties involved, R_t has 2^k vertices. For k=10 and 10 constraint functions to be evaluated a total of 2^{10}x10 = 10,240 constraint functions need to be evaluated, in general, to test all the vertices.

3. Previous Work

A classified bibliography is appended. The aim is to bring the microwave engineer up to date with developments, mostly in the circuit theory and design area, relevant to sensitivity and tolerance analysis and optimization and to briefly review the work of some authors.

Central to circuit design subject to tolerances is the efficient calculation of first- and higher-order sensitivities [1-7] which may be used, for example, in gradient minimization algorithms or in the approximation of the performance function due to changes in parameter values.

Useful work in circuit and system theory related to changes in network functions due to small or large changes in parameter values is available in the literature [8-15]. The bilinear property of network functions [9,13,15], for example, is an important concept.

Efficient computational schemes for the evaluation of large-change sensitivities or the evaluation of tolerance effects [16-22] are useful in both analysis and design.

Optimization methods [23-27] which either have found application in this area or should find use are referenced. Included are methods of linear programming [27], nonlinear programming [23-25] and a highly efficient unconstrained optimization method [26].

Numerous references to work on the optimization of tolerances are cited [28-46]. Most authors attempt to achieve minimum cost designs. Bandler and Liu [29] as well as Pinel [43] have tried examples in which the nominal point was allowed to move subject only to the constraints of the given problem.

Finally, some applications or related work are referenced [47-49]. Of particular interest is the work by Pinel [49]. The problems of designing tunable circuits or circuits that are designed to permit tuning to facilitate alignment or correction for parasitic effects not accounted for in the design theory are obviously closely related to the tolerance problem.

4. Conclusions

Much useful work has been done in this area. A drawback is that extensive use of intuitive or ad hoc techniques seems to be made. Badly needed are automated, efficient, and reliable methods of design subject to tolerances. The problem, in general, is formidable.

5. Acknowledgement

The assistance of P.C. Liu of McMaster University is gratefully acknowledged.

6. References

6.1 Computation of Sensitivities

[1] J.W. Bandler and R.E. Seviora, "Current trends in network optimization", IEEE Trans. Microwave Theory and Techniques, vol. MTT-18, Dec. 1970, pp. 1159-1170.

[2] M.L. Blostein, "Sensitivity analysis of parasitic effects in resistance-terminated LC two-ports", IEEE Trans. Circuit Theory, vol. CT-14, March 1967, pp. 21-25.

[3] S.W. Director and R.A. Rohrer, "Automated network design - the frequency-domain case", IEEE Trans. Circuit Theory, vol. CT-16, Aug. 1969, pp. 330-337.

[4] S.W. Director and D.A. Wayne, "Computational efficiency in the determination of Thevenin and Norton equivalents", IEEE Trans. Circuit Theory, vol. CT-19, Jan. 1972, pp. 96-98.

[5] K. Géher, Theory of Network Tolerances. Budapest, Hungary: Akademiai Kiadó, 1971.

[6] J.F. Pinel and M.L. Blostein, "Computer techniques for the frequency analysis of linear electrical networks", Proc. IEEE, vol. 55, Nov. 1967, pp. 1810-1819.

[7] A.K. Seth, "Electrical network tolerance optimization", Ph.D. Thesis, University of Waterloo, Waterloo, Ontario, Canada, 1972.

6.2 Theory of Circuit Sensitivities

[8] M.L. Blostein [2].

[9] J.K. Fidler and C. Nightingale, "Differential - incremental - sensitivity relationships", Electronics Letters, vol. 8, Dec. 1972, pp. 626-627.

[10] K. Géher [5].

[11] S.L. Hakimi and J.B. Cruz, Jr., "Measures of sensitivity for linear systems with large multiple parameter variations", IRE WESCON Conv. Rec., Aug. 1960, part 2, pp. 109-115.

[12] G. Martinelli and M. Poggelli, "Bounds on magnitude of sensitivity to variations of component values of passive transfer-voltage ratios", Electronics Letters, vol. 4, March 1968, pp. 98-99.

[13] S.R. Parker, E. Peskin and P.M. Chirlian, "Application of a bilinear theorem to network sensitivity", IEEE Trans. Circuit Theory", vol. CT-12, Sept. 1965, pp. 448-450.

[14] J. Shekel, "Some properties of networks with one variable element", IEEE Trans. Circuit Theory, vol. CT-14, March 1967, pp. 89-92.

[15] E.V. Sorensen, "General relations governing the exact sensitivity of linear networks", Proc.IEE, vol. 114, Sept. 1967, pp. 1209-1212.

6.3 Computation of Tolerances

[16] R.N. Gadenz, G. Rezai-Fakhr and G.C. Temes, "A method for the fast computation of large tolerance effects", Proc. IEEE Int. Symp. Circuit Theory (Toronto, Canada, April, 1973), pp. 220-222.

[17] P.J. Goddard, P.A. Villalaz and R. Spence, "Method for the efficient computation of the large-change sensitivity of linear nonreciprocal networks", Electronics Letters, vol. 7, Feb. 1971, pp. 112-113.

[18] J.F. Pinel and K.A. Roberts, "Tolerance assignment in linear networks using nonlinear programming", IEEE Trans. Circuit Theory, vol. CT-19, Sept. 1972, pp. 475-479.

[19] A.K. Seth [7].

[20] A.K. Seth and P.H. Roe, "Selection of component tolerances for optimum circuit reproducibility", Proc. IEEE Int. Symp. Circuit Theory (London, England, Sept. 1971), pp. 105-106.

[21] R. Spence, P. Villalaz and P. Goddard, "The efficient prediction of tolerance regions, with application to circuit design and device modelling", Proc. IEEE Int. Symp. Circuit Theory (London, England, Sept. 1971), pp. 107-108.

[22] G.C. Temes and H.J. Orchard, "First-order sensitivity and worst-case analysis of doubly terminated reactance two-ports", IEEE Trans. Circuit Theory, vol. CT-20, Sept. 1973.

6.4 Optimization Methods

[23] J.W. Bandler and C. Charalambous, "A new approach to nonlinear programming", Proc. 5th Hawaii Int. Conf. System Sciences (Honolulu, Hawaii, Jan. 1972), pp. 127-129.

[24] J.W. Bandler and C. Charalambous, "Nonlinear programming using minimax techniques", J. Optimization Theory and Applications, to be published.

[25] A.V. Fiacco and G.P. McCormick, Nonlinear Programming: Sequential Unconstrained Minimization Techniques. New York: Wiley, 1968.

[26] R. Fletcher, "A new approach to variable metric algorithms", Computer J., vol. 13, Aug. 1970, pp. 317-322.

[27] S.I. Gass, Linear Programming Methods and Applications (3rd Ed.). New York: McGraw-Hill, 1969.

6.5 Optimization of Tolerances

[28] J.W. Bandler, "Optimization of design tolerances using nonlinear

programming", Proc. 6th Princeton Conf. on Information Sciences and Systems (Princeton, N.J., March 1972), pp. 655-659. To be published in J. Optimization Theory and Applications.

[29] J.W. Bandler and P.C. Liu, "Automated network design with optimal tolerances", Proc. IEEE Int. Symp. Circuit Theory (Toronto, Canada, April 1973), pp. 181-184.

[30] P.W. Becker and B. Jarkler, "A systematic procedure for the generation of cost-minimized designs", IEEE Trans. Reliability, vol. R-21, Feb. 1972, pp. 41-45.

[31] S. Ben-Yaakov, "Application of linear programming to the economical optimization of electrical networks", Proc. IEEE, vol. 56, Sept. 1968, pp. 1619-1621.

[32] P. Bodharamik and R. Newcomb, "Network design with tolerance interval analysis approach," Proc. IEEE Int. Symp. Circuit Theory (Los Angeles, Calif., April 1972), pp. 112-115.

[33] P.W. Broome and F.J. Young, "The selection of circuit components for optimum circuit reproducibility", IRE Trans. Circuit Theory, vol. CT-9, March 1962, pp. 18-23.

[34] E.M. Butler, "Realistic design using large-change sensitivities and performance contours", IEEE Trans. Circuit Theory, vol. CT-18, Jan. 1971, pp. 58-66.

[35] E.M. Butler, "Large change sensitivities for statistical design", BSTJ, vol. 50, April 1971, pp. 1209-1224.

[36] E. de Castro, G. Iuculano and V.A. Monaco, "Component value spread and network function tolerances: an optimal design procedure", Alta Frequenza, vol. 40, Nov. 1971, pp. 867-872.

[37] S.W. Director and A.R. Thorbjornsen, "Computer-aided tolerance assignment for linear circuits with correlated elements", IEEE Trans. Circuit Theory, vol. CT-20, Sept. 1973.

[38] K. Géher [5].

[39] M.H. Hamza and W. Prater, "Economical optimization of electrical networks using separable programming", Proc. IEEE, vol. 60, March 1972, pp. 332-333.

[40] B.J. Karafin, "The optimum assignment of component tolerances for electrical networks", BSTJ., vol. 50, April 1971, pp. 1225-1242.

[41] G. Kjellström, "Optimization of electrical networks with respect to tolerance costs", Ericsson Technics, 1970, pp. 157-175.

[42] J.F. Pinel and K.A. Roberts [18].

[43] J.F. Pinel, "Tolerance assignment and network alignment of linear networks in the frequency domain", IEEE Short Course, Course Notes 73-SC-06, 1973, pp. 17-25.

[44] A.K. Seth [7].

[45] A.K. Seth and P.H. Roe [20].

[46] P.A. Villalaz and R. Spence, "Scheme for the elimination of redundant model complexity", Electronics Letters, vol. 8, Jan. 1972, pp. 38-40.

6.6 Applications or Extensions

[47] P.W. Becker and B. Jarkler [30].

[48] S. Klapp, "Empirical parameter variation analysis for electronic circuits", IEEE Trans. Reliability, vol. R-13, March 1964, pp. 34-39.

[49] J.F. Pinel, "Computer-aided network tuning", IEEE Trans. Circuit Theory, vol. CT-18, Jan. 1971, pp. 192-194.

PART II: THE TOLERANCE PROBLEM

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Optimization of Design Tolerances Using Nonlinear Programming[1]

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Abstract. A possible mathematical formulation of the practical problem of computer-aided design of electrical circuits (for example) and systems and engineering designs in general, subject to tolerances on k independent parameters, is proposed. An automated scheme is suggested, starting from arbitrary initial acceptable or unacceptable designs and culminating in designs which, under reasonable restrictions, are acceptable in the worst-case sense. It is proved, in particular, that, if the region of points in the parameter space for which designs are both feasible and acceptable satisfies a certain condition (less restrictive than convexity), then no more than 2^k points, the vertices of the tolerance region, need to be considered during optimization.

Key Words. Engineering design, nonlinear programming, convex programming, optimization theorems, approximation of functions.

1. Introduction

An extremely important practical problem is the problem of optimal design subject to tolerances. Recently published work (Refs. 1 6) has yielded some practical insight into the nature of the problem. Indeed, it

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suggests immediately the possibility of formulating the complete worst-case design of circuits or systems as a nonlinear programming problem.

An automated scheme would start from an arbitrary acceptable or unacceptable design and, under appropriate restrictions, stop at an acceptable design which is optimum in the worst-case sense for specified tolerances. The most suitable objective function to be minimized would also seem to be one that best describes the cost of fabrication of the circuit or system, as suggested by some authors (Refs. 1–6).

It is the purpose of this paper to propose possible formulations and to discuss this problem generally. It is not claimed that a complete solution has been obtained. However, a number of interesting objective functions (more appropriately, perhaps, cost functions) have been investigated.

Many types of objective functions can be formulated. A number of variations on the sum of the inverses of the absolute tolerances or the sum of the inverses of the tolerances relative to the respective nominal parameter values can be obtained. Furthermore, the nominal parameter values may or may not be variable. The relative merits of these and other functions which attempt in some way to maximize the size of the region of possible designs (namely, the tolerance region) are discussed.

For the purposes of this paper, it is assumed that the parameter tolerances can be specified independently. Furthermore, it is assumed that the design parameters and tolerances can be varied continuously. The tolerance region, in this case, will be defined by simple upper and lower bounds on the parameters. Of course, the region will contain an infinite number of acceptable designs, assuming that it is a subregion of the intersection of regions of acceptable and feasible designs. It is proved that, if this region satisfies a certain condition (less restrictive than convexity), then only the (finite) number of vertices of the tolerance region need, at most, to be investigated.

2. Feasible and Acceptable Designs

A wide range of design problems can be formulated as nonlinear programming problems. One usually defines a scalar objective function $U(\phi)$, where

$$\phi \triangleq \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{bmatrix} \tag{1}$$

represents the k independent design parameters. Design constraints can be assembled into a column vector $g(\phi)$ and the problem can be stated as finding $\check{\phi}$ such that

$$U(\check{\phi}) = \min_{\phi \in R_c} U(\phi), \tag{2}$$

where

$$R_c \triangleq \{\phi \mid g(\phi) \geqslant 0\}. \tag{3}$$

For the purposes of the present discussion, let us assume that two kinds of constraint functions are present, those that determine the *feasibility* of a design [designated $g_f(\phi)$] and those that determine the *acceptability* of a design [designated $g_a(\phi)$]. Therefore, we will define a feasible region of points R_f as

$$R_f \triangleq \{\phi \mid g_f \geqslant 0\} \tag{4}$$

and an acceptable region of points R_a as

$$R_a \triangleq \{\phi \mid g_a \geqslant 0\}. \tag{5}$$

Thus, $R_c = R_f \cap R_a$. It is assumed that all sets are nonempty. Note that R_a is not necessarily a subset of R_f.

The objective function is usually set up so that a feasible solution is obtained at an interior point of the acceptable region and as far as possible (in some sense) from its boundary. The reasoning behind this is the hope that, when the design is fabricated, inevitable errors in the design parameters might yield, nevertheless, an acceptable design. It is this flexibility which can be exploited in the optimization of tolerances. Often,

$$U(\phi) = -\min_{i \in I_a} g_i(\phi), \tag{6}$$

where the index set I_a relates to constraints defining R_a. It follows then that

$$R_a = \{\phi \mid U(\phi) \leqslant 0\}. \tag{7}$$

3. Tolerance Region

Given a *nominal* point ϕ^0 and a set of nonnegative *tolerances* ϵ, where

$$\epsilon \triangleq \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_k \end{bmatrix} \geqslant 0, \tag{8}$$

we can define a region of *possible* designs R_t as

$$R_t \triangleq \{\phi \mid \phi_i^0 - \epsilon_i \leqslant \phi_i \leqslant \phi_i^0 + \epsilon_i\,, \qquad i = 1, 2, \ldots, k\} \tag{9}$$

or, equivalently,

$$R_t \triangleq \{\phi \mid \phi_i = \phi_i^0 + t_i \epsilon_i\,, -1 \leqslant t_i \leqslant 1, i = 1, 2, \ldots, k\}. \tag{10}$$

Obviously, depending on the location of ϕ^0 and the value of ϵ, R_t may or may not be a subset of R_c.

The tolerance problem is beginning to take shape: R_t should be placed inside R_c in some optimal manner by adjusting ϕ^0 and ϵ to optimal values $\check{\phi}^0$ and $\check{\epsilon}$. A serious development, however, is that all points $\phi \in R_t$ must satisfy $g \geqslant 0$. We have, effectively, to deal with an infinite number of constraints.

For any given point ϕ^0, we can view the functions $g(\phi)$ with respect to ϵ as follows. We let the origin of the ϵ-space correspond to ϕ^0 (transla-

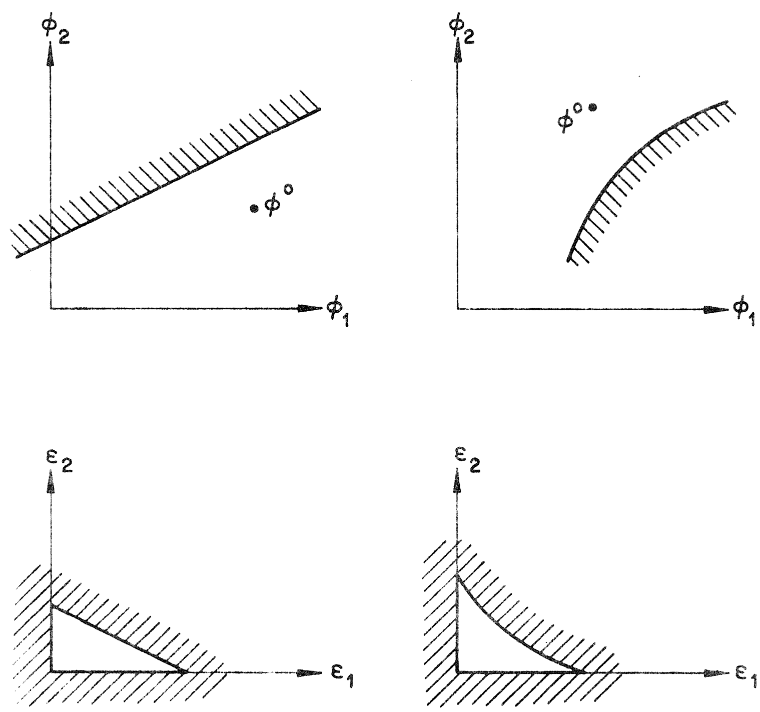

Fig. 1. Allowable tolerances corresponding to particular constraints and particular nominal points.

tion). Then, we consider all the possible linear parameter transformations [from (10)]

$$\epsilon = T(\phi - \phi^0)$$

suggested by the transformation matrix (magnification and reflection)

$$T \triangleq \begin{bmatrix} 1/t_1 & 0 & \cdots & 0 \\ 0 & 1/t_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/t_k \end{bmatrix}, \qquad -1 \leqslant t_i \leqslant 1, \quad i = 1, 2, \ldots, k. \tag{11}$$

Two-dimensional examples of allowable tolerances in the tolerance space corresponding to particular constraints and particular nominal points in the parameter space are shown in Figs. 1–2.

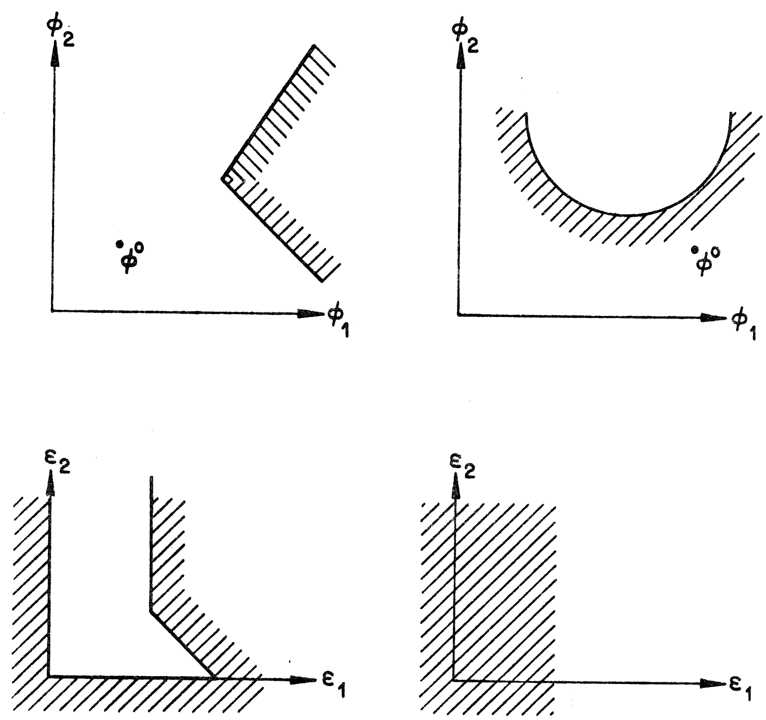

Fig. 2. Allowable tolerances corresponding to particular constraints and particular nominal points.

4. Restrictions on R_c

For obvious reasons, it is impractical to consider an infinite number of constraints. In order to make the problem tractable, a number of simplifying assumptions could be made to try to obtain a solution to the problem with reasonable computational effort.

It can be shown that, if R_c is convex, then from Refs. 7 or 8,

$$\phi^i \in R_c, \qquad i = 1, 2, \ldots, n, \tag{12}$$

implies that

$$\phi = \sum_{i=1}^{n} \lambda_i \phi^i \in R_c \tag{13}$$

for all λ_i satisfying

$$\sum_{i=1}^{n} \lambda_i = 1 \qquad \text{and} \qquad \lambda_i \geqslant 0, \quad i = 1, 2, \ldots, n. \tag{14}$$

For example, given a finite number of points ϕ^i in a finite-dimensional Euclidean space, it is easy to visualize that the ϕ^i are vertices of a polytope (the intersection of a finite number of closed halfspaces) and that ϕ is any interior or boundary point. If R_c is itself a polytope (all constraints linear), it is clearly convex.

The polytope R_t has 2^k vertices. Let the ith vertex be denoted by ϕ^i and let

$$\phi^i = \phi^0 - \epsilon + 2Ev_{i-1} \in R_c, \qquad i = 1, 2, \ldots, 2^k, \tag{15}$$

where

$$E \triangleq \begin{bmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \epsilon_k \end{bmatrix} \tag{16}$$

and where v_i is a k-element vector whose elements reflect the subscript i in binary notation, i.e.,

$$v_0 \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad v_1 \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad v_2 \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad v_3 \triangleq \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots. \tag{17}$$

The vector v_{i-1} can be formed as follows:

$$v_{i-1} = \sum_{j=1}^{k} \mu_j(i)\, u_j\,, \tag{18}$$

where

$$\mu_1\,, \mu_2\,,..., \mu_k \in \{0, 1\} \tag{19}$$

must satisfy (see Table 1)

$$i = 1 + \sum_{j=1}^{k} \mu_j(i)\, 2^{j-1}, \tag{20}$$

and where the k-element vectors u_j are given by

$$u_1 \triangleq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad u_2 \triangleq \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, ..., u_k \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{21}$$

Figure 3 illustrates an example in three dimensions. Observe that

$$Ev_{i-1} = \sum_{j=1}^{k} \mu_j(i)\, \epsilon_j u_j\,. \tag{22}$$

Table 1. Numbering scheme for the vertices of R_t.

i	$\mu_1(i)$	$\mu_2(i)$	$\mu_3(i)$	$\cdots$	$\mu_k(i)$	$\sum_{j=1}^{k} \mu_j(i)\, \epsilon_j u_j$
1	0	0	0		0	0
2	1	0	0		0	$\epsilon_1 u_1$
3	0	1	0		0	$\epsilon_2 u_2$
4	1	1	0		0	$\epsilon_1 u_1 + \epsilon_2 u_2$
5	0	0	1		0	$\epsilon_3 u_3$
6	1	0	1		0	$\epsilon_1 u_1 + \epsilon_3 u_3$
7	0	1	1		0	$\epsilon_2 u_2 + \epsilon_3 u_3$
8	1	1	1		0	$\epsilon_1 u_1 + \epsilon_2 u_2 + \epsilon_3 u_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
2^k	1	1	1		1	ϵ

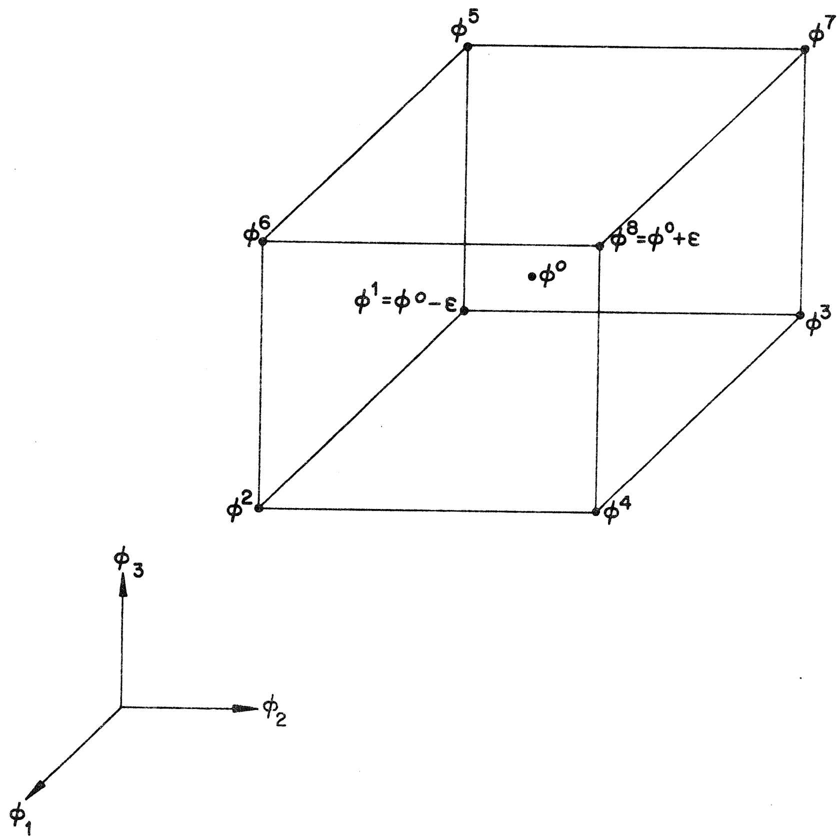

Fig. 3. Three-dimensional example of points defining the vertices of R_t.

Using (12)–(14), we have

$$\phi = \phi^0 - \epsilon + 2 \sum_{i=1}^{2^k} \left(\lambda_i \sum_{j=1}^{k} \mu_j(i)\, \epsilon_j u_j \right) \in R_c \tag{23}$$

if R_c is convex and the vertices of R_t are elements of R_c. Equation (23) generates the set R_t. Therefore, $R_t \subset R_c$. See Fig. 4.

It will now be shown that the assumption that R_c is convex is unnecessarily restrictive.

Theorem 4.1. If the vertices of R_t are in R_c, then $R_t \subset R_c$ if, for all $j = 1, 2,..., k$,

$$\phi^a, \phi^{b(j)} = \phi^a + \alpha u_j \in R_c, \tag{24}$$

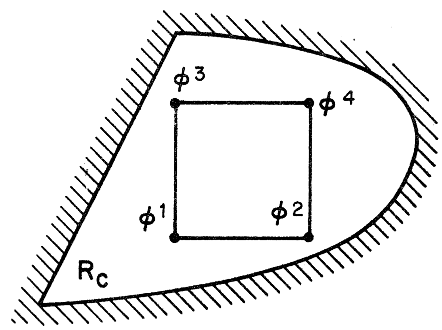

Fig. 4. Possible region R_c.

where α is a scalar, implies that

$$\phi = \phi^a + \lambda(\phi^{b(j)} - \phi^a) \in R_c \tag{25}$$

for all λ satisfying

$$0 \leqslant \lambda \leqslant 1. \tag{26}$$

See, for example, Fig. 5.

Proof. Let ϕ_l denote some point, in general, in an l-dimensional linear manifold generated by the first 2^l vertices as

$$\phi_l = \phi^0 - \epsilon + 2 \sum_{i=1}^{2^l} \left(p_i \sum_{j=1}^{l} \mu_j(i)\, \epsilon_j u_j \right), \tag{27}$$

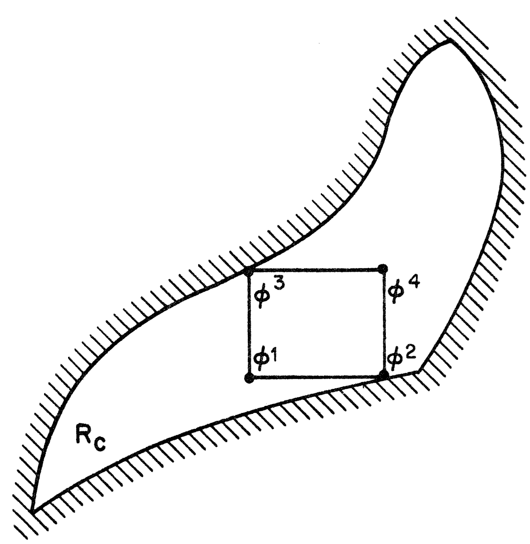

Fig. 5. Possible region R_c.

with p_i satisfying

$$\sum_{i=1}^{2^l} p_i = 1 \quad \text{and} \quad p_i \geq 0, \quad i = 1, 2, ..., 2^l. \tag{28}$$

Note that, since max $i = 2^l$, we can deduce from (20) that

$$\mu_j = 0, \qquad j \geq l, \tag{29}$$

in (22), so that the relevant summation need be taken only up to l and not k.

Assume that $\phi_l \in R_c$ for all $\phi^i \in R_o$ given in (22). Now, consider

$$\phi_{l+1} = \phi^0 - \epsilon + 2 \sum_{i=1}^{2^{l+1}} \left(q_i \sum_{j=1}^{l+1} \mu_j(i)\, \epsilon_j u_j \right), \tag{30}$$

with q_i satisfying

$$\sum_{i=1}^{2^{l+1}} q_i = 1 \quad \text{and} \quad q_i \geq 0, \quad i = 1, 2, ..., 2^{l+1}. \tag{31}$$

After some manipulation, we find that

$$\phi_{l+1} = \phi^0 - \epsilon + 2 \sum_{i=1}^{2^l} \left[(q_i + q_{2^l+i}) \sum_{j=1}^{l} \mu_j(i)\, \epsilon_j u_j \right] + 2 \left(\sum_{i=2^l+1}^{2^{l+1}} q_i \right) \epsilon_{l+1} u_{l+1} \,. \tag{32}$$

Let

$$\lambda = \sum_{i=2^l+1}^{2^{l+1}} q_i \tag{33}$$

and

$$p_i = q_i + q_{2^l+i}, \qquad i = 1, 2, ..., 2^l. \tag{34}$$

Hence, (32) becomes

$$\phi_{l+1} = \phi_l + 2\lambda \epsilon_{l+1} u_{l+1} \,. \tag{35}$$

With $\lambda = 0$,

$$\phi_{l+1} = \phi_l \in R_c \,,$$

by assumption. If $\lambda = 1$,

$$\phi_{l+1} = \phi_l + 2\epsilon_{l+1} u_{l+1} \,,$$

which represents a translation of the l-dimensional manifold. Thus, $\phi_{l+1} \in R_c$, by assumption. For $0 < \lambda < 1$, we note that $\phi_{l+1} \in R_c$ if (24)–(26) hold for $j = l + 1$.

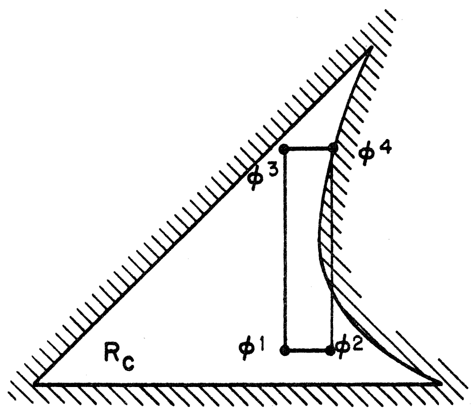

Fig. 6. Possible region R_c.

It is easy to verify that $\phi_1 \in R_c$ and, furthermore, that $\phi_2 \in R_c$ if (24)–(26) hold for $j = 1$ and $j = 2$, respectively. It follows by the foregoing inductive reasoning that $\phi_k = \phi$, as defined by (23), is in R_c under the conditions of the theorem.

The theorem allows both Figs. 4–5, but not Fig. 6.

5. Some Objective Functions

A number of potentially useful and fairly well-behaved objective functions which might be used to represent the cost of a design can be formulated. In practice, of course, a suitable modelling problem would first have to be solved to determine the significant parameters involved partially or totally in the actual cost. Here, we will assume that either absolute or relative tolerances are the main variables and, furthermore, that the total cost $C(\phi^0, \epsilon)$ of the design is just the sum of the cost of the individual components.

It is intuitively reasonable to assume that

$$C(\phi^0, \epsilon) \to c \geqslant 0 \quad \text{as} \quad \epsilon \to \infty, \tag{36}$$

$$C(\phi^0, \epsilon) \to \infty \quad \text{for any} \quad \epsilon_i \to 0. \tag{37}$$

Two out of many possible functions which fulfil these requirements are, for $c = 0$,

$$C_a = \sum_{i=1}^{k} (c_i/\epsilon_i), \tag{38}$$

subject to $\epsilon \geqslant 0$ as stated in (8), and

$$C_r = \sum_{i=1}^{k} c_i \log_e(\phi_i^0/\epsilon_i), \tag{39}$$

subject to

$$\phi^0 \geqslant \epsilon \geqslant 0. \tag{40}$$

In both cases,

$$c_i \geqslant 0, \qquad i = 1, 2,\ldots, k. \tag{41}$$

6. Examples

It is interesting to consider C_a and C_r for the different regions R_c sketched in Figs. 7–10. We will let $c_1 = c_2 = 1$. Figure 7 depicts a situation where $\check{\phi}^0$ has relatively little variation in going from C_a to C_r. Figure 8 has $\check{\phi}_1^0 > \check{\epsilon}_1$ and $\check{\phi}_2^0 = \check{\epsilon}_2$; for C_a, $\check{\phi}_2^0 > 0$ but, for C_r, $\check{\phi}_2^0 = 0$ which (physics permitting) indicates that one parameter may be ***removed.*** It can be shown (see Fig. 11) that min C_r is given by $\check{\phi}_2^0 = 0$, at $\check{\phi}_1^0 = 2.5$, $\check{\epsilon}_1 = 1.5$. Figure 9 allows the possibility of removing ϕ_1 if C_r is optimized. The minimum cost is then $\log_e 9$. However, it is easily shown that, to minimize the cost, ϕ_1 should not be removed (see, for example, Fig. 12). Using C_r in Fig. 10 would indicate that $\check{\phi}_1^0$ and $\check{\phi}_2^0$ may be zero. Using C_a in all the cases of Figs. 7–10, we would find $\check{\phi}^0$ to be an interior point of R_c.

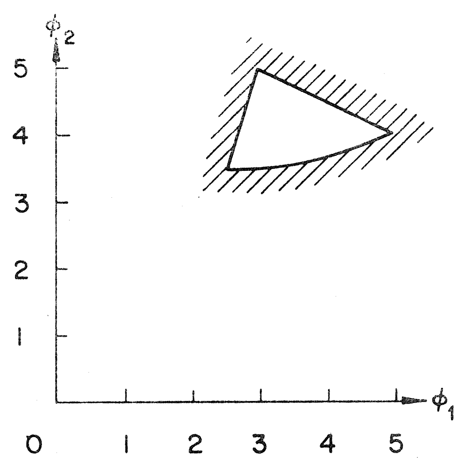

Fig. 7. Example used in the discussion of objective functions.

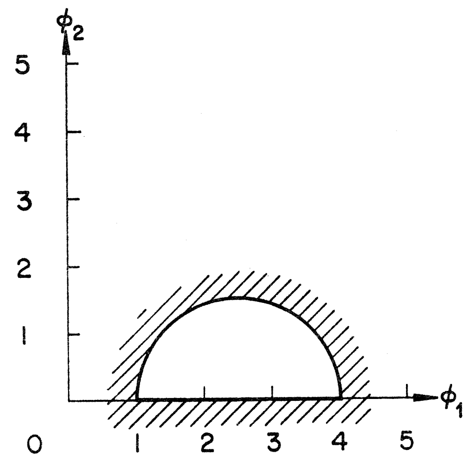

Fig. 8. Example used in the discussion of objective functions.

A number of corresponding observations to those made above can be made if, for the cases sketched in Figs. 7–10, we take (for example) $\phi_1' = 1/\phi_1$ and $\phi_2' = \phi_2$ as parameters.

7. Conclusions

If, as is usual in the design of circuits or systems, the optimal design is obtained by solving an approximation problem, then a fairly

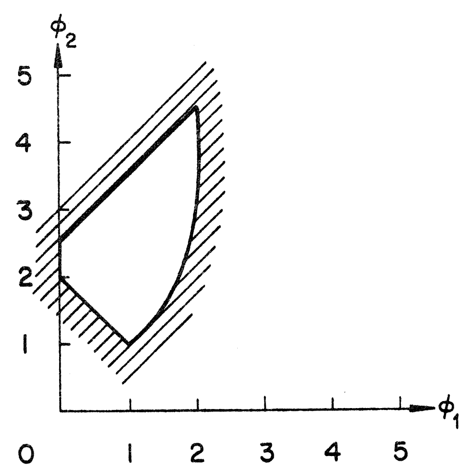

Fig. 9. Example used in the discussion of objective functions.

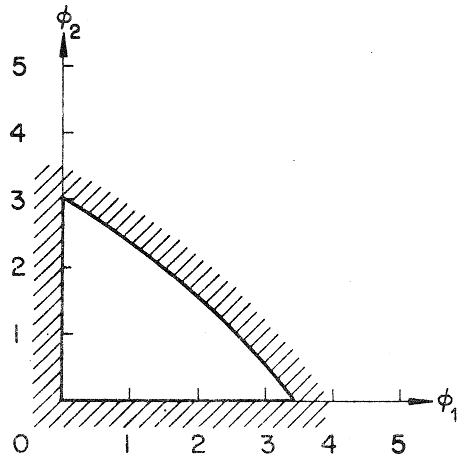

Fig. 10. Example used in the discussion of objective functions.

large number of inequality constraints usually define the acceptable region. For any particular set of reasonable tolerances, one could exploit the likelihood of the worst case (point most likely to violate a given constraint) being predictable by a local linearization or higher-order approximation of the constraints to greatly reduce the computational effort over the computational effort implied by the 2^k vertices of the tolerance region. Further study of these ideas from a nonlinear programming

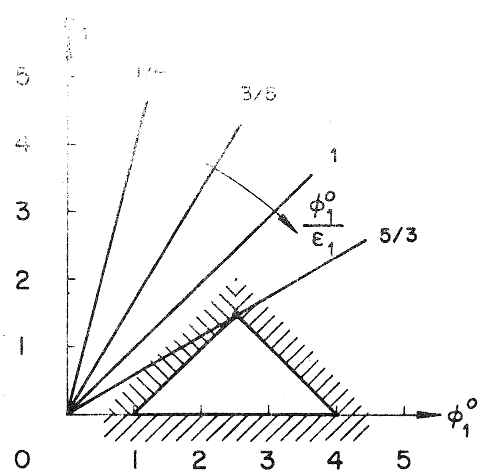

Fig. 11. Example corresponding to Fig. 8 with $\check{\phi}_2^0 = \check{\epsilon}_2 = 0$.

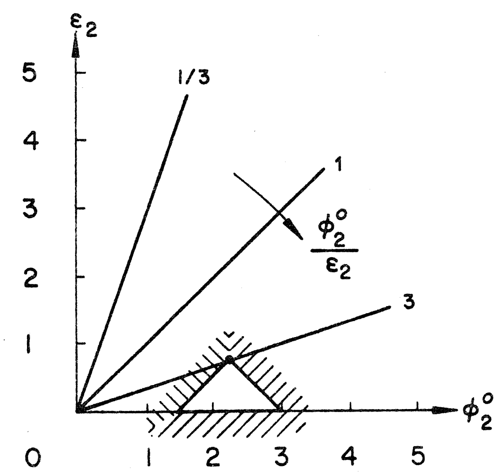

Fig. 12. Example corresponding to Fig. 9 with $\phi_1{}^0 = 1$ and $\epsilon_1 = 0.5$. The best value of C_r is, in this case, $\log_e 6$.

point of view should yield more insight into the possible success or failure of particular tolerance optimization algorithms that might suggest themselves.

References

1. Butler, E. M., *Realistic Design Using Large-Change Sensitivities and Performance Contours*, IEEE Transactions on Circuit Theory, Vol. CT-18, No. 1, 1971.
2. Butler, E. M., *Large Change Sensitivities for Statistical Design*, Bell System Technical Journal, Vol. 50, No. 4, 1971.
3. Karafin, B. J., *The Optimum Assignment of Component Tolerances for Electrical Networks*, Bell System Technical Journal, Vol. 50, No. 4, 1971.
4. Seth, A. K., *Electrical Network Tolerance Optimization*, University of Waterloo, Waterloo, Canada, Ph.D. Thesis, 1972.
5. Pinel, J. F., and Roberts, K. A., *Tolerance Assignment in Linear Networks Using Nonlinear Programming*, IEEE Transactions on Circuit Theory, Vol. CT-19, No. 5, 1972.
6. De Castro, E., Iuculano, G., and Monaco, V. A., *Component Value Spread and Network Function Tolerances: An Optimal Design Procedure*, Alta Frequenza, Vol. 40, No. 11, 1971.

7. MANGASARIAN, O. L., *Nonlinear Programming*, McGraw-Hill Book Company, New York, New York, 1969.
8. ZANGWILL, W. I., *Nonlinear Programming: A Unified Approach*, Prentice-Hall, Englewood Cliffs, New Jersey, 1969.

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Automated Network Design with Optimal Tolerances

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Abstract—A new approach to network design to obtain optimal parameter values simultaneously with an optimal set of component tolerances is proposed. An automated scheme could start from an arbitrary initial acceptable or unacceptable design and under appropriate restrictions stop at an acceptable design which is optimum in the worst case sense for the obtained tolerances.

I. Introduction

IT IS the purpose of this paper to present a new concept in the network design and tolerance selection problem. The concept of a "floating and expanding polytope" suggests that the two procedures of finding an acceptable nominal point and an optimal set of tolerances be replaced by one automated scheme. Using a suitable nonlinear programming technique, any arbitrary initial acceptable or unacceptable design may be used as a starting point. The scheme would stop at an acceptable design which is optimal in the worst case sense of obtained tolerances. The most suitable objective function to be minimized would seem to be one that best describes the cost of fabrication of the circuit, as suggested by some authors [1]–[6]. Several objective functions have been investigated and the results are discussed.

II. Theoretical Considerations

The Tolerance Region

A point $\phi \triangleq [\phi_1 \; \phi_2 \cdots \phi_k]^T$ is a vector of k elements and corresponds to the component values of the network. A nominal point $\phi^0 \triangleq [\phi_1^0 \; \phi_2^0 \cdots \phi_k^0]^T$ is a point associated with a set of nonnegative tolerances $\epsilon \triangleq [\epsilon_1 \; \epsilon_2 \cdots \epsilon_k]^T \geqslant \mathbf{0}$ such that the tolerance region R_t is given by

$$R_t \triangleq \{\phi \mid \phi_i^0 - \epsilon_i \leqslant \phi_i \leqslant \phi_i^0 + \epsilon_i, \quad i = 1, 2, \cdots, k\}. \tag{1}$$

Obviously, R_t is a polytope of k dimensions with sides of length $2\epsilon_i$, $i = 1, 2, \cdots, k$, and centered at ϕ^0. The polytope has 2^k vertices. Each vertex will be indexed from an index set $H \triangleq \{1, 2, \cdots, 2^k\}$ such that

$$\phi^1 \triangleq \begin{bmatrix} \phi_1^0 - \epsilon_1 \\ \phi_2^0 - \epsilon_2 \\ \vdots \\ \phi_k^0 - \epsilon_k \end{bmatrix}, \quad \phi^2 \triangleq \begin{bmatrix} \phi_1^0 + \epsilon_1 \\ \phi_2^0 - \epsilon_2 \\ \vdots \\ \phi_k^0 - \epsilon_k \end{bmatrix}, \quad \phi^3 \triangleq \begin{bmatrix} \phi_1^0 - \epsilon_1 \\ \phi_2^0 + \epsilon_2 \\ \vdots \\ \phi_k^0 - \epsilon_k \end{bmatrix}, \cdots, \quad \phi^{2^k} \triangleq \begin{bmatrix} \phi_1^0 + \epsilon_1 \\ \phi_2^0 + \epsilon_2 \\ \vdots \\ \phi_k^0 + \epsilon_k \end{bmatrix}. \tag{2}$$

A possible outcome of a circuit with a nominal design ϕ^0 and tolerance ϵ falls somewhere in or on the polytope. Depending on the location of ϕ^0 and the size of ϵ, a circuit with parameters ϕ may or may not be acceptable.

The Acceptable Region

The following discussion refers to the frequency-domain design of linear time-invariant circuits, but the results can be applied to the time domain as well. Let the set of frequency points under consideration be $\Omega = \{\omega_1, \omega_2, \cdots, \omega_u, \omega_{u+1}, \cdots, \omega_{u+l}\}$. Upper specifications $S_u(\omega_i)$, $i = 1, 2, \cdots, u$ are assigned to the first u frequency points and lower specifications $S_l(\omega_i)$, $i = u + 1, \cdots, u + l$ to the rest. Frequency points that have both upper and lower specifications may appear twice in the set. Let the response of the network at frequency ω_i be $F(\phi, \omega_i)$.

An acceptable region R_a is given by

$$R_a \triangleq \{\phi \mid S_u(\omega_i) - F(\phi, \omega_i) \geqslant 0, \quad i = 1, 2, \cdots, u$$
$$F(\phi, \omega_j) - S_l(\omega_j) \geqslant 0, \quad j = u + 1, \cdots, u + l\}. \tag{3}$$

Obviously, a design $\{\phi^0, \epsilon\}$ is an acceptable design only if $R_t \subseteq R_a$.

A Theorem

It is impossible to test all the points in R_t to see whether they are in the acceptable region R_a. In order to make the problem tractable, a number of simplifying assumptions could be made to obtain a solution to the problem with reasonable computational effort. Obviously, if R_a is convex and if all the vertices of R_t are interior or boundary points of R_a, then $R_t \subseteq R_a$. It can be shown that the assumption of convexity is unnecessarily restrictive.

Theorem [1]: If the vertices of R_t are in R_a, then $R_t \subseteq R_a$ if, for all $j = 1, 2, \cdots, k$, the assumption that

$$\phi^a, \phi^{b(j)} = \phi^a + \alpha u_j \in R_a \tag{4}$$

where α is a scalar and

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$$u_1 \triangleq \begin{bmatrix}1\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{bmatrix}, \quad u_2 \triangleq \begin{bmatrix}0\\1\\ \cdot\\ \cdot\\ \cdot\\0\end{bmatrix}, \cdots, \quad u_k \triangleq \begin{bmatrix}0\\0\\ \cdot\\ \cdot\\ \cdot\\1\end{bmatrix}$$

implies that

$$\phi = \phi^a + \lambda(\phi^{b(j)} - \phi^a) \in R_a \tag{5}$$

for all λ satisfying

$$0 \leqslant \lambda \leqslant 1. \tag{6}$$

Under such assumptions, only the vertices of the polytope need be tested to ensure that $R_t \subseteq R_a$. It is easy to verify that the theorem holds for $k = 1$ and 2. The proof of the theorem follows by mathematical induction. A complete proof is presented by Bandler [1].

Other constraints such as parameter constraints can be considered. These constraints define a feasible region R_f. Then it is required that $R_t \subseteq (R_a \cap R_f) = R_c$.

The Nonlinear Programming Problem

A function $C_1(\phi^0, \epsilon)$ to be minimized may be

$$C_1 = \sum_{i=1}^{k} \frac{c_i \phi_i^0}{\epsilon_i} \tag{7}$$

where c_i is a weighting factor. See, for example, Pinel and Roberts [4].

Other possibilities are [1]

$$C_2 = \sum_{i=1}^{k} \frac{c_i}{\epsilon_i} \tag{8}$$

and

$$C_3 = \sum_{i=1}^{k} c_i \log_e \frac{\phi_i^0}{\epsilon_i}. \tag{9}$$

In (9) we would be minimizing the ratio of the volume of the polytope defined by the space diagonal ϕ^0 and the volume of the polytope defined by ϵ if the $c_i = 1$.

Let

$$g_{ij}(\phi^i, \omega_j) \triangleq \begin{cases} S_u(\omega_j) - F(\phi^i, \omega_j), & \text{for } 1 \leqslant j \leqslant u \\ F(\phi^i, \omega_j) - S_l(\omega_j), & \text{for } u+1 \leqslant j \leqslant u+l \end{cases} \tag{10}$$

for $i \in H$. That is, at each vertex ϕ^i, there are $l + u$ frequency constraints. There are 2^k vertices for a polytope of k dimensions. A total of $2^k(l + u)$ constraints have to be considered. Other constraints can be added.[1]

A suitable method for solving the nonlinear programming problem is to define [7]

$$B(\phi^0, \epsilon, r) = C(\phi^0, \epsilon) + \sum_{j=1}^{u+l} \sum_{i=1}^{2^k} \frac{r}{g_{ij}(\phi^i, \omega_j)} \tag{11}$$

and minimize B with respect to ϕ^0 and ϵ for appropriately decreasing values of r. Another more recent and efficient method of handling constrained minimization is by the least pth optimization [8], [9] of

$$V(\phi^0, \epsilon, \alpha) = \max_{i,j} \left[C(\phi^0, \epsilon), C(\phi^0, \epsilon) - \alpha_{ij} g_{ij}(\phi^i, \omega_j)\right], \quad \alpha_{ij} > 0. \tag{12}$$

For sufficiently large constant values α_{ij}, the unconstrained minimization of V with respect to ϕ^0 and ϵ yields exactly the constrained minimum of C. This nonlinear programming technique makes it possible to have any initial starting point, acceptable or otherwise, as shown by Bandler and Charalambous [8], [9].

III. Examples

A Low-Pass Filter

A normalized 3-component LC low-pass ladder network, terminated with equal load and source resistances of 1 Ω, is considered. An insertion loss of 0.53 dB in the passband 0–1 rad/s and 26.0 dB in the stopband (band edge is 2.5 rad/s is realized by a minimax design without taking tolerances into account. The parameter values are $\phi_1^0 = L_1 = 1.6280$, $\phi_2^0 = C = 1.0897$, and $\phi_3^0 = L_2 = 1.6280$. The chosen set of frequency points is $\Omega = \{0.45, 0.50, 0.55, 1.0, 2.5\}$. $S_u = 1.5$ dB for the passband and $S_l = 25$ dB for the stopband are assigned. Two starting values $\phi_1^0 = 2$, $\phi_2^0 = 1$, $\phi_3^0 = 2$, and $\phi_1^0 = \phi_2^0 = \phi_3^0 = 1.5$ with 1-percent tolerances, have been studied. The first starting point is inside the acceptable region.

The sequential unconstrained minimization techniques (SUMT) method using C_1 of (7) and $c_i = 1$, $i = 1, 2, 3$, yields a solution of $\phi_1^0 = 1.9990$, $\phi_2^0 = 0.9058$, $\phi_3^0 = 1.9990$, and the corresponding tolerances are 9.89, 7.60, and 9.89 percent. Initially, $r = 1$. It is reduced by a factor of ten after each cycle of optimization. The adjoint network technique [10] and the Fletcher method [11] are used in the optimization process. A total of 185 function evaluations were performed to reduce C_1 from 300 to 33.38 for 6 complete cycles. One-hundred thirty-six function evaluations are needed to get the same results by the new nonlinear programming technique. The constants α_{ij}, $i = 1, \cdots, 8$, $j = 1, \cdots, 5$, are set uniformly to 100. p is increased from a starting value of 10–1000 for 2 cycles of optimization.

The SUMT method is not directly applicable with the second starting point which is outside the acceptable region. The same optimal point as before is reached with 105 function evaluations for 1 optimization by the new method. p is 1000 and α_{ij} is 100 for all i and j.

In contrast, if the nominal point is fixed, tolerances of 3.45, 3.18, and 3.45 percent are obtained for the three components.

A Bandpass Filter

The bandpass filter shown in Fig. 1 was studied by Butler [2], Karafin [3], and Pinel and Roberts [4]. An upper specification of 3 dB for the passband and a lower specification of

[1] Selecting, on physical or other grounds, constraints which are likely to be active at the solution to a nonlinear programming problem and discarding the rest can result in faster solution times, as is well known. Ultimately, all the constraints have to be satisfied.

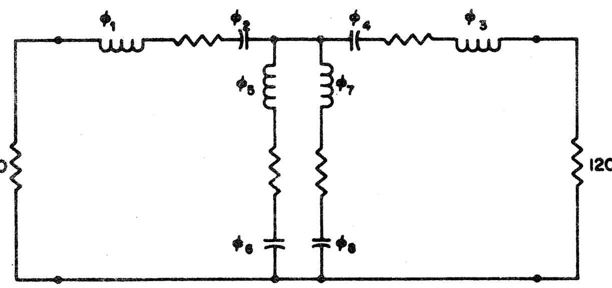

Fig. 1. Bandpass-filter example.

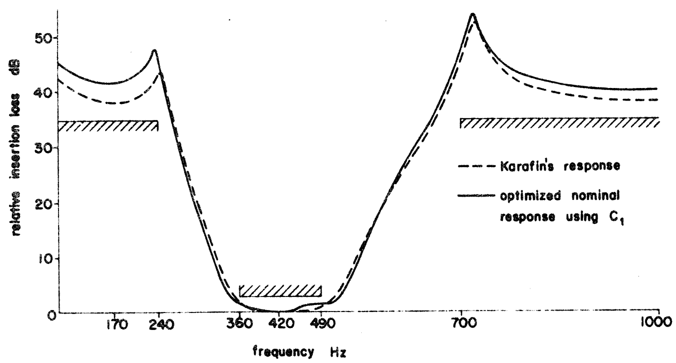

Fig. 2. Bandpass-filter response.

TABLE I
RESULTS FOR THE BANDPASS FILTER

	Karafin [3] Pinel and Roberts [4]	C_1	C_2	C_3
ϕ_1^o	1.824×10^{0}	3.0142×10^{0}	2.3206×10^{0}	2.7682×10^{0}
ϕ_2^o	7.870×10^{-8}	4.9750×10^{-8}	6.3694×10^{-8}	5.2611×10^{-8}
ϕ_3^o	1.824×10^{0}	2.9020×10^{0}	2.3206×10^{0}	2.7682×10^{0}
ϕ_4^o	7.870×10^{-8}	5.0729×10^{-8}	6.3694×10^{-8}	5.2611×10^{-8}
ϕ_5^o	4.272×10^{-1}	8.2836×10^{-1}	6.0517×10^{-1}	7.7895×10^{-1}
ϕ_6^o	9.880×10^{-7}	5.5531×10^{-7}	7.7708×10^{-7}	5.8726×10^{-7}
ϕ_7^o	1.437×10^{-1}	3.0319×10^{-1}	2.1677×10^{-1}	2.5438×10^{-1}
ϕ_8^o	3.400×10^{-7}	1.6377×10^{-7}	2.2630×10^{-7}	1.8981×10^{-7}
t_1	3 , 3.32	6.99	2.29	7.67
t_2	5 , 2.41	6.52	11.26	6.53
t_3	5 , 3.30	6.97	2.29	7.67
t_4	3 , 2.41	6.55	11.26	6.53
t_5	2 , 1.14	4.36	3.30	4.33
t_6	2 , 1.89	5.69	3.02	8.10
t_7	3 , 7.80	6.80	6.61	5.85
t_8	5 , 2.07	5.25	4.40	2.71
Cost	2.60 3.45	1.34	2.06	1.46

35 dB for the stopband relative to 0 dB at a central frequency at 420 Hz are assigned. See Fig. 2. $\Omega = \{360, 490, 170, 240, 700, 1000\}$ in which the first 2 frequencies are assigned to the upper specification and the last 4 to the lower specification. The frequency point of 420 Hz is not included as it is kept at zero. A constant Q is assumed for the four inductors and, therefore, the four corresponding resistances are dependent variables.

Nominal values used by Pinel and Roberts and a $\frac{1}{2}$-percent tolerance for each component are used as a starting point. Parameter values are scaled by normalizing with respect to the central frequency and the load resistance such that the inductors and capacitors will have the same order of magnitude to avoid ill-conditioning. Components ϕ_3 and ϕ_4 are assumed equal to ϕ_1 and ϕ_2, respectively, for the objective function C_2 and C_3. Only 2^6 vertices are taken. Initially, the same assumptions are made for the objective function C_1, but because of some violations a selection of the 2^8 vertices are subsequently taken.[2]

Using the SUMT method, initially, $r = 1$. r was reduced successively by a factor of ten. The adjoint network technique and the Fletcher method are again used in the optimization process. See Table I and Fig. 2 for some results. No more than 10 min on a CDC 6400 are needed to obtain the results for 2^6 vertices. Note that $c_i = 1$, $t_i \triangleq 100\epsilon_i/\phi_i^0$, and the cost is $\sum_{i=1}^{8} 1/t_i$. There are no violations observed for both the Monte Carlo and the worst case analyses at the specified test frequencies assuming 2^8 vertices. The relative insertion loss, however, becomes negative in some instances in the passband. The same assumptions were made as Pinel and Roberts [4] that the component distribution is uniformly concentrated within 5 percent of the extremes of the relative tolerances and 1000 simulations were made for the Monte Carlo analysis.

[2]The algorithm currently being used selects, for each vertex ϕ^j at a particular frequency, another vertex ϕ^i such that the signs of the components of $\phi^i - \phi^0$ are all opposite to the corresponding signs of the components of the gradient vector of the constraint evaluated at ϕ^j and that frequency. This usually leads to a substantially smaller number of constraints to be considered at each frequency during optimization. Periodic updating of the selected vertices and restarting of the optimization process is generally required.

IV. CONCLUSIONS

It has been shown that, by moving the nominal point, a set of larger tolerances can usually be obtained, and that an arbitrary initial design may be used to start the automated scheme. A drawback of this basic scheme is, of course, that a large number of constraints are used. Future work should, it is felt, be concentrated on methods of reducing them. Some preliminary ideas of reducing the number of constraints are currently being tested.[2] A complete solution to the problem is not claimed; however, it may be concluded that our approach is a promising one in network design subject to tolerance considerations.

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REFERENCES

[1] J. W. Bandler, "Optimization of design tolerances using nonlinear programming," in *Proc. 6th Princeton Conf. Information Sciences and Systems* (Princeton, N. J.), Mar. 1972, pp. 655–659; also in

J. Optimiz. Theory Appl., to be published.

[2] E. M. Butler, "Realistic design using large-change sensitivities and performance contours," *IEEE Trans. Circuit Theory (Special Issue on Computer-Aided Circuit Design)*, vol. CT-18, pp. 58–66, Jan. 1971.

[3] B. J. Karafin, "The optimum assignment of component tolerances for electrical networks," *Bell Syst. Tech. J.*, vol. 50, pp. 1225–1242, Apr. 1972.

[4] J. F. Pinel and K. A. Roberts, "Tolerance assignment in linear networks using nonlinear programming," *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 475–479, Sept. 1972.

[5] A. K. Seth, "Electrical network tolerance optimization," Ph.D. dissertation, Univ. Waterloo, Waterloo, Ont., Canada, 1972.

[6] E. De Castro, G. Iuculano, and V. A. Monaco, "Component value spread and network function tolerances: An optimal design procedure," *Alta Freq.*, vol. 40, pp. 867–872, 1971.

[7] A. V. Fiacco and G. P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques.* New York: Wiley, 1968.

[8] J. W. Bandler and C. Charalambous, "A new approach to nonlinear programming," in *Proc. 5th Hawaii Int. Conf. System Sciences* (Honolulu, Hawaii), Jan. 1972, pp. 127–129.

[9] —, "Nonlinear programming using minimax techniques," *J. Optimiz. Theory Appl.*, vol. 13, 1974.

[10] J. W. Bandler and R. E. Seviora, "Current trends in network optimization," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 1159–1170, Dec. 1970.

[11] R. Fletcher, "A new approach to variable metric algorithms," *Comput. J.*, vol. 13, pp. 317–322, Aug. 1970.

[12] J. W. Bandler and P. C. Liu, "Automated network design with optimal tolerances," in *Proc. 1973 Int. Symp. Circuit Theory* (Toronto, Ont., Canada), Apr. 1973, pp. 181–184.

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Some Implications of Biquadratic Functions in the Tolerance Problem

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***Abstract*—The usual assumptions for the tolerance problem in the frequency domain are that the worst cases occur at boundary points of a tolerance region, and that the acceptable region is simply connected. These assumptions are investigated and conditions for validity are given for the class of networks which have bilinear dependence on the parameter of interest. This paper elaborates on an underlying assumption made in a theorem proposed by Bandler.**

I. Introduction

LARGE change sensitivities and worst-case tolerance problems dealing with linear networks in the frequency domain have attracted much attention recently [1]–[5]. The workers in these areas usually assume that the worst cases occur at the vertices or the surfaces of the tolerance region and that the acceptable region is simply connected. Although the assumptions may be true if the tolerances are small certain conditions have to be met.

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The purpose of this paper is to justify these assumptions and state the conditions for the assumptions to be valid. We are interested in the effect of variation of a single parameter on the overall network function. We shall be concerned with the class of networks for which the network function can be expressed as a bilinear function of the parameter of interest [6]–[8]. We use some mathematical concepts [9] to elaborate on an underlying assumption made in a theorem proposed by Bandler [10].

II. The Biquadratic Function

General Properties

Consider the biquadratic function

$$F(\phi) = \frac{N(\phi)}{M(\phi)} = \frac{c\phi^2 + 2d\phi + e}{\phi^2 + 2a\phi + b}. \tag{1}$$

The first derivative of $F(\phi)$ is

$$F'(\phi) = 2\,\frac{(c\phi + d)M(\phi) - (\phi + a)N(\phi)}{M^2(\phi)}. \tag{2}$$

It may be noted that the numerator of (2) is a quadratic function of ϕ which implies that the derivative has, at most,

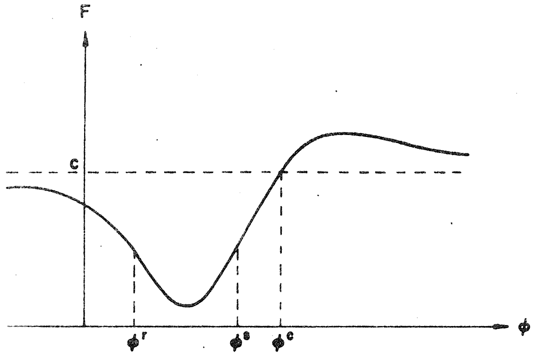


Fig. 1. A general biquadratic function.

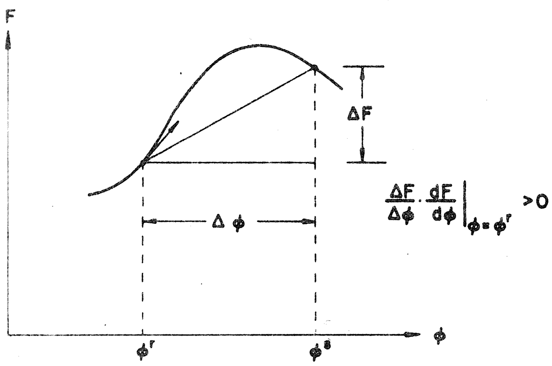


Fig. 2. Illustration of pseudoconcavity on an interval.

two changes of sign for finite values of ϕ. Furthermore, the function value approaches the value of c as $\phi \to \pm\infty$.

Take any two points ϕ^r and ϕ^s and let $\Delta\phi = \phi^s - \phi^r$. $F(\phi^s)$ may be expressed in terms of ϕ^r, $\Delta\phi$, and the coefficients of $N(\phi)$ and $M(\phi)$ as follows:

$$F(\phi^s) = \frac{N(\phi^s)}{M(\phi^s)} = \frac{N(\phi^r) + 2\Delta\phi(c\phi^r + d) + c\Delta\phi^2}{M(\phi^r) + 2\Delta\phi(\phi^r + a) + \Delta\phi^2}. \quad (3)$$

The large change sensitivity

$$\frac{\Delta F}{\Delta\phi} \triangleq \frac{F(\phi^s) - F(\phi^r)}{\phi^s - \phi^r} \quad (4)$$

may be related to the first differential sensitivity $F'(\phi^r)$. We have

$$F(\phi^s) - F(\phi^r) = \frac{2\Delta\phi\{(c\phi^r + d)M(\phi^r) - (\phi^r + a)N(\phi^r)\} - \Delta\phi^2\{N(\phi^r) - cM(\phi^r)\}}{M(\phi^r)M(\phi^s)}$$

$$= \Delta\phi F'(\phi^r)\frac{M(\phi^r)}{M(\phi^s)} - \Delta\phi^2\frac{(F(\phi^r) - c)}{M(\phi^s)}$$

therefore,

$$M(\phi^s)\frac{\Delta F}{\Delta\phi} = F'(\phi^r)M(\phi^r) - \Delta\phi(F(\phi^r) - c). \quad (5)$$

Given a fixed value ϕ^r, we can find uniquely one other point ϕ^s such that $F(\phi^s) = F(\phi^r)$, except when the function $F(\phi^r) = c$, $F'(\phi^r) = 0$, or $M(\phi^r) = 0$. The point ϕ^s is given, using (5) with $\Delta F = 0$, by

$$\phi^s = \phi^r + \frac{F'(\phi^r)M(\phi^r)}{F(\phi^r) - c}. \quad (6)$$

For the case $F'(\phi^r) = 0$, the point ϕ^r is either at the maximum or at the minimum of the function. There is only one finite point ϕ^c such that $F(\phi^c) = c$. The other points with the same value can only be at infinity. See, for example, Fig. 1.

Assumptions

In the following discussion, we shall assume that $M(\phi)$ does not change sign on $[\phi^r,\phi^s]$. We shall also exclude points where $M(\phi) = 0$ since the derivative of $F(\phi)$ is not defined at such points.

III. Lemmas and Theorems

Lemma 1: $F(\phi^r + \lambda(\phi^s - \phi^r)) > \min[F(\phi^r),F(\phi^s)]$ for all λ satisfying $0 < \lambda < 1$ provided that

$$\frac{\Delta F}{\Delta\phi} \cdot \frac{dF}{d\phi}\bigg|_{\phi=\check{\phi}} > 0 \quad (7)$$

where $\Delta F/\Delta\phi$ is given in (4), $\check{\phi}$ is ϕ^r or ϕ^s, whichever corresponds to the lower function value. (Fig. 2 illustrates this lemma.)

Proof: The case $F(\phi^s) > F(\phi^r)$ will be considered first. From (5), we have

$$M(\phi)\frac{F(\phi) - F(\phi^r)}{\lambda\Delta\phi} = F'(\phi^r)M(\phi^r) - \lambda\Delta\phi(F(\phi^r) - c) \quad (8)$$

where

$$\phi = \phi^r + \lambda(\phi^s - \phi^r), 0 < \lambda < 1. \quad (9)$$

If (7) is satisfied, $F'(\phi^r) = dF/d\phi|_{\phi=\phi^r} > 0$ then

$$\frac{1}{M(\phi^s)}[F'(\phi^r)M(\phi^r) - \Delta\phi(F(\phi^r) - c)] > 0$$

implies, since $M(\phi)$ must not change sign, that

$$\frac{1}{M(\phi)}[F'(\phi^r)M(\phi^r) - \lambda\Delta\phi(F(\phi^r) - c)] > 0.$$

Therefore,

$$F(\phi) - F(\phi^r) > 0. \quad (10)$$

Similarly, for the case when $F(\phi^r) > F(\phi^s)$, it is required from (7) that $F'(\phi^s) = dF/d\phi|_{\phi=\phi^s} < 0$. The equations corresponding to (5) and (8) are, respectively,

$$M(\phi^r)\frac{F(\phi^s) - F(\phi^r)}{\Delta\phi} = F'(\phi^s)M(\phi^s) + \Delta\phi(F(\phi^s) - c) \quad (11)$$

and

$$M(\phi)\frac{F(\phi^s) - F(\phi)}{(1-\lambda)\Delta\phi} = F'(\phi^s)M(\phi^s) + (1-\lambda)\Delta\phi(F(\phi^s) - c). \quad (12)$$

Since $\Delta F/\Delta\phi < 0$,

$$\frac{1}{M(\phi^r)}[F'(\phi^s)M(\phi^s) + \Delta\phi(F(\phi^s) - c)] < 0$$

implies, since $M(\phi)$ must not change sign, that

$$\frac{1}{M(\phi)}[F'(\phi^s)M(\phi^s) + (1-\lambda)\Delta\phi(F(\phi^s) - c)] < 0$$

and hence that

$$F(\phi) - F(\phi^s) > 0. \quad (13)$$

Inequalities (10) and (13) are true for all $0 < \lambda < 1$, hence the lemma is proved.

Corollary:

$$F(\phi^r + \lambda(\phi^s - \phi^r)) < \max[F(\phi^r), F(\phi^s)],$$

where $0 < \lambda < 1$, provided that

$$\frac{\Delta F}{\Delta\phi}\cdot\frac{dF}{d\phi}\bigg|_{\phi=\hat{\phi}} > 0 \quad (14)$$

where $\hat{\phi}$ is ϕ^r or ϕ^s whichever corresponds to the higher function value.

The corollary may be proved by defining a new function $G(\phi) = -F(\phi)$ and applying Lemma 1. See Fig. 3 for an illustration. Fig. 4 shows an example where both the lemma and its corollary apply.

Lemma 2: The function $F(\phi)$ is pseudoconcave [9] on the interval $[\phi^r, \phi^s]$ except where $M(\phi) = 0$ if the conditions of Lemma 1 are satisfied.

Proof: Consider the case $F(\phi^s) > F(\phi^r)$. The other case follows a similar argument. Let us assume that the function has more than one turning point in the interval. By the nature of the biquadratic function, there are at most two turning points. If we assume that there are two turning points on $[\phi^r, \phi^s]$, there exist two points $\phi^\alpha = \phi^r + \alpha\Delta\phi$ and $\phi^\beta = \phi^r + \beta\Delta\phi$, where $0 < \alpha < \beta < 1$ such that the following inequalities hold:

$$F(\phi^\alpha) > F(\phi^\beta) \quad (15)$$

and

$$F'(\phi^\beta) > 0. \quad (16)$$

As a direct consequence of Lemma 1 and inequality (16), the following inequalities can be made to hold:

$$F(\phi^s) > F(\phi^\beta) \quad (17)$$

and

$$F(\phi^\beta) > F(\phi^r). \quad (18)$$

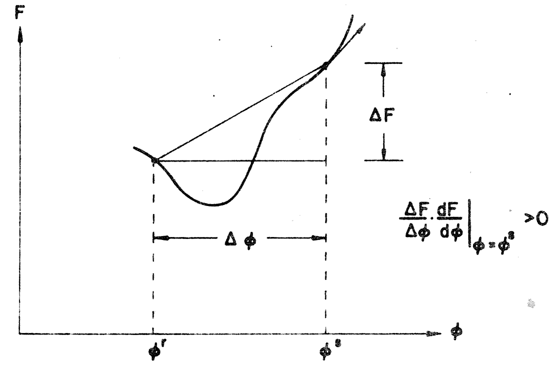


Fig. 3. Illustration of pseudoconvexity on an interval.

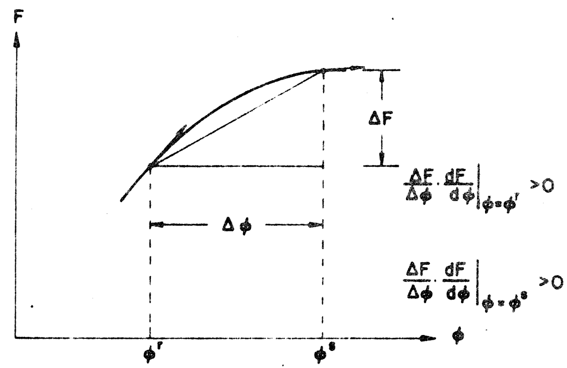


Fig. 4. Illustration of monotonicity on an interval.

Rewriting the function values in terms of $F'(\phi^\beta)$, $F(\phi^\beta)$, and $M(\phi^\beta)$ as in (5), we obtain

$$\frac{1}{M(\phi^\alpha)}[F'(\phi^\beta)M(\phi^\beta) + (\beta - \alpha)\Delta\phi(F(\phi^\beta) - c)] < 0 \quad (19)$$

and

$$\frac{1}{M(\phi^r)}[F'(\phi^\beta)M(\phi^\beta) + \beta\Delta\phi(F(\phi^\beta) - c)] > 0 \quad (20)$$

and

$$\frac{1}{M(\phi^s)}[F'(\phi^\beta)M(\phi^\beta) - (1-\beta)\Delta\phi(F(\phi^\beta) - c)] > 0. \quad (21)$$

Multiply (19) by $M(\phi^\alpha)$, (20) by $M(\phi^r)$, and (21) by $M(\phi^s)$. Subtracting appropriately, we have

$$\alpha\Delta\phi(F(\phi^\beta) - c)\begin{cases} > 0 \text{ for } M > 0 \\ < 0 \text{ for } M < 0 \end{cases}$$

and

$$-(1-\alpha)\Delta\phi(F(\phi^\beta) - c)\begin{cases} > 0 \text{ for } M > 0 \\ < 0 \text{ for } M < 0. \end{cases}$$

The last two pairs of inequalities are inconsistent, therefore the assumption that there are two turning points on the interval is false. $F(\phi)$, $\phi \in [\phi^r, \phi^s]$, is unimodal with a positive derivative at ϕ^r.

Given any two points ϕ^a and ϕ^b, such that $F(\phi^b) > F(\phi^a)$, we will consider that 1) $F'(\phi^a) > 0$, then $\phi^b > \phi^a$

because F is an increasing function between ϕ^r and ϕ^a, and 2) $F'(\phi^a) < 0$, then $\phi^b < \phi^a$ because F is a decreasing function between ϕ^a and ϕ^s. Therefore, in both cases, $F(\phi^b) > F(\phi^a)$ implies $F'(\phi^a)(\phi^b - \phi^a) > 0$, which proves the lemma.

Corollary: The function $F(\phi)$ is pseudoconvex on the interval $[\phi^r, \phi^s]$ except where $M(\phi) = 0$ if the conditions of the corollary to Lemma 1 are satisfied.

Theorem 1: The minimum/maximum of $F(\phi)$, $\phi \in [\phi^r, \phi^s]$, lies on the boundary of the interval if one of the following conditions is satisfied.

$$F'(\phi^r) \gtrless 0 \tag{22a}$$

$$F'(\phi^s) \lessgtr 0 \tag{22b}$$

or

$$F'(\phi^r) > 0,\ F'(\phi^s) > 0 \text{ and } F(\phi^r) < F(\phi^s) \tag{23}$$

or

$$F'(\phi^r) < 0,\ F'(\phi^s) < 0 \text{ and } F(\phi^r) > F(\phi^s). \tag{24}$$

See, for example, Figs. 2–4.

Proof: We will prove the case for the minimum of $F(\phi)$ to be on the boundary of an interval for the conditions of (22a), (23), and (24).

1) Take $\check{\phi} = \phi^r$, then $F(\phi^s) > F(\phi^r)$ and $\Delta F/\Delta\phi > 0$. Using Lemma 1, $F(\phi^r + \lambda(\phi^s - \phi^r)) > \min[F(\phi^r), F(\phi^s)]$, $0 < \lambda < 1$, will hold if $F'(\phi^r) > 0$. This is satisfied in (22a) and (23).

2) Take $\check{\phi} = \phi^s$, then $F(\phi^r) > F(\phi^s)$ and $\Delta F/\Delta\phi < 0$. Using Lemma 1 again, the requirement that $F'(\phi^s) < 0$ will be met in (22a) and (24).

3) Suppose $F(\phi^r) = F(\phi^s)$ and hence $\Delta F/\Delta\phi = 0$. We can find one point ϕ^a such that $F(\phi^a) > F(\phi^r) = F(\phi^s)$. Two subintervals are thus obtained, each of which may be considered under cases 1) and 2).

It should be noted that, from Lemma 2, (22a), (23), and (24) imply pseudoconcavity. From its corollary, (22b), (23), and (24) imply pseudoconvexity.

Let us define the upper and lower specifications by S_{ui}, $i \in I_u$, and S_{li}, $i \in I_l$, respectively, where I_u and I_l are disjoint index sets. An acceptable interval I_a may be defined as

$$I_a \triangleq \{\phi \mid S_{ui} - F_i(\phi) \geq 0,\ i \in I_u,\ F_j(\phi) - S_{lj} \geq 0, j \in I_l\}. \tag{25}$$

Theorem 2: I_a is convex if the condition (22a), (23), or (24) is satisfied by $F_i(\phi)$, for all $i \in I_l$, and condition (22b), (23), or (24) is satisfied by $F_i(\phi)$, for all $i \in I_u$.

Proof: Consider the case $i \in I_l$ and let

$$I_i \triangleq \{\phi \mid F_i(\phi) - S_{li} \geq 0\}, \qquad i \in I_l. \tag{26}$$

Take two different points $\phi^r, \phi^s \in I_i$. If the condition (22a), (23), or (24) is satisfied, then, from Theorem 1,

$$F_i(\phi^\lambda) = F_i(\phi^r + \lambda(\phi^s - \phi^r)) > \min[F_i(\phi^r), F_i(\phi^s)]$$
$$0 < \lambda < 1.$$

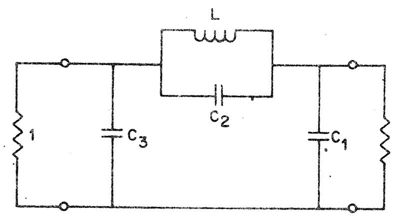


Fig. 5. An LC low-pass example.

Thus

$$F_i(\phi^\lambda) - S_{li} > \min[F_i(\phi^r) - S_{li}, F_i(\phi^s) - S_{li}]$$
$$0 < \lambda < 1.$$

Since

$$\phi^r, \phi^s \in I_i$$

and

$$F_i(\phi^\lambda) - S_{li} > 0 \tag{27}$$

therefore,

$$\phi^\lambda = \phi^r + \lambda(\phi^s - \phi^r) \in I_i. \tag{28}$$

Hence I_i, $i \in I_l$, is a convex interval by definition of a convex set. Similarly, for the case $i \in I_u$, if the condition (22b), (23), or (24) is satisfied, using Theorem 1, we may prove that I_i, $i \in I_u$, is convex.

The intersection of convex sets is convex, and since, by definition,

$$I_a = \bigcap_{\substack{i \in I_l \\ i \in I_u}} I_i,$$

I_a is convex. If any $F(\phi)$ has both upper and lower specifications, the required conditions for a convex acceptable interval are restricted to (23) and (24).

IV. The Network Tolerance Problem

We consider a bilinear network function [6]–[8] of the form $(A + \phi B)/(C + \phi D)$ where A, B, C, and D are, in general, complex and frequency dependent. Thus we assume a function of the form

$$F(\phi) = \left|\frac{A + \phi B}{C + \phi D}\right|^2 = \frac{N(\phi)}{M(\phi)}.$$

In this case, $N, M \geq 0$. The coefficients of (1) are related to the bilinear function as follows:

$$a = \frac{C_r D_r + C_i D_i}{|D|^2},\ b = \frac{|C|^2}{|D|^2},\ c = \frac{|B|^2}{|D|^2},$$

$$d = \frac{A_r B_r + A_i B_i}{|D|^2},\ \text{and } e = \frac{|A|^2}{|D|^2}$$

where the subscripts i and r denote the imaginary and real parts of the complex number.

We have studied the behavior of $|\rho|^2$, the modulus squared of the reflection coefficient ρ, for the LC low-pass

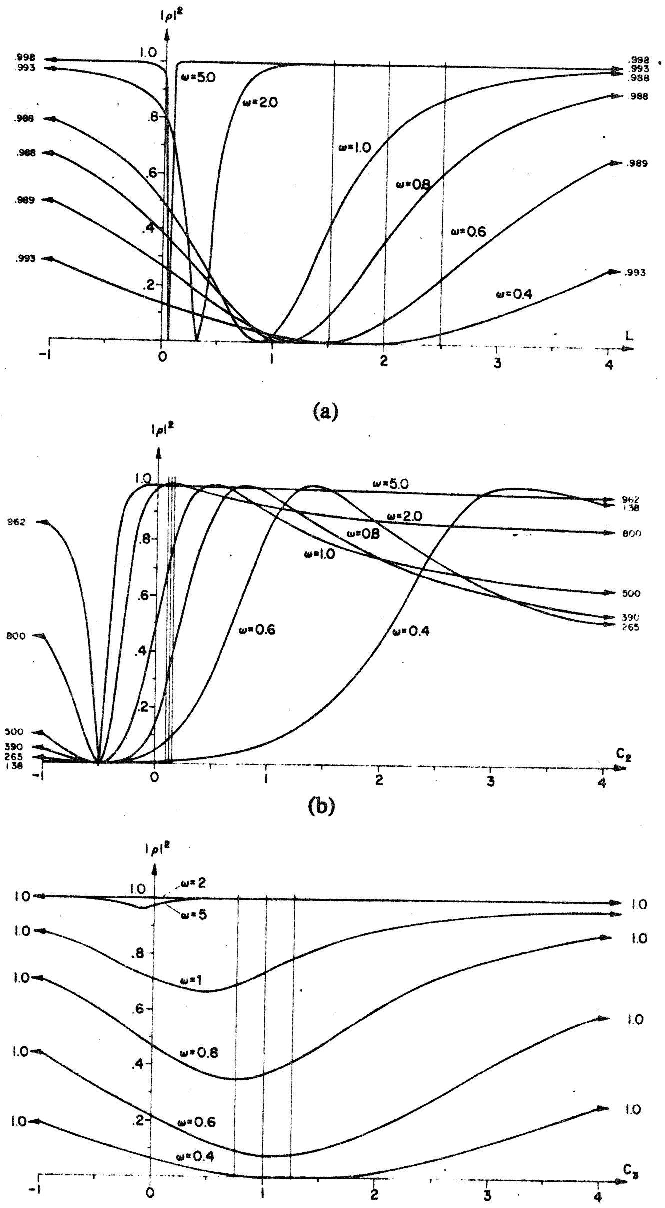


Fig. 6. (a) $|\rho|^2$ vs. L for the example. (b) $|\rho|^2$ vs. C_2 for the example. (c) $|\rho|^2$ vs. C_3 for the example.

filter (Fig. 5) with respect to the variations of L, C_2, and C_3, respectively. Fig. 6 shows some of the curves for different values of frequency. The three vertical lines on each drawing represent the nominal values and the extreme values of ± 25 percent relative tolerance. The nominal values for L, C_2, and C_3 are 2, 0.125, and 1, respectively. $C_1 = C_3$ for reasons of symmetry.

The curves for L and C_2 have two turning points each. For example, at $\omega = 1$,

$$|\rho(L)|^2 = \frac{81L^2 - 144L + 64}{82L^2 - 160L + 128}.$$

The turning points are at $L = 0.889$ and $L = 8.0$ corresponding to the minimum of $|\rho|^2 = 0$ and the maximum of $|\rho|^2 = 1$, respectively. Setting $|\rho|^2 = 81/82 = c$, we can solve for one unique point $L = 4.44$ at which the curve is divided into two parts: $|\rho|^2 \geq 0.988$ for $L \geq 4.44$ and $|\rho|^2 \leq 0.988$ for $L \leq 4.44$. The maximum and minimum function values occur separately in the two parts. The derivatives at the boundary of the tolerance region are both positive, indicating that the curve is monotonic in the region (both pseudoconvex and pseudoconcave).

For parameter C_2 at $\omega = 1$,

$$|\rho(C_2)|^2 = \frac{4C_2^{\,2} + 4C_2 + 1}{8C_2^{\,2} + 2}.$$

The maximum and minimum occur at values of 0.5 and -0.5. At $C_2 = 0$, the curve is again divided into two parts for $|\rho|^2 \geq 0.5$ and $|\rho|^2 \leq 0.5$ for positive or negative C_2 values, respectively.

The curves for C_3 have only one turning point. The biquadratic function is of the form

$$|\rho(C_3)|^2 = \frac{C_3{}^2 + 2aC_3 + e}{C_3{}^2 + 2aC_3 + b}.$$

The minimum occurs at $C_3 = -a$. The curves are pseudoconvex on $(-\infty,\infty)$ for frequencies in both the passband and stopband. For the worst case at stopband frequencies to occur at the boundary of an interval, it is required that the curves corresponding to these frequencies also be pseudoconcave on the interval, i.e., the curves should be monotonic on the interval.

V. Conclusions

The present work deals with a one-dimensional case. Conditions for the worst case to occur at the boundary of an interval are given. The conditions may be used at least to partially justify the usual assumptions for the tolerance problem. The analysis presented here is exact unlike an approximation procedure which makes use of the first- and second-order sensitivities at the nominal point. Bandler [10] has already related a one-dimensional convexity assumption for the acceptable interval to that of the k-dimensional case. It was proven [10] that only vertices of the tolerance region need be tested for the worst case problem if the one-dimensional assumption holds everywhere. Thus Theorem 1 in the present paper involves necessary conditions for the vertices of a k-dimensional region. That networks exist where the vertices do not give worst case results is seen, for example, by studying the $\omega = 2.0$ curve of Fig. 6(a) for L between 0 and 1.

References

[1] S. L. Hakimi and J. B. Cruz, Jr., "Measure of sensitivity for linear systems with large multiple parameter variations," *IRE WESCON Conv. Rec.*, Part 2, pp. 109–115, Aug. 1960.

[2] R. Spence, P. Villalaz, and P. Goddard, "The efficient prediction of tolerance regions, with application to circuit design and device modelling," *Proc. IEEE Int. Symp. Circuit Theory*, London, England, Sept. 1971, pp. 107–108.

[3] E. M. Butler, "Realistic design using large-change sensitivities and performance contours," *IEEE Trans. Circuit Theory*, vol. CT-18, pp. 58–66, Jan. 1971.

[4] J. W. Bandler and P. C. Liu, "Automated network design with optimal tolerances," *IEEE Trans. Circuits and Systems*, vol. CAS-21, pp. 219–222, Mar. 1974.

[5] G. C. Temes and H. J. Orchard, "First-order sensitivity and worst-case analysis of doubly terminated reactance two-ports," *IEEE Trans. Circuit Theory*, vol. CT-20, pp. 567–571, Sept. 1973.

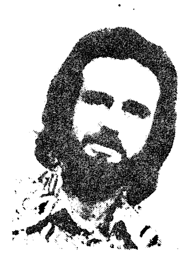
[6] S. R. Parker, E. Peskin, and P. M. Chirlian, "Application of a bilinear theorem to network sensitivity," *IEEE Trans. Circuit Theory*, vol. CT-12, pp. 448–450, Sept. 1965.

[7] E. V. Sorensen, "General relations governing the exact sensitivity of linear networks," *Proc. IEE* (London), vol. 114, pp. 1209–1212, Sept. 1967.

[8] J. K. Fidler and C. Nightingale, "Differential-incremental sensitivity relationships," *Elect. Lett.*, vol. 8, pp. 626–627, Dec. 1972.

[9] O. L. Mangasarian, *Nonlinear Programming.* New York: McGraw-Hill, 1969.

[10] J. W. Bandler, "Optimization of design tolerances using nonlinear programming," *J. Optimiz. Theory Appl.*, vol. 14, pp. 99–114, July 1974.



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Worst Case Network Tolerance Optimization

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***Abstract*—The theory and its implementation in a new user-oriented computer program package is described for solving continuous or discrete worst case tolerance assignment problems simultaneously with the selection of the most favorable nominal design. Basically, the tolerance problem is to ensure that a design subject to specified tolerances will meet performance or other specifications. Our approach, which is believed to be new to the microwave design area, can solve a variety of tolerance and related problems. Dakin's tree search, a new quasi-Newton minimization method, and least *p*th approximation are used. The program itself is organized such that future additions and deletions of performance specifications and constraints, and replacement of cost functions and optimization methods are readily realized. Options and default values are used to enhance flexibility. The full Fortran listing of the program and documentation will be made available.**

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I. INTRODUCTION

A NEW user-oriented computer program package called TOLOPT (TOLerance OPTimization) is presented which can solve continuous or discrete worst case tolerance assignment problems simultaneously with the selection of the most favorable nominal design, taking full advantage of the most recent developments in optimization practice. Our approach, it is believed, is new to the microwave design area. Previous design work has usually been concentrated on obtaining a best nominal design, disregarding the manufacturing tolerances and material uncertainties. Basically, the tolerance assignment problem is to ensure that a design, when fabricated, will meet performance or other specifications.

The package is designed to handle the objective functions, performance specifications, and parameter constraints in a unified manner such that any of the nominal values or tolerances (relative or absolute) can be fixed or varied automatically at the user's discretion. Time-saving techniques for choosing constraints (vertices selection) are incorporated. The routine involved also checks assumptions and performs worst case analyses. The paper also contains a brief discussion of network symmetry and how it may be implemented to further reduce the number of constraints.

The continuous and (optional) discrete optimization methods are programmed in such a way that they may be used as a separate unit. This part, called DISOP2 and incorporating several optional features, is an updated version of DISOPT, which has been applied successfully in many different areas [1]–[3]. Dakin's tree search for discrete problems [4], efficient gradient minimization of functions of many variables by a recent quasi-Newton method [5], and the latest developments in least *p*th approximation by Bandler and Charalambous [6]–[9] are employed. Extrapolation is also featured [10].

Another practical problem which is analogous to the tolerance assignment problem is to determine the optimum component values to a certain number of significant figures, which can be done with DISOP2.

The TOLOPT program is organized in such a way that future additions and deletions of performance specifications and constraints, and replacement of cost functions and optimization methods are readily realized. Any of the two different vertices elimination schemes can be bypassed or replaced by the user. It is felt that the program is particularly flexible in the way that the user may enter at any stage of the problem's solution. The user supplies the network analysis subroutines. With an arbitrary initial acceptable or unacceptable design as a starting point, the program would output the set of nominal component parameters together with a set of optimal tolerances satisfying all the specifications in the worst case sense. The user decides on a continuous solution and/or discrete solutions.

The package, written in Fortran IV and run on a CDC 6400 digital computer, will be made available. Several test examples are presented here to illustrate the theory and practice of the approach.

II. THE TOLERANCE PROBLEM

Introduction [11]–[15]

A design consists of design data of the nominal design point $\boldsymbol{\phi}^0 \triangleq [\phi_1^0\phi_2^0\cdots\phi_k^0]^T$ and a set of associated tolerances $\boldsymbol{\varepsilon} \triangleq [\epsilon_1\epsilon_2\cdots\epsilon_k]^T$, where k is the number of network parameters. Let $I_\phi \triangleq \{1,2,\cdots,k\}$ be the index set for these parameters. We take the ith absolute tolerance as ϵ_i in the discussion in this section; however, the discussion applies also to the relative tolerance $t_i \triangleq \epsilon_i/\phi_i^0$ without any conceptual difference. An outcome of a circuit is any point $\boldsymbol{\phi} \triangleq [\phi_1\phi_2\cdots\phi_k]^T$ in the tolerance region $R_t \triangleq \{\boldsymbol{\phi} \mid \phi_i^0 - \epsilon_i \leq \phi_i \leq \phi_i^0 + \epsilon_i,\ i \in I_\phi\}$. The constraint region R_c is the region of points $\boldsymbol{\phi}$ such that all performance specifications and constraints are satisfied by the circuit. The worst case design requires that $R_t \subseteq R_c$. The optimal worst case design can, therefore, be stated as: minimize some cost function C subject to $R_t \subseteq R_c$.

We need the following assumptions on R_c in order to make the problem tractable.

Assumptions on R_c

1) R_c is not empty.
2) R_c is bounded and simply connected.
3) R_c is at least one-dimensionally convex.

Assumption 1) guarantees there is at least one feasible solution, and assumption 2) is a computational safeguard against infinite parameter values.

We say that R_c is one-dimensionally convex if for all $j \in I_\phi$ [11]

$$\boldsymbol{\phi}^a, \boldsymbol{\phi}^{b(j)} \triangleq \boldsymbol{\phi}^a + \alpha \boldsymbol{u}_j \in R_c \tag{1}$$

where α is some constant and $\boldsymbol{u}_j$ is the jth unit vector, implies that

$$\boldsymbol{\phi} = \boldsymbol{\phi}^a + \lambda(\boldsymbol{\phi}^{b(j)} - \boldsymbol{\phi}^a) \in R_c \tag{2}$$

for all $0 \leq \lambda \leq 1$.

Let us also define the set of vertices $R_v \triangleq \{\boldsymbol{\phi}^1, \boldsymbol{\phi}^2, \cdots, \boldsymbol{\phi}^{2^k}\}$, and the corresponding index set I_v, where

$$\boldsymbol{\phi}^r \triangleq \boldsymbol{\phi}^0 + \boldsymbol{E}\boldsymbol{\mu}(r) \tag{3}$$

$\mu_j(r) \in \{-1,1\}$ and satisfies the relation

$$r = 1 + \sum_{j=1}^{k} \left(\frac{\mu_j(r) + 1}{2}\right) 2^{j-1} \tag{4}$$

where $\boldsymbol{E}$ is a diagonal matrix with ϵ_i as the ith element. Under the foregoing assumptions

$$R_v \subseteq R_c \Rightarrow R_t \subseteq R_c. \tag{5}$$

See [11] for the proof, and Fig. 1 for an illustration of the concepts.

Assumptions on the Constraints

R_c may be defined specifically by a set of constraint functions, namely,

$$R_c \triangleq \{\boldsymbol{\phi} \mid g_i(\boldsymbol{\phi}) \geq 0, i \in I_c\} \tag{6}$$

where I_c is the index set for the functions. Concave constraint functions or, more generally, quasi-concave functions will satisfy assumption 3). The function $g(\boldsymbol{\phi})$ (dropping the subscript i, $i \in I_c$) is said to be quasi-concave in a region if, for all $\boldsymbol{\phi}^a, \boldsymbol{\phi}^b$ in the region,

$$g(\boldsymbol{\phi}^a + \lambda(\boldsymbol{\phi}^b - \boldsymbol{\phi}^a)) \geq \min\left[g(\boldsymbol{\phi}^a), g(\boldsymbol{\phi}^b)\right] \tag{7}$$

for all $0 \leq \lambda \leq 1$. An immediate consequence of (7) is that a region defined as $\{\boldsymbol{\phi} \mid g(\boldsymbol{\phi}) \geq 0\}$ is convex [16]. The intersection of convex regions is also convex, and the multidimensional convexity implies the one-dimensional convexity of assumption 3).

If the point $\boldsymbol{\phi}^b$ in (7) is defined as in (1), then the function $g(\boldsymbol{\phi})$ satisfying (7) will be called a one-dimensional quasi-concave function. The region defined by these functions is one-dimensionally convex. Assumption 3) is satisfied [17]. Throughout the following discussions, we will assume the functions to have this less restrictive property.

Under the foregoing assumptions we have the nonlinear programming problem: minimize C subject to $g_i(\boldsymbol{\phi}^r) \geq 0$ for all $\boldsymbol{\phi}^r \in R_v$, $i \in I_c$.

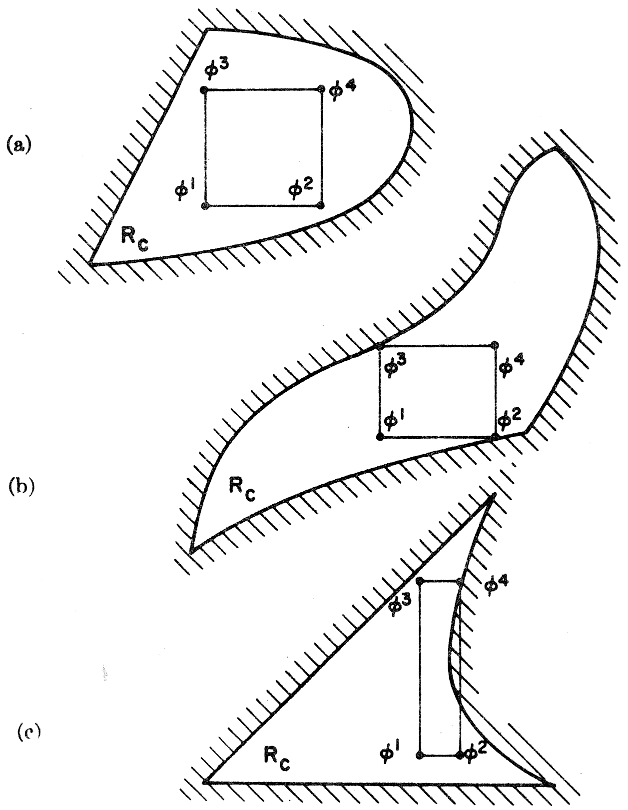

Fig. 1. Possible regions R_c. (a) R_v is a subset of R_c implies that R_t is a subset of R_c. (b) R_v is a subset of R_c implies that R_t is a subset of R_c. (c) R_v is a subset of R_c does not imply that R_t is a subset of R_c.

Conditions for Monotonicity

Given a differentiable one-dimensional quasi-concave function $g(\phi)$ (see, for example, Fig. 2), the function is monotonic with respect to ϕ on an interval $[\phi^a,\phi^b]$ if $\operatorname{sgn}(g'(\phi^a)) = \operatorname{sgn}(g'(\phi^b))$. Furthermore, the minimum of $g(\phi)$ is at $\phi = \frac{1}{2}[\phi^a + \phi^b - \operatorname{sgn}(g'(\phi^a))(\phi^b - \phi^a)]$. This may be proved as follows.

Consider the case $\operatorname{sgn}(g'(\phi^a)) = \operatorname{sgn}(g'(\phi^b)) > 0$. Suppose $g(\phi)$ is not monotonic. Then there exist two points $\phi^1,\phi^2 \in (\phi^a,\phi^b),\phi^2 > \phi^1$ such that $g'(\phi^1) < 0$ and $g(\phi^2) > g(\phi^1)$. Thus $g(\phi^1 + \lambda(\phi^2 - \phi^1))$ for some $0 < \lambda < 1$ is smaller than $g(\phi^1)$, which contradicts (7). The assumption that $g(\phi)$ is not monotonic is wrong, hence $g(\phi)$ is monotonic. Furthermore, it is nondecreasing on $[\phi^a,\phi^b]$. The minimum is at ϕ^a.

Similarly, it may be proved that the case $\operatorname{sgn}(g'(\phi^a)) = \operatorname{sgn}(g'(\phi^b)) < 0$ implies monotonicity with $g(\phi)$ nonincreasing on $[\phi^a,\phi^b]$. The minimum is at ϕ^b.

Implications of Monotonicity

Suppose g_i is monotonic in the same direction with respect to ϕ_j throughout R_t. Then the minimum of g_i is on the hyperplane $\phi_j = \phi_j^0 - \epsilon_j \operatorname{sgn}(\partial g_i/\partial \phi_j)$. Hence only those vertices which lie on that hyperplane need to be constrained. In general, if there are l variables with respect to which the function g_i is monotonic in this way, the 2^{k-l} vertices lying on the intersection of the hyperplanes are constrained. In the case where $l = k$, the vertex of minimum g occurs at $\boldsymbol{\phi}^r$ where

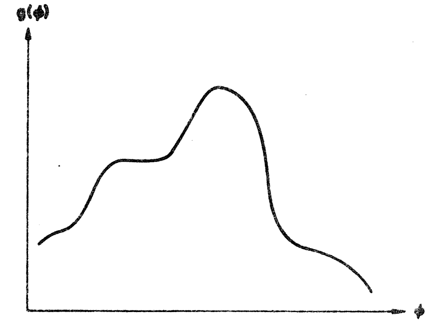

Fig. 2. A one-dimensional quasi-concave function.

$$\phi_j^r = \phi_j^0 - \epsilon_j \operatorname{sgn}\left(\frac{\partial g_i}{\partial \phi_j}\right) \quad \text{for all} \quad j \in I_\phi. \tag{8}$$

Let the set that contains the critical vertices be denoted by $R_v'(i) \subseteq R_v$. The modified problem is: minimize C subject to $g_i(\boldsymbol{\phi}^r) \geq 0$, for all $\boldsymbol{\phi}^r \in R_v'(i)$, $i \in I_c$.

The Vertices Elimination Schemes

Various schemes may be developed to identify or to predict the most critical vertices that are likely to give rise to active constraints. Our proposed schemes will eliminate all but one vertex for each constraint function in the most favorable conditions. In this case, the subsequent computational effort for the optimization procedure is comparable to the linearization technique commonly used. When monotonicity assumptions are not sufficient to describe the function behavior, our scheme will increase the number of vertices until, at worst, all the 2^k vertices are included.

In principle, our schemes may be stated as follows:

Step 1) systematic generation, for positive α, of sets of points

$$\boldsymbol{\phi}^a,\boldsymbol{\phi}^{b(j)} = \boldsymbol{\phi}^a + \alpha \mathbf{u}_j$$

Step 2) evaluation of the function values and the partial derivatives at these points.

Step 3) If

$$\operatorname{sgn}\left(\left.\frac{\partial g_i}{\partial \phi_j}\right|_{\phi=\phi^a}\right) = \operatorname{sgn}\left(\left.\frac{\partial g_i}{\partial \phi_j}\right|_{\phi=\phi^{b(j)}}\right)$$

eliminate the vertices $\boldsymbol{\phi}^r \in R_v$ on the hyperplane

$$\phi_j = \phi_j^0 + \epsilon_j \operatorname{sgn}\left(\frac{\partial g_i}{\partial \phi_j}\right).$$

If

$$\operatorname{sgn}\left(\left.\frac{\partial g_i}{\partial \phi_j}\right|_{\phi=\phi^a}\right) < 0 \quad \text{and} \quad \operatorname{sgn}\left(\left.\frac{\partial g_i}{\partial \phi_j}\right|_{\phi=\phi^{b(j)}}\right) > 0$$

note that the quasi-concavity assumption is violated.

Comments

1) We have investigated and implemented two methods for step 1), involving: a) $\boldsymbol{\phi}^a = \boldsymbol{\phi}^0 - \epsilon_j \mathbf{u}_j$ and $\boldsymbol{\phi}^b = \boldsymbol{\phi}^0 + \epsilon_j \mathbf{u}_j$, for all $j \in I_\phi$; b) all the vertices of R_t. A special case which we do not consider in this paper is for $\boldsymbol{\phi}^a = \boldsymbol{\phi}^b$ in

step 1), in which case the first part of step 3) is applicable. $R_v'(i)$ contains only one vertex.

2) It is possible to further eliminate some vertices by considering the relative magnitudes of $g_i(\boldsymbol{\phi}^r)$.

3) For method b), a worst case check can be accomplished as a by-product of the vertices elimination scheme since function values are computed at each vertex.

4) The schemes are based on local information. R_v' should be updated at suitable intervals.

Symmetry

A circuit designer should exploit symmetry to reduce computation time. The following is an example of how it may be done in the tolerance problem.

A function is said to be symmetrical with respect to $\boldsymbol{S}$ in a region if

$$g(\boldsymbol{S\phi}) = g(\boldsymbol{\phi}) \tag{9}$$

where S is a matrix obtained by interchanging suitable rows of a unit matrix [18]. It has exactly one entry of 1 in each row and in each column, all other entries being 0.

A common physical symmetry configuration is a mirror-image symmetry with respect to a center line. The $\boldsymbol{S}$ matrix in this case is

$$\boldsymbol{S} = \begin{bmatrix} 0 & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & 0 \end{bmatrix}. \tag{10}$$

Postmultiplication of a matrix $\boldsymbol{A}$ by any $\boldsymbol{S}$ simply permutes the columns of A, and premultiplication of $\boldsymbol{A}$ permutes the rows of $\boldsymbol{A}$. $\boldsymbol{SS}^T = \boldsymbol{1}$, and $\boldsymbol{S}^T\boldsymbol{DS} = \boldsymbol{D}_s$, where D is a diagonal matrix and $\boldsymbol{D}_s$ is also a diagonal matrix with diagonal entries permuted.

Consider symmetrical $\boldsymbol{S}$, $\boldsymbol{\phi}^0$, and $\boldsymbol{\varepsilon}$. By this we imply

$$\boldsymbol{S}(\boldsymbol{SA}) = \boldsymbol{A} \tag{11}$$

$$\boldsymbol{S\phi}^0 = \boldsymbol{\phi}^0 \tag{12}$$

and

$$\boldsymbol{S}^T\boldsymbol{ES} = \boldsymbol{E}. \tag{13}$$

Let us premultiply the rth vertex from (3) by $\boldsymbol{S}$, giving

$$\begin{aligned} \boldsymbol{S\phi}^r &= \boldsymbol{S\phi}^0 + \boldsymbol{S}(\boldsymbol{E\mu}(r)), \quad r \in I_v \\ &= \boldsymbol{\phi}^0 + \boldsymbol{S}(\boldsymbol{S}^T\boldsymbol{ES\mu}(r)) \\ &= \boldsymbol{\phi}^0 + \boldsymbol{ES\mu}(r). \end{aligned} \tag{14}$$

Now $\boldsymbol{S\mu}(r)$ is another vector with $+1$ and -1 entries. Let it be denoted by $\boldsymbol{\mu}(s)$, $s \in I_v$. In some cases $\boldsymbol{\mu}(r)$ is identical to $\boldsymbol{\mu}(s)$ if the vector is symmetrical. In other cases $\boldsymbol{\mu}(r) \neq \boldsymbol{\mu}(s)$. In all cases

$$\boldsymbol{S\phi}^r = \boldsymbol{\phi}^s. \tag{15}$$

Making use of the symmetrical property of g

$$g(\boldsymbol{S\phi}^r) = g(\boldsymbol{\phi}^r) = g(\boldsymbol{\phi}^s). \tag{16}$$

Let the number of symmetrical vectors $\boldsymbol{\mu}(r)$ and the number of pairs of nonsymmetrical $\boldsymbol{\mu}(r)$ and $\boldsymbol{\mu}(s)$ be denoted by $N(r = s)$ and $N(r \neq s)$, respectively. Then

$$N(r = s) = 2^{k-k_s}, \qquad 2k_s \leq k \tag{17}$$

and

$$N(r \neq s) = (2^k - 2^{k-k_s})/2, \qquad 2k_s \leq k \tag{18}$$

where k_s is the number of pairs of symmetrical components. Therefore, there are $N(r = s) + N(r \neq s)$ effective vertices as compared to 2^k topological vertices. Take, for example, $k = 6$ and $k_s = 3$; only 36 function evaluations are required for all the 64 vertices.

The above discussion and results may be used to reduce computation time. However, in general, it is not certain that a nominal design without tolerances yielding a symmetrical solution will imply a symmetrical optimal solution with tolerances either in the continuous or in the discrete cases.

III. OPTIMIZATION METHODS

Nonlinear Programming Problem

After eliminating the inactive vertices and constraints as discussed in Section II, the tolerance problem takes the form

$$\text{minimize } f \triangleq f(\boldsymbol{x}) \tag{19}$$

subject to

$$g_i(\boldsymbol{x}) \geq 0, \qquad i = 1,2,\cdots,m \tag{20}$$

where f is the chosen objective function (see Section IV). The vector $\boldsymbol{x}$ represents a set of up to $2k$ design variables which include the nominal values, and the relative and/or absolute tolerances of the network components. The constraint functions $g_1(\boldsymbol{x}),g_2(\boldsymbol{x}),\cdots,g_m(\boldsymbol{x})$ comprise the selected response specifications, component bounds, and any other constraints. The constraints are renumbered from 1 to m for simplicity.

Constraint Transformation

Recently, Bandler and Charalambous have proposed a minimax approach [8] to transform a nonlinear programming problem into an unconstrained objective. The method involves minimizing the function

$$V(\boldsymbol{x},\alpha) = \max_{1 \leq i \leq m} [f(\boldsymbol{x}), f(\boldsymbol{x}) - \alpha g_i(\boldsymbol{x})] \quad \text{where} \quad \alpha > 0. \tag{21}$$

A sufficiently large value of α must be chosen to satisfy the inequality

$$\frac{1}{\alpha}\sum_{i=1}^{m} u_i < 1 \tag{22}$$

where the u_i's are the Kuhn–Tucker multipliers at the optimum. This approach compares favorably with the well-regarded Fiacco–McCormick technique [19].

Several least pth optimization algorithms are available for solving the resulting minimax problem. The function to be minimized is computed in the present paper as

$$U(x) \leftarrow (M(x) - \epsilon)\left(\sum_{j \in J}\left(\frac{e_j(x) - \epsilon}{M(x) - \epsilon}\right)^q\right)^{1/q} \qquad (23)$$

where

$$M(x) \leftarrow \max_{j \in J} e_j(x)$$

$$\epsilon \leftarrow \begin{cases} 0 & \text{for } M(x) \neq 0 \\ \text{small positive number for } M(x) = 0 \end{cases}$$

$$q \leftarrow p \operatorname{sgn}(M(x) - \epsilon)$$

$$p > 1$$

and if

$$M(x) \begin{cases} >0, J \leftarrow \{ j \mid e_j(x) > 0 \} \\ <0, J \leftarrow \{1,2,\cdots,m+1\}. \end{cases}$$

The definition of the e_j, the appropriate value(s) of p, and the convergence features of the algorithms are summarized in Table I (algorithms 1–4).

Another approach to nonlinear programming which utilizes a least pth objective is also detailed in Table I (algorithm 5). It is a modification of an existing nonparametric exterior-point algorithm described by Lootsma [20].

Existence of a Feasible Solution

The existence of a feasible solution can be detected by minimizing (23) when

$$e_j \leftarrow \begin{cases} -g_j, & j = 1,2,\cdots,m \\ f - \bar{f}, & j = m+1 \end{cases}$$

where $\bar{f}$ is an upper bound on f. A nonpositive value of M at the minimum, or even before the minimum is reached indicates that a feasible solution exists. Otherwise, no feasible solution satisfying the current set of constraints and the upper bound on the objective function value is perceivable. Only one single optimization with a small value of p greater than unity is required.

Unconstrained Minimization Method

Gradient unconstrained minimization methods have become very popular because of their reported efficiency.

TABLE I
THE OPTIONAL LEAST pTH ALGORITHMS

Algorithm	Definition of e_i	Convergence feature	Value(s) of p	Number of optimizations
1	$e_i \leftarrow \begin{cases} f - \alpha g_i, i=1,2,\ldots,m \\ f, i = m+1 \end{cases}$		Large	1
2	where $\alpha > 0$	Increment of p	Increasing	Implied by the sequence but superceded by the stopping quantity
3		Extrapolation	Geometrically increasing	
4	$e_i \leftarrow \begin{cases} f - \alpha g_i - \xi^r, i=1,2,\ldots,m \\ f - \xi^r, i = m+1 \end{cases}$ where $\alpha > 0$ $\xi^r \leftarrow \begin{cases} \min[0, M^0 + \gamma], r=1 \\ \check{M}^{r-1} + \gamma, r > 1 \end{cases}$ r indicates the optimization number γ is a small positive quantity	Updating of ξ^r	Finite	Depend on the stopping quantity
5	$e_i \leftarrow \begin{cases} -g_i, i=1,2,\ldots,m \\ f - t^r, i = m+1 \end{cases}$ where $t^r \leftarrow \begin{cases} \text{optimistic estimate of } \check{f}, r = 1 \\ t^{r-1} + \check{U}^{r-1}, r > 1 \end{cases}$ r is defined as in 4	Updating of t^r		

One such program is the Fortran subroutine, which utilizes first derivatives, implemented by Fletcher [5]. The method used belongs to the class of quasi-Newton methods. The direction of search s^j at the jth iteration is calculated by solving the set of equations

$$B^j s^j = -\nabla U(x^j) \quad (24)$$

where B^j is an approximation to the Hessian matrix G of U, ∇U is the gradient vector, and x^j is the estimate of the solution at the jth iteration.

As proposed by Gill and Murray [21], the matrix B^j is factorized as

$$B^j = L^j D^j L^{jT} \quad (25)$$

where L is a lower unit triangular matrix and D is a diagonal matrix. It is important that B^j must always be kept positive definite, and with the above factorization, it is easy to guarantee this by ensuring $d_{ii} > 0$ for all i.

A modification of the earlier switching strategy of Fletcher [22] is used to determine the choice of the correction formula for the approximate Hessian matrix. The Davidon–Fletcher–Powell (DFP) formula is used if

$$\delta^T L D L^T \delta < \delta^T (\nabla U(x^{j+1}) - \nabla U(x^j))$$

where

$$\delta = x^{j+1} - x^j.$$

Otherwise, the complementary DFP formula is used.

The minimization will be terminated when $|x_i^{j+1} - x_i^j|$ is less than a prescribed small quantity for all i.

Discrete Optimization

In practical design, a discrete solution may be more realistic than a continuous solution. In network tolerance-optimization problems, very often only components of certain discrete values, or having certain discrete tolerances are available on the market. At present, a general strategy for solving a nonlinear discrete programming problem is the tree-search algorithm due to Dakin [4].

Dakin's integer tree-search algorithm first finds a solution to the continuous problem. If this solution happens to be integral, the integer problem is solved. If it is not, then at least one of the integer variables, e.g., x_i, is nonintegral and assumes a value x_i^*, say, in this solution. The range

$$[x_i^*] < x_i < [x_i^*] + 1$$

where $[x_i^*]$ is the largest integer value included in x_i^*, is inadmissible, and consequently we may divide all solutions to the given problem into two nonoverlapping groups, namely, 1) solutions in which

$$x_i \leq [x_i^*]$$

2) solutions in which

$$x_i \geq [x_i^*] + 1.$$

Each of the constraints is added to the continuous problem sequentially, and the corresponding augmented problems are solved. The procedure is repeated for each of the two solutions so obtained. Each resulting nonlinear programming problem thus constitutes a node, and from each node two branches may emanate. A node will be fathomed if the following happen: 1) the solution is integral; 2) no feasible solution for the current set of constraints is achievable; 3) the current optimum solution is worse than the best integer solution obtained so far. The search stops when all the nodes are fathomed.

It seems, then, that the most efficient way of searching would be to branch, at each stage, from the node with the lowest $f(x)$ value. This would minimize the searching of unlikely subtrees. To do this, all information about a node has to be retained for comparison; this may require cumbersome housekeeping and excessive storage for computer implementation. One way of compromising is to search the tree in an orderly manner; each branch is followed until it is fathomed.

The tree is not, in general, unique for a given problem. The tree structure depends on the order of partitioning on the integer variables used. The amount of computation may be vastly different for different trees.

For the case of discrete programming problems subject to uniform quantization step sizes, the Dakin algorithm is modified as follows: let x_i be the discrete variable which assumes a nondiscrete solution x_i^*; and let q_i be the corresponding quantization step; then the two variable constraints added sequentially after each node become

$$x_i \geq [x_i^*/q_i]q_i + q_i$$

and

$$x_i \leq [x_i^*/q_i]q_i.$$

The integer problem is thus a special case of the discrete problem with $q_i = 1$, $i = 1,2,\cdots,n$, where n is the number of discrete variables.

If, however, a finite set of discrete values given by

$$D_i = \{d_{i1}, d_{i2}, \cdots, d_{ij}, d_{i(j+1)}, \cdots, d_{iu}\}, \qquad i = 1,2,\cdots,n$$

is imposed upon each of the discrete variables, the variable constraints are then added according to the following rules.

1) If

$$d_{ij} < x_i^* < d_{i(j+1)}$$

then add the two constraints

$$x_i \leq d_{ij}$$

and

$$x_i \geq d_{i(j+1)}$$

sequentially.

2) If

$$x_i^* < d_{i1}$$

only add the constraint

$$x_i \geq d_{i1}.$$

3) If

$$x_i^* > d_{iu}$$

only add the constraint

$$x_i \leq d_{iu}.$$

The resulting nonlinear programming problem at each node is solved by one of the algorithms described earlier. The feasibility check is particularly useful here since the additional variable constraints may conflict with the original constraints on the continuous problem. An upper bound $\bar{f}$, on $f(\boldsymbol{x})$, if not specified, may be taken as the current best discrete solution. For a discrete problem, the best solution among all the discrete solutions given by letting variables assume combinations of the nearest upper and lower discrete values (if they exist) may be taken as the initial upper bound on $f(\boldsymbol{x})$.

The new variable constraint added at each node excludes the preceding optimum point from the current solution space and the constraint is therefore active if the function is locally unimodal. Thus the value of the variable under the new constraint may be optionally fixed on the constraint boundary during the next optimization. See Fig. 3 for illustrations of the search procedure and a tree structure.

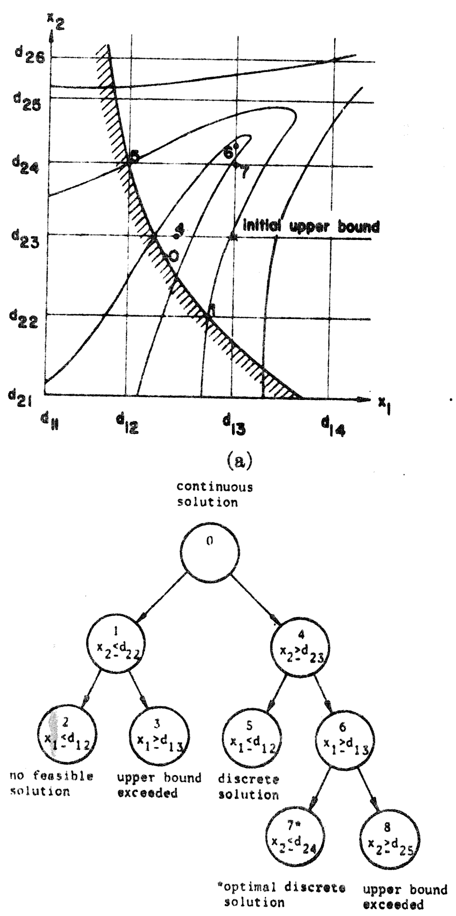

Fig. 3. An illustration of the search for discrete solutions. (a) Contours of a function of two variables with grid and intermediate solutions. (b) The tree structure.

IV. IMPLEMENTATION OF THE TOLERANCE PROBLEM

The Overall Structure of TOLOPT

Fig. 4 displays a block diagram of the principal subprograms comprising the TOLOPT program. A brief description of these subprograms is given in this section.

TOLOPT is the subroutine called by the user. It organizes input data and coordinates other subprograms. Subroutine DISOP2 is a general program for continuous and discrete nonlinear programming problems. Subroutine VERTST eliminates the inactive vertices of the tolerance region. Subroutine CONSTR sets up the constraint functions based on the response specifications, component bounds, and other constraints supplied in the user subroutine USERCN. Subroutine COSTFN computes the cost function. The user has the option of supplying his own subroutine to define other cost functions. The user-supplied subroutine NETWRK returns the network responses and the partial derivatives.

Table II is a summary of the features and options currently incorporated in TOLOPT.

Some components of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\phi}^0$ may be fixed which do not enter into the optimization parameters $\boldsymbol{x}$. The user supplies the initial values of the tolerances (relative or absolute) and the nominals with an appropriate vector to indicate whether they are fixed or variable, relative or absolute. The program will assign those variable components to vector $\boldsymbol{x}$.

The Objective Function

The objective function we have investigated and implemented is [11]–[13]

$$C = \sum_i \frac{c_i}{x_i} \qquad (26)$$

where x_i is either ϵ_i or t_i, and the c_i are some suitable weighting factors supplied by the user. The default value is one. To avoid negative tolerances we let $x_i = x_i'^2$, where x_i' is taken as a new variable replacing x_i.

Vertices Selection and Constraints

Two schemes of increasing complexity are programmed in the subroutine. The user decides on the maximum allowable calls for each scheme, starting with the simple one. He may, if he wishes, bypass either one or even bypass

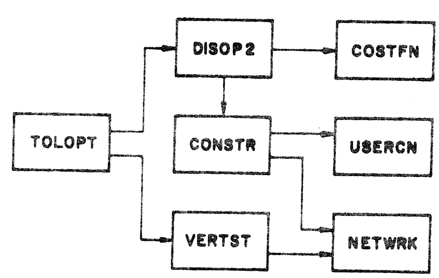

Fig. 4. The overall structure of TOLOPT. The user is responsible for NETWRK and USERCN.

TABLE II

SUMMARY OF FEATURES, OPTIONS, PARAMETERS, AND SUBROUTINES REQUIRED

Features	Type	Options	Parameters†/subroutines
Design parameters	Nominal and tolerance	Variable or fixed Relative or absolute tolerances	Number of parameters Starting values Indication for fixed or variable parameters and relative or absolute tolerances
Objective function	Cost	Reciprocal of relative and/or absolute tolerances	Weighting factors
		Other	Subroutine to define the objective function and its partial derivatives
Vertices selection*	Gradient direction strategy		Maximum allowable number of calls of the vertices selection subroutine
Constraints	Specifications on functions of network parameters	Upper and/or lower	Sample points (e.g., frequency) Specifications Subroutine to calculate, for example, the network response and its partial derivatives (NETWRK)
	Network parameter bounds		Upper and lower bounds
	Other constraints	As many as required	Subroutine to define the constraint functions and their partial derivatives (USERCN)
Nonlinear programming	Bandler-Charalambous minimax	Least pth optimization algorithms See Table I	Controlling parameter α Value(s) of p Test quantities for termination
	Exterior-point		Optimistic estimate of objective function Value of p
Solution feasibility check*	Least pth	Discrete problem Continuous and discrete problem	Constraint violation tolerance Value of p
Unconstrained minimization method	Quasi-Newton	Gradient checking at starting point by numerical perturbation	Number of function evaluations allowed Estimate of lower bound on least pth objective Test quantities for termination
Discrete optimization*	Dakin tree-search	Reduction of dimensionality User supplied or program determined initial upper bound on objective function Single or multiple optimum discrete solution Uniform or nonuniform quantization step sizes	Upper bound on objective function Maximum permissible number of nodes Discrete values on step sizes Number of discrete variables Discrete value tolerance Order of partitioning Indication for discrete variables

† Parameters associated with the options are not explicitly listed.
* These features are optional and may be bypassed.

the whole routine by supplying his own vertices, or set up his own strategy of vertices selection routine.

The user supplies three sets of numbers, the elements of which correspond to the controlling parameter ψ_i, the specification S_i, and the weighting factor w_i. ψ_i is an independent parameter, e.g., frequency, or any number to identify a particular function. w_i is given by

$$w_i = \begin{cases} +1 & \text{if } S_i \text{ is an upper specification} \\ -1 & \text{if } S_i \text{ is a lower specification.} \end{cases}$$

If both upper and lower specifications are assigned to one point, the user can treat it as two points, one with an upper specification and the other with a lower specification. The theory presented earlier will apply in this case under the monotonicity restrictions.

The scheme will, for each i, select a set of appropriate μ. Corresponding to each μ, the values ψ_i, S_i, and w_i are stored. This information is outputted and used for forming the constraint functions.

The constraints associated with response specifications are of the form

$$g = w(S - F) \geq 0 \tag{27}$$

with appropriate subscripts, where F is the circuit response function of ϕ and ψ, and w and S are as before.

The parameter constraints are

$$\phi_j^0 - \epsilon_j - \phi_{lj} \geq 0 \quad (28)$$

and

$$\phi_{uj} - \phi_j^0 - \epsilon_j \geq 0 \quad (29)$$

where ϕ_{uj} and ϕ_{lj}, $j \in I_\phi$ are the user-supplied upper and lower bounds.

Updating Procedure

Once the constraints have been selected, optimization is started with a small value of p and α ($p = \alpha = 10$ as default values). We have decided to call the routine for updating constraints whenever the α value is updated or the optimization with the initial value of p is completed, until the maximum number of calls is exceeded, or when there is no change of values for consecutive calls. For updating the values, we add new values of μ to the existing ones without any eliminations. This may not be the most efficient way, but it will be stable.

V. EXAMPLES

Example 1

To illustrate the basic ideas of different cost functions, variable nominal point, and continuous and discrete solutions, a two-section 10:1 quarter-wave transformer is considered [23]. Table III shows the specifications of the design and the result of a minimax solution without tolerances. Fig. 5 shows the contours of $\max_i |\rho_i|$ over the range of sample points. The region R_c satisfies all the assumptions. Two cost functions, namely, $C_1 = 1/t_{Z_1} + 1/t_{Z_2}$ and $C_2 = 1/\epsilon_{Z_1} + 1/\epsilon_{Z_2}$ are optimized for the continuous case. The optimal solution with a fixed nominal point at **a** yields a continuous tolerance set of 8.3 percent and 7.7 percent for C_1. For the same function with a variable nominal point, the set is {12.8,12.8} percent with nominal solution at **b**. The tolerance set for C_2 is {15.0,9.1} percent with nominal solution at **c**. **d** and **e** correspond to the two discrete solutions with tolerance 10 percent and 15 percent. This example depicts an important fact that an optimal discrete solution cannot always be obtained by rounding or truncating the continuous tolerances to the discrete values. The nominal points must be allowed to move.

Example 2

To illustrate the branch and bound strategy, a 3-component LC low-pass filter is studied [12]. The circuit is shown in Fig. 6. Table IV summarizes the specifications and Table V lists the results for both the continuous and the discrete solutions. Two different tree structures are shown in Figs. 7 and 8. This example illustrates that the tree structure, and hence the computational effort, is dependent upon the order of partitioning on the discrete variables. An asterisk attached to the node denotes an optimum discrete solution. It may be noted that one of the discrete solutions, as well as the continuous solution, yields symmetrical results, although symmetry is not assumed in the formulation of the problem.

TABLE III
TWO-SECTION 10:1 QUARTER-WAVE TRANSFORMER

Relative Bandwidth	Sample Points (GHz)	Reflection Coefficient Specification	Type
100%	0.5, 0.6, ..., 1.5	0.55	upper
Minimax solution (no tolerances) $\|\rho\| = 0.4286$			

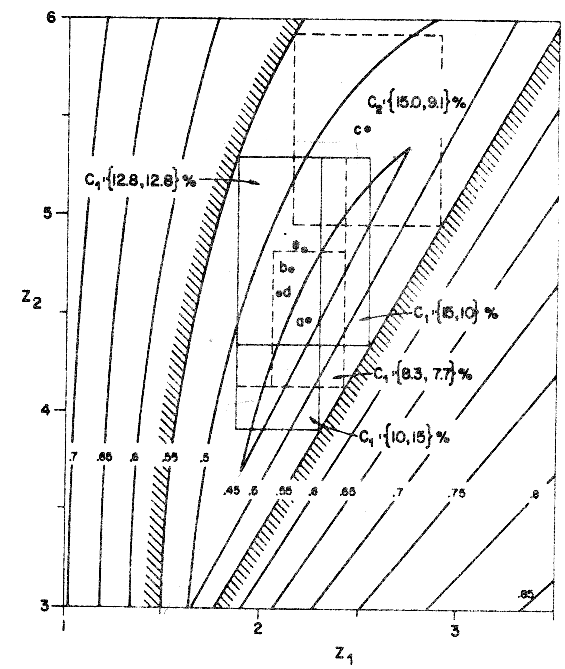

Fig. 5. Contours of max $|\rho_i|$ with respect to Z_1 and Z_2 for example 1 indicating a number of relevant solution points (see text).

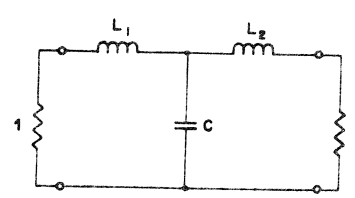

Fig. 6. The circuit for example 2.

Example 3

Consider a five-section cascaded transmission-line low-pass filter with characteristic impedances fixed at the values

$$Z_1^0 = Z_3^0 = Z_5^0 = 0.2$$

$$Z_2^0 = Z_4^0 = 5.0$$

and terminated in unity resistances [1], [6]. See Table VI for the specifications. The length units are normalized with respect to $l_q = c/4f_0$, where $f_0 = 1$ GHz. Two problems are presented here.

1) A uniform 1-percent relative tolerance is allowed for each impedance. Maximize the absolute tolerances on the

TABLE IV

LC LOW-PASS FILTER

Frequency Range (rad/s)	Sample Points (rad/s)	Insertion Loss Specification (dB)	Type
0 - 1	0.5, 0.55, 0.6, 1.0	1.5	upper (passband)
2.5	2.5	25	lower (stopband)

Minimax solution (no tolerances)

passband 0.53 dB
stopband 26 dB

TABLE V

LC LOW-PASS FILTER TOLERANCE OPTIMIZATION (C_1)

Parameters	Continuous Solution Fixed Nominal	Continuous Solution Variable Nominal	Discrete Solution From {1,2,5,10,15}% 1	2	3
$x_2 = t_{L_1}$	3.5 %	9.9 %	5 %	10 %	10 %
$x_1 = t_C$	3.2 %	7.6 %	10 %	5 %	10 %
$x_3 = t_{L_2}$	3.5 %	9.9 %	10 %	10 %	5 %
$x_5 = L_1^o$	1.628		1.999		
$x_4 = C^o$	1.090		0.906		
$x_6 = L_2^o$	1.628		1.999		

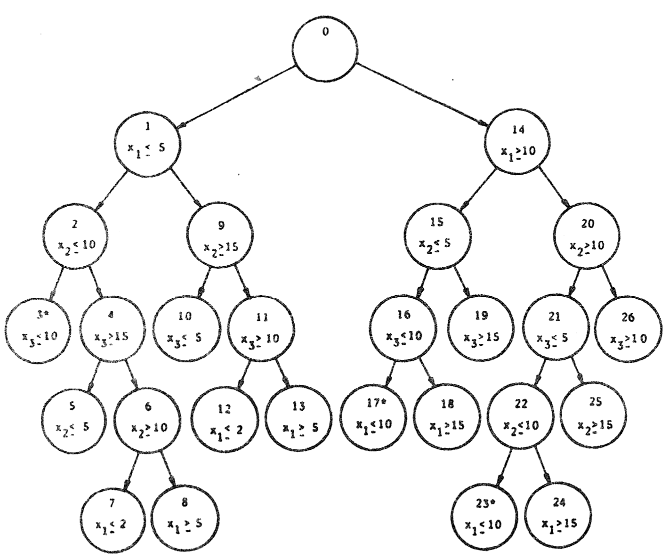

Fig. 7. Tree structure for example 2, partitioning on x_1 first (see Table V). Asterisk denotes optimal discrete solutions.

section lengths and find the corresponding nominal lengths. Let the cost function be

$$C_2 = \sum_{i=1}^{5} \frac{1}{\epsilon_{l_i}}.$$

2) A uniform length tolerance of 0.001 is given. Maximize the relative tolerances on the impedances and obtain the corresponding nominal lengths. Let the cost function be

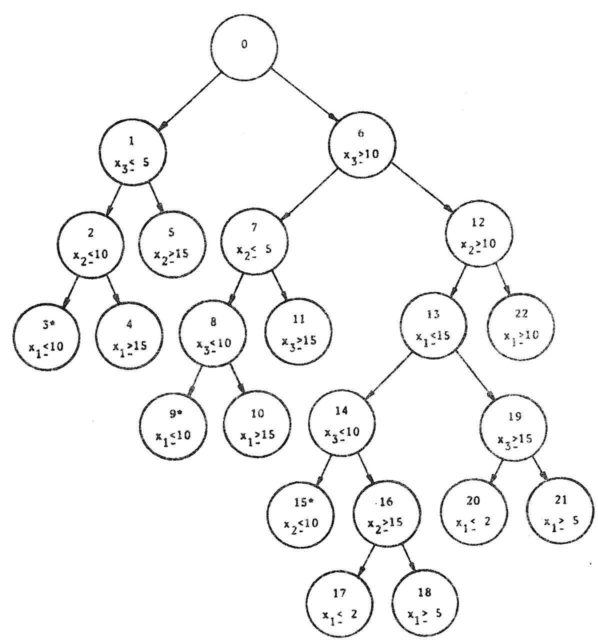

Fig. 8. Tree structure for example 2, partitioning on x_3 first (see Table V). Asterisk denotes optimal discrete solutions.

$$C_1 = \sum_{i=1}^{5} \frac{1}{t_{Z_i}}.$$

The filter has 10 circuit parameters which may be arranged in the order $Z_1, Z_2, \cdots, Z_5, l_1, l_2, \cdots, l_5$. To simplify the problem, symmetry with respect to a center line

TABLE VI
FIVE-SECTION TRANSMISSION-LINE LOW-PASS FILTER

Frequency Range (GHz)	Sample Points (GHz)	Insertion Loss Specification (dB)	Type
0 - 1	.35,.4,.45,.75,.8,.85,1.0	.02	upper (passband)
2.5 - 10	2.5, 10	25	lower (stopband)

through the circuit is assumed. The matrix S is given by

$$S = \begin{bmatrix} & & & & 1 & & & & & 0 \\ & & & 1 & & & & & & \\ & & 1 & & & & & & & \\ & 1 & & & & & & & & \\ 1 & & & & & & & & & \\ & & & & & & & & & 1 \\ & & & & & & & & 1 & \\ & & & & & & & 1 & & \\ & & & & & & 1 & & & \\ 0 & & & & & 1 & & & & \end{bmatrix}$$

which also implies that $l_1^0 = l_5^0$, and $l_2^0 = l_4^0$. The same kind of equalities is applied to the tolerances.

The first vertices elimination scheme is applied with values at the optimal nominal values without tolerances, and the relative impedance tolerance and the absolute length tolerances at 2 percent and 0.002, respectively. A total of 46 vertices corresponding to all the frequency points were selected from a possible set of 9×2^{10}. Fourteen were further eliminated by symmetry. A final total of 15 constraints were chosen after comparing relative magnitudes. These 15 constraints were used throughout the optimization. The continuous and discrete solutions to the two problems are shown in Tables VII and VIII.

TABLE VII
FIVE-SECTION TRANSMISSION-LINE LOW-PASS FILTER TOLERANCE OPTIMIZATION (C_1)

Parameters	Continuous Solution	Discrete Solution From {.5, 1, 1.5, 2, 3, 5}%
$t_{Z_1} = t_{Z_5}$	3.56 %	3 %
$t_{Z_2} = t_{Z_4}$	2.27 %	2 %
t_{Z_3}	1.98 %	2 %
$\ell_1^o = \ell_5^o$	0.0786	
$\ell_2^o = \ell_4^o$	0.1415	
ℓ_3^o	0.1736	

$Z_1^o = Z_3^o = Z_5^o = 0.2, \quad Z_2^o = Z_4^o = 5$

$\epsilon_{\ell_i} = 0.001, \quad i = 1,2,\ldots,5$

TABLE VIII
FIVE-SECTION TRANSMISSION-LINE LOW-PASS FILTER TOLERANCE OPTIMIZATION (C_2)

Parameters	Continuous Solution	Discrete Solution .0005 Step Size
$\epsilon_{\ell_1} = \epsilon_{\ell_5}$	0.0033	0.0030
$\epsilon_{\ell_2} = \epsilon_{\ell_4}$	0.0028	0.0030
ϵ_{ℓ_3}	0.0027	0.0025
$\ell_1^o = \ell_5^o$	0.0788	
$\ell_2^o = \ell_4^o$	0.1414	
ℓ_3^o	0.1738	

$Z_1^o = Z_3^o = Z_5^o = 0.2, \; Z_2^o = Z_4^o = 5$

$t_{Z_i} = 1\%, \; i = 1,2,\ldots,5$

VI. DISCUSSION AND CONCLUSIONS

We have described an efficient user-oriented program for circuit design with worst case tolerance considerations embodying a number of new ideas and recent algorithms. The automated scheme could start from an arbitrary initial acceptable or unacceptable design to obtain continuous and/or discrete optimal nominal parameter values and tolerances simultaneously. However, optimization of the nominal values without tolerances should preferably be done first to obtain a suitable starting point. The effort is small compared with the complete tolerance problem when a small value of p greater than unity, e.g., $p = 2$, is used. An exact minimax solution is not needed. This also serves as a feasibility check. If R_c is indicated to be empty, the designer has to relax some specifications or change his circuit. The solution process may also provide valuable information to the designer, e.g., parameter or frequency symmetry.

The problem without tolerances may be solved easily by available programs such as CANOPT [24]. The user may alternatively utilize the optimization part, namely DISOP2, of the present package.

It is good practice first to obtain a continuous solution before attempting the discrete problem. A useful feature of the program is that, for example, depending on information obtained from prior runs, the user can reenter at a number of different stages of the solution process.

The assumptions on the constraints may be difficult to test. For this reason, a Monte Carlo simulation of the final solution is usually carried out.

We have presented results for two basic types of cost function. A more realistic cost-tolerance model should be established from known component cost data, if these are unsuitable in particular cases.

The complete Fortran listing and documentation for TOLOPT will be made available [25]. It is very important that the user-provided routine for network function computation and the respective sensitivities be efficient. Typical running time for a small and medium size problem (less than 10 network parameters or 20 optimization parameters) will be from 2 to 20 min. The execution time on a CDC 6400, taking the *LC* low-pass filter as an example, was less than 10 s for the continuous case, and a total of 80–100 s for the entire problem, depending on the order of partitioning. The five-section transmission-line example needed about 300–400 s.

REFERENCES

[1] J. W. Bandler, P. C. Liu, and J. H. K. Chen, "Computer-aided tolerance optimization applied to microwave circuits," in *IEEE Int. Microwave Symp. Dig.* (Atlanta, Ga., June 1974), pp. 275–277.

[2] J. W. Bandler, B. L. Bardakjian, and J. H. K. Chen, "Design of recursive digital filters with optimum word length coefficients," presented at the 8th Princeton Conf. Information Sciences and Systems, Princeton, N. J., Mar. 1974.

[3] a) J. W. Bandler and J. H. K. Chen, "DISOPT—a general program for continuous and discrete nonlinear programming problems," *Int. J. Syst. Sci.*, to be published.
b) J. H. K. Chen, McMaster Univ., Hamilton, Ont., Canada, Int. Rep. Simulation, Optimization, and Control SOC-29, Mar. 1974.

[4] R. J. Dakin, "A tree-search algorithm for mixed integer programming problems," *Comput. J.*, vol. 8, pp. 250–255, 1966.

[5] R. Fletcher, "FORTRAN subroutines for minimization by quasi-Newton methods," Atomic Energy Research Establishment, Harwell, Berks., England, Rep. AERE-R7125, 1972.

[6] J. W. Bandler and C. Charalambous, "Practical least *p*th optimization of networks," *IEEE Trans. Microwave Theory Tech.* (*1972 Symp. Issue*), vol. MTT-20, pp. 834–840, Dec. 1972.

[7] C. Charalambous and J. W. Bandler, "New algorithms for network optimization," *IEEE Trans. Microwave Theory Tech.* (*1973 Symp. Issue*), vol. MTT-21, pp. 815–818, Dec. 1973.

[8] J. W. Bandler and C. Charalambous, "Nonlinear programming using minimax techniques," *J. Optimiz. Theory Appl.*, vol. 13, pp. 607–619, June 1974.

[9] C. Charalambous, "A unified review of optimization," *IEEE Trans. Microwave Theory Tech.* (*Special Issue on Computer-Oriented Microwave Practices*), vol. MTT-22, pp. 289–300, Mar. 1974.

[10] W. Y. Chu, "Extrapolation in least *p*th approximation and nonlinear programming," McMaster Univ., Hamilton, Ont., Canada, Int. Rep. Simulation, Optimization, and Control, SOC-71, Dec. 1974.

[11] J. W. Bandler, "Optimization of design tolerances using nonlinear programming," *J. Optimiz. Theory Appl.*, vol. 14, pp. 99–114, 1974.

[12] J. W. Bandler and P. C. Liu, "Automated network design with optimal tolerances," *IEEE Trans. Circuits Syst.*, vol. CAS-21, pp. 219–222, Mar. 1974.

[13] J. F. Pinel and K. A. Roberts, "Tolerance assignment in linear networks using nonlinear programming," *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 475–479, Sept. 1972.

[14] E. M. Butler, "Realistic design using large-change sensitivities and performance contours," *IEEE Trans. Circuit Theory* (*Special Issue on Computer-Aided Circuit Design*), vol. CT-18, pp. 58–66, Jan. 1971.

[15] B. J. Karafin, "The optimum assignment of component tolerances for electrical networks," *Bell Syst. Tech. J.*, vol. 50, pp. 1225–1242, Apr. 1971.

[16] O. L. Mangasarian, *Nonlinear Programming.* New York: McGraw-Hill, 1969.

[17] J. W. Bandler and P. C. Liu, "Some implications of biquadratic functions in the tolerance problem," in *Proc. IEEE Int. Symp. Circuits and Systems* (San Francisco, Calif., Apr. 1974), pp. 740–744.

[18] P. Lancaster, *Theory of Matrices.* New York: Academic, 1969.

[19] A. V. Fiacco and G. P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques.* New York: Wiley, 1968.

[20] F. A. Lootsma, "A survey of methods for solving constrained minimization problems via unconstrained minimization," in *Numerical Methods for Nonlinear Optimization*, F. A. Lootsma, Ed. New York: Academic, 1972.

[21] P. E. Gill and W. Murray, "Quasi-Newton methods for unconstrained optimization," *J. Inst. Math. Its Appl.*, vol. 9, pp. 91–108, 1972.

[22] R. Fletcher, "A new approach to variable metric algorithms," *Comput. J.*, vol. 13, pp. 317–322, 1970.

[23] J. W. Bandler and P. A. Macdonald, "Cascaded noncommensurate transmission-line networks as optimization problems," *IEEE Trans. Circuit Theory* (Corresp.), vol. CT-16, pp. 391–394, Aug. 1969.

[24] J. W. Bandler, J. R. Popović, and V. K. Jha, "Cascaded network optimization program," *IEEE Trans. Microwave Theory Tech.* (*Special Issue on Computer-Oriented Microwave Practices*), vol. MTT-22, pp. 300–308, Mar. 1974.

[25] J. W. Bandler, J. H. K. Chen, P. Dalsgaard, and P. C. Liu, "TOLOPT—A program for optimal, continuous or discrete, design centering and tolerancing," McMaster Univ., Hamilton, Ont., Canada, Int. Rep. Simulation, Optimization, and Control.

TOLOPT - A PROGRAM FOR OPTIMAL, CONTINUOUS OR DISCRETE, DESIGN CENTERING AND TOLERANCING

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ABSTRACT

This paper presents a user-oriented computer program package called TOLOPT (TOLerance OPTimization), which can solve continuous and/or discrete worst-case tolerance assignment problems. Worst-case vertices can be automatically selected and optimization will lead to the most favorable nominal design simultaneously with the largest possible tolerances on specified toleranced components. The program, which is available, contains recent techniques and algorithms for nonlinear programming.

INTRODUCTION

TOLOPT is a package of subroutines which can solve continuous and discrete worst-case tolerance assignment problems simultaneously with the selection of the most favorable nominal design [1-3]. The package is designed to handle the objective function, performance specifications, and parameter constraints in a unified manner such that any of the nominal values or tolerances (relative or absolute) can be fixed or varied automatically at the user's discretion. Time-saving techniques for choosing constraints (vertices selection) are incorporated. The routine involved also checks assumptions and performs worst-case analyses.

The continuous and (optional) discrete optimization methods are programmed in such a way that they may be used as a separate unit. This part, called DISOP2 and incorporating several optional features, is an updated version of DISOPT, which has been successfully applied in many different areas [3 - 6]. Dakin's tree search for discrete problems [7], efficient gradient minimization of functions of many variables by a recent quasi-Newton method [8] and recent developments in least pth approximation by Bandler and Charalambous [9 - 12] are employed. Extrapolation is also featured [13]. The Fortran IV package typically requires 64000 octal words on the CDC 6400.

FEATURES OF TOLOPT

TOLOPT organizes input data and coordinates other subprograms such

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as DISOP2, VERTST, CONSTR, USERCN, COSTFN and NETWRK. Subroutine DISOP2 is a general program for continuous and discrete non-linear programming problems. Subroutine VERTST eliminates the inactive vertices of the tolerance region. Subroutine CONSTR sets up the constraint functions based on the response specifications, component bounds and other constraints supplied in the user subroutine USERCN. Subroutine COSTFN computes the cost function. The user has the option of supplying his own subroutine to define other cost functions. The user supplied subroutine NETWRK returns the network responses and the partial derivatives. In the user supplied subroutine USERCN the user has to define whatever extra constraints he needs and the corresponding partial derivatives. It should be noted that the constraints given in USERCN are not checked against the worst-case vertices.

Table I is a summary of the features and options currently incorporated in TOLOPT.

The objective function we have investigated and implemented [1 - 3] is the weighted summation of the inverses of the relative or the absolute tolerances. The weighting factors may (as default values) be taken as one, but the user can specify his own set of weighting factors.

Various schemes have been developed to identify or to predict the most critical vertices that are likely to give rise to active constraints. Our proposed schemes will eliminate all but one vertex for each constraint function in the most favourable conditions. When monotonicity assumptions [2, 14] are not sufficient to describe the function behaviour, our scheme will increase the number of vertices until, at worst, all vertices are included.

Two major schemes of increasing complexity are programmed in the subroutine VERTST [3]. One involves vertices $\phi^a = \phi^0 - \varepsilon_j u_j$ and $\phi^b = \phi^0 + \varepsilon_j u_j$. Here, ϕ^0 is the nominal point and ε_j the tolerance on the jth component. u_j is the jth unit vector and $j \in I_\phi$, where $I_\phi \triangleq \{1,2,\ldots,k\}$ is the index set for the network components. Another involves all vertices. Also, the special case which occurs for $\phi^a = \phi^b$, has been programmed. In this case only one vertex is considered for each sample point.

The user decides on which vertices selection scheme he wants to use as well as the maximum number of allowable calls for the scheme selected for the updating procedure. He may, if he wishes, bypass the whole subroutine by supplying his own vertices or set up his own strategy of vertices selection. Furthermore, the user decides on the maximum number of vertices allowable at each sample point. If more than the maximum allowable numbers are detected, the subroutine selects the ones corresponding to the lowest constraint value arranged in ascending order.

After printing out the detected vertices and the value of the corresponding constraints, the user has the possibility of eliminating further vertices by considering the relative magnitude of the constraints.

As an option the TOLOPT program can be used for vertices detection only. The program will print out the detected vertices and the value of the corresponding constraints such that the user has the possibility manually to eliminate vertices using his own judgement. The user has the possibility of supplying his own set of active vertices in two different ways.

Before using the automated vertex selection an initial feasibility check is performed to check the feasibility of the nominal design. The outcome from this feasibility check is used as a starting point in the tolerance assignment problem. If a feasible nominal point is not attainable, the user has to relax some specifications or change his design.

The different optimization methods incorporated [9 - 13] employ the least pth approach. Once the constraints have been selected, optimization is started with a small value of p and α (a parameter associated with the minimax approach to nonlinear programming [11]). The routine for updating constraints is called whenever the α value is updated and/or each time new constraints have been added. We add new values of $\underset{\sim}{\mu}$ (a vector identifying a vertex) to existing ones without any eliminations, for stability. When the maximum number of calls is exceeded or when there is no change of values for consecutive calls the program goes to the final optimization with the set of vertices chosen.

Using all the detected vertices could, depending on the problem under investigation, easily involve so many constraints that the optimization would be very time consuming. This could, however, for some problems, be overcome by specifying a sufficiently large but reasonable limit. In such cases the updating and optimization procedure will converge if the vertices, which are active at the solution, are not discarded during updating. The same convergence should occur if manual elimination by the user is performed without discarding vertices which are active at the solution.

It should be pointed out that vertices which are detected at an early stage of the updating procedure need not be active at the solution and vice-versa. The final solution is worst-case only at the chosen sample points.

The solution process may provide valuable information to the user, e.g., parameter or frequency symmetry, which could be useful in order to reduce the number of active vertices.

EXAMPLE

We consider a simple voltage divider [4, 15] with resistances of ϕ_1 and ϕ_2, a transfer function of $\phi_2/(\phi_1+\phi_2)$ and input resistance $\phi_1+\phi_2$. The design specifications are $0.46 \leq \phi_2/(\phi_1+\phi_2) \leq 0.53$ and $1.85 \leq \phi_1+\phi_2 \leq 2.15$. In the case of the discrete problem the set of obtainable discrete values for the tolerances of ϕ_1 and ϕ_2 are 1,3,5,10,15 percent.

A typical main program to supply the values and proper dimensioning for the parameters in the argument list of subroutine TOLOPT and the common statements /TOL/ and /DEFAULT/ is displayed in Fig. 1. Fig. 2 shows the subroutine NETWRK and Fig. 3 illustrates USERCN for a constraint inactive at the solution. Typical printouts of data and the gradient check are shown in Figs. 4 and 5, respectively. Results of continuous and discrete optimizations are shown in Fig. 6.

In this example all four known vertices are supplied and TOLOPT goes directly to the final optimization.

CONCLUSIONS

We have presented an efficient user-oriented program for worst-case tolerance optimization, particularly suited to circuit design. It is based on work carried out by Chen [5], Liu [2] and Bandler, Liu and Chen [3]. The package has been under continuous development to make it sufficiently user-oriented. This has been to some extent at the expense of the greater efficiency which can be realized by a more specialized program. Running times of the package can vary significantly according to the various termination and error criteria used as data. This is particularly true in the generation of the tree structure in a discrete optimization and the interpretation of the solutions as being feasible, discrete, etc.

A detailed report with a complete documented listing is available from the first author at a nominal charge [16].

REFERENCES

[1] J.W. Bandler, "Optimization of design tolerances using nonlinear programming", J. Optimization Theory and Applications, vol. 14, 1974, pp. 99-114.

[2] P.C. Liu, "A theory of optimal worst-case design embodying centering, tolerancing and tuning, with circuit applications", Faculty of Engineering, McMaster University, Hamilton, Canada, Report SOC-87, May 1975.

[3] J.W. Bandler, P.C. Liu and J.H.K. Chen, "Worst case network tolerance optimization", IEEE Trans. Microwave Theory Tech., vol. MTT-23, Aug. 1975, pp. 630-641.

[4] J.W. Bandler and J.H.K. Chen, "DISOPT - a general program for continuous and discrete nonlinear programming problems", Int. J. Systems Science, vol. 6, 1975, pp. 665-680.

[5] J.H.K. Chen, "DISOPT- a general program for continuous and discrete nonlinear programming problems", Faculty of Engineering, McMaster University, Hamilton, Canada, Report SOC-29, March 1974 (Revised June 1975).

[6] J.W. Bandler, B.L. Bardakjian and J.H.K. Chen, "Design of recursive digital filters with optimized word length coefficients", Computer Aided Design, vol. 7, July 1975, pp. 151-156.

[7] R.J. Dakin, "A tree-search algorithm for mixed integer programming problems", Computer J., vol. 8, 1966, pp. 250-255.

[8] R. Fletcher, "FORTRAN subroutines for minimization by quasi-Newton methods", Atomic Energy Research Establishment, Harwell, Berkshire, England, Report AERE-R7125, 1972.

[9] J.W. Bandler and C. Charalambous, "Practical least pth optimization of networks", IEEE Trans. Microwave Theory Tech., vol. MTT-20, Dec. 1972, pp. 834-840.

[10] C. Charalambous and J.W. Bandler, "New algorithms for network optimization", IEEE Trans. Microwave Theory Tech., vol. MTT-21, Dec. 1973, pp. 815-818.

[11] J.W. Bandler and C. Charalambous, "Nonlinear programming using minimax techniques", J. Optimization Theory and Applications, vol. 13, June 1974, pp. 607-619.

[12] C. Charalambous, "A unified review of optimization", IEEE Trans. Microwave Theory Tech., vol. MTT-22, March 1974, pp. 289-300.

[13] W.Y. Chu, "Extrapolation in least pth approximation and nonlinear programming", Faculty of Engineering, McMaster University, Hamilton, Canada, Report SOC-71, Dec. 1974.

[14] J.W. Bandler and P.C. Liu, "Some implications of biquadratic functions in the tolerance problem", IEEE Trans. Circuits and Systems, vol. CAS-22, May 1975, pp. 385-390.

[15] B.J. Karafin, "The optimum assignment of component tolerances for electrical networks", B.S.T.J., vol. 50, April 1971, pp. 1225-1242.

[16] J.W. Bandler, J.H.K. Chen, P. Dalsgaard and P.C. Liu, "TOLOPT- a program for optimal, continuous or discrete, design centering and tolerancing", Faculty of Engineering, McMaster University, Hamilton, Canada, Report SOC-105, Sept. 1975.

TABLE I
SUMMARY OF FEATURES AND OPTIONS

Features	Type	Options
Design parameters	Tolerance and nominal	Variable or fixed. Relative or absolute tolerances
Objective function	Cost	Reciprocal of relative and/or absolute tolerances
		Other
Vertices selection*	Gradient direction strategy	
Constraints	Specifications on functions of network parameters	Upper and/or lower
	Network parameter bounds	
	Other constraints	As many as required
Nonlinear programming	Bandler-Charalambous minimax	Least pth optimization algorithms
	Exterior-point	
Solution feasibility check*	Least pth	Discrete problem. Continuous and discrete problem
Unconstrained minimization method	Quasi-Newton	Gradient checking at starting point by numerical perturbation
Discrete optimization*	Dakin tree-search	Reduction of dimensionality. User supplied or program determined initial upper bound on objective function. Single or multiple optimum discrete solution. Uniform or nonuniform quantization step sizes

* These features are optional and may be bypassed.

```
      PROGRAM TESTVOL(INPUT, OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT)
      DIMENSION X(4), EPS(4), G(4), PS(1), XB(4), IX(4), X1(4), X2(4), W(16),
     *    H(10), XE(4,1,1), INDX(4), GF(4)
      DIMENSION IAA(50), IBB(50), A(50), T1(50), T1P(50)
      DIMENSION NSTEP(2), QSTEP(2), DISCR(2,5), XU(2), XL(2), ID(25), IB(2,25)
     *   ,ICHECK(25), IVAR(25), P1(25), P2(25), ESTD(25), AL(25)
      DIMENSION Z(4), I1(4), I2(4), AZ(2), AX(2), MU(2,10), NV(10), SAMPT(3,10)
     *   ,GRAD(2), PL(2), PU(2), W1(4), CW(2), IB1(2), SG(10)
      DIMENSION GPHI(4,24), PHI(24), I3(10), I4(10)
      COMMON /TOL/ IUPD, ISCEME, IWORST, IPRINT, IDATA, IOPT1, IOPT2, IOPT3,
     *   IOPT4, IOPT5, IOPT6, IOPT7, ND2, ND3, ND4, ND5, MAX, MAXNOD, ICON, NDIM,
     *   NSP, MAXVN, NVSUM, NEC, ND1, ND6
      COMMON /DEFAULT/ EST, EST1, AO, AI, XMAL, ZERO, ETA, INSOLN, BSOLN
      .
      .
      .
      KP=2*KT
      READ 3, ((SAMPT(I,J), I=1,3), J=1,NVSUM)
    3 FORMAT(3F5.2)
      READ 4, ((MU(I,J), J=1,NVSUM), I=1, KT)
      READ 4, (NV(I), I=1,NVSUM)
    4 FORMAT(4I3)
      CALL TOLOPT (NR,KT,KR,KD,KP,NP,Z,I1,I2,AZ,AX,MU,NV,SAMPT,
     *   GRAD,PL,PU,W1,CW,IB1,SG,I3,I4,X,EPS,G,PS,XB,IX,X1,X2,
     *   W,H,XE,INDX,GF,IAA,IBB,A,T1,T1P,NSTEP,QSTEP,DISCR,XU,XL,ID,
     *   IB,ICHECK,IVAR,P1,P2,ESTD,AL,GPHI,PHI)
      STOP
      END
```

Fig. 1. Main program for the example.

```
      SUBROUTINE NETWRK (AX,OM,RSP,GR,IG)
      DIMENSION AX(1), GR(1)
      A=AX(1) + AX(2)
      A2=A**2
      T=AX(2)/A
      KV=IFIX(OM)
      GO TO (1,2,2,1), KV
    1 RSP=A
      IF(IG.EQ.0) RETURN
      GR(1)=1.
      GR(2)=1.
      RETURN
    2 RSP=T
      IF(IG.EQ.0) RETURN
      GR(1)=-AX(2)/A2
      GR(2)=AX(1)/A2
      RETURN
      END
```

Fig. 2. Subroutine NETWRK.

```
      SUBROUTINE USERCN (Z,G,GG,NR,KP)
      DIMENSION Z(1), G(1), GG(KP,1)
      G(1)=Z(3)+Z(4)
      GG(1,1)=0.
      GG(2,1)=0.
      GG(3,1)=1.
      GG(4,1)=1.
      RETURN
      END
```

Fig. 3. Subroutine USERCN.

```
INPUT DATA

DISCRETE VALUES FOR THE VARIABLES
Z( 1)  .10000000E+01  .30000000E+01  .50000000E+01  .10000000E+02  .15000000E+02
Z( 2)  .10000000E+01  .30000000E+01  .50000000E+01  .10000000E+02  .15000000E+02
USER SUPPLIED COMPONENTS......TOLERANCE, DISCRETE..........Z( 1)=  .10000000E-01
                              TOLERANCE, DISCRETE..........Z( 2)=  .10000000E-01
                              NOMINAL, CONTINUOUS..........Z( 3)=  .10000000E+01
                              NOMINAL, CONTINUOUS..........Z( 4)=  .10000000E+01
ERROR TOLERANCE IN CONSTRAINTS.............................ZERO= -.10000000E-05
TEST QUANTITIES TO BE USED IN FLETCHER METHOD............EPS( 1)=  .10000000E-05
                                                         EPS( 2)=  .10000000E-05
                                                         EPS( 3)=  .10000000E-05
                                                         EPS( 4)=  .10000000E-05
ESTIMATE OF LOWER BOUND ON ARTIFICIAL OBJECTIVE FUNCTION.....EST= 0.
INITIAL VALUE OF THE PARAMETER ALPHA.........................AO=  .10000000E+03
MAXIMUM ALLOWABLE VALUE OF THE PARAMETER ALPHA.............XMAL=  .10000000E+08
MULTIPLYING FACTOR IN ALPHA VALUE............................AI=  .10000000E+02
TEST QUANTITY TO BE USED IN NLP ALGORITHM 2/4/5.............ETA=  .10000000E-02
NUMBER OF P VALUES...........................................NP=  1
VALUE(S) OF P USED IN NLP ALGORITHM.......................PS( 1)=  .60000000E+01
FOLLOWING OPTIONS USED
NLP ALGORITHM 4 EMPLOYED-SEQUENCE OF LEAST PTH OPTIMIZATION WITH FINITE VALUES
OF P
(N-1) VARIABLE OPTIMIZATION PERFORMED IN DISCRETE PROBLEM
VERTICES CHECKED ABOUT CONTINUOUS SOLUTION TO OBTAIN AN INITIAL UPPER BOUND
IN DISCRETE PROBLEM
FEASIBILITY CHECKED IN FINAL OPTIMIZATION
PARTITIONING STARTS ON LAST DISCRETE VARIABLE
DATA GIVEN FOR SPECIFIC PROBLEM

     IF      SAMPT(1,IF)     SAMPT(2,IF)      SAMPT(3,IF)
     1      .101000E+01     .185000E+01     -.100000E+01
     2      .201000E+01     .460000E+00     -.100000E+01
     3      .301000E+01     .530000E+00      .100000E+01
     4      .401000E+01     .215000E+01      .100000E+01
```

Fig. 4. Printout of data.

GRADIENT CHECK AT NOMINAL STARTING POINT

THE GRADIENTS FROM THE USER SUPPLIED NETWRK HAVE BEEN CHECKED AT THE FIRST SAMPLE POINT

ANALYTICAL GRADIENTS	NUMERICAL GRADIENTS	PERCENTAGE ERRORS
.100000E+01	.100000E+01	.378577E-08
.100000E+01	.100000E+01	.378577E-08

THE GRADIENTS FROM NETWRK HAVE BEEN CHECKED AT ALL SAMPLE POINTS
THE LARGEST OVERALL DETECTED ERRORS ARE AS FOLLOWS

ANALYTICAL GRADIENTS	NUMERICAL GRADIENTS	PERCENTAGE ERRORS	SAMPLE POINT
-.250000E+00	-.250000E+00	.244904E-06	2
.250000E+00	.250000E+00	.244904E-06	2

THE GRADIENTS FROM THE USER SUPPLIED USERCN HAVE BEEN CHECKED
FOR EACH GIVEN EXTRA CONSTRAINT THE ERRORS ARE AS FOLLOWS

ANALYTICAL GRADIENTS	NUMERICAL GRADIENTS	PERCENTAGE ERRORS	CONSTRAINT
.100000E-13	.100000E-13	0.	1
.100000E-13	.100000E-13	0.	1
.100000E+01	.100000E+01	.378577E-08	1
.100000E+01	.100000E+01	.378577E-08	1

Fig. 5. Check of user supplied gradients.

RESULTS OF THE FEASIBILITY CHECK

NODE NUMBER = 0

INEQUALITY CONSTRAINTS	OCCURRING AT
G(1) = .13000000E+00	SAMPLE POINT 1
G(2) = .35000000E-01	SAMPLE POINT 2
G(3) = .25000000E-01	SAMPLE POINT 3
G(4) = .13000000E+00	SAMPLE POINT 4
G(5) = .19000000E+00	LOWER BOUND 1
G(6) = .19000000E+00	UPPER BOUND 1
G(7) = .19000000E+00	LOWER BOUND 2
G(8) = .19000000E+00	UPPER BOUND 2
G(9) = .20000000E+01	EXTRA CONST 1

NUMBER OF CONSTRAINTS USED = 9
NUMBER OF VIOLATED CONSTRAINTS = 0
NUMBER OF FUNCTION EVALUATIONS = 1

FOLLOWING IS RESULT OF OPTIMIZATION

NODE NUMBER = 0
ARTIFICIAL UNCONSTRAINED FUNCTION U = -.11592424E-01
ACTUAL OBJECTIVE FUNCTION F = .28569099E+02

X(1) = .26458592E+00	GU(1) =	.15487643E-03
X(2) = .26458592E+00	GU(2) =	.16120715E-03
X(3) = .10139413E+01	GU(3) =	.10367310E-03
X(4) = .99376532E+00	GU(4) =	-.10577793E-03

INEQUALITY CONSTRAINTS	OCCURRING AT
G(1) = .17155658E-01	SAMPLE POINT 1
G(2) = .66321421E-06	SAMPLE POINT 2
G(3) = .66395748E-06	SAMPLE POINT 3
G(4) = .17425017E-02	SAMPLE POINT 4
G(5) = .14295958E+00	LOWER BOUND 1
G(6) = .11507707E+00	UPPER BOUND 1
G(7) = .12419608E+00	LOWER BOUND 2
G(8) = .13666543E+00	UPPER BOUND 2
G(9) = .20077066E+01	EXTRA CONST 1

NUMBER OF CONSTRAINTS USED = 9
NUMBER OF VIOLATED CONSTRAINTS = 0
NUMBER OF FUNCTION EVALUATIONS = 192
FINAL VALUE OF THE PARAMETER ALPHA = .10000000E+05

FOLLOWING IS THE OPTIMUM SOLUTION

Z(1) = .70005708E-01
Z(2) = .70005708E-01
Z(3) = .10139413E+01
Z(4) = .99376532E+00

EXECUTION TIME IN SECONDS = 4.20300

Fig. 6. Results of the continuous optimization.

BEST DISCRETE SOLUTION FOUND SO FAR

F = .40000000E+02

```
X( 1) =  .50000000E-01
X( 2) =  .50000000E-01
X( 3) =  .10139413E+01
X( 4) =  .99376532E+00
INEQUALITY CONSTRAINTS        OCCURRING AT
G( 1) =  .57321249E-01        SAMPLE POINT  1
G( 2) =  .99904561E-02        SAMPLE POINT  2
G( 3) =  .10014581E-01        SAMPLE POINT  3
G( 4) =  .41908093E-01        SAMPLE POINT  4
G( 5) =  .16324419E+00        LOWER BOUND   1
G( 6) =  .13536168E+00        UPPER BOUND   1
G( 7) =  .14407706E+00        LOWER BOUND   2
G( 8) =  .15654641E+00        UPPER BOUND   2
G( 9) =  .20077066E+01        EXTRA CONST   1
NUMBER OF FUNCTION EVALUATIONS =  198
```

RESULTS OF THE FEASIBILITY CHECK

```
NODE NUMBER =  8
INEQUALITY CONSTRAINTS        OCCURRING AT
G( 1) =  .94253713E-02        SAMPLE POINT  1
G( 2) = -.34594404E-02        SAMPLE POINT  2
G( 3) =  .68230156E-02        SAMPLE POINT  3
G( 4) =  .26401155E-01        SAMPLE POINT  4
G( 5) =  .16418646E+00        LOWER BOUND   1
G( 6) =  .13432023E+00        UPPER BOUND   1
G( 7) =  .95238913E-01        LOWER BOUND   2
G( 8) =  .14208092E+00        UPPER BOUND   2
G( 9) =  .19915121E+01        EXTRA CONST   1
NUMBER OF CONSTRAINTS USED =  10
NUMBER OF VIOLATED CONSTRAINTS =  2
NUMBER OF FUNCTION EVALUATIONS =  10
EXECUTION TIME IN SECONDS =  .27500
```

OPTIMUM DISCRETE SOLUTION FOUND

MINIMUM F = .40000000E+02

```
X( 1) =  .50000000E-01
X( 2) =  .50000000E-01
X( 3) =  .10139413E+01
X( 4) =  .99376532E+00
INEQUALITY CONSTRAINTS        OCCURRING AT
G( 1) =  .57321249E-01        SAMPLE POINT  1
G( 2) =  .99904561E-02        SAMPLE POINT  2
G( 3) =  .10014581E-01        SAMPLE POINT  3
G( 4) =  .41908093E-01        SAMPLE POINT  4
G( 5) =  .16324419E+00        LOWER BOUND   1
G( 6) =  .13536168E+00        UPPER BOUND   1
G( 7) =  .14407706E+00        LOWER BOUND   2
G( 8) =  .15654641E+00        UPPER BOUND   2
G( 9) =  .20077066E+01        EXTRA CONST   1
NUMBER OF FUNCTION EVALUATIONS =  620
```

FOLLOWING IS THE OPTIMUM SOLUTION

```
Z( 1) =  .50000000E-01
Z( 2) =  .50000000E-01
Z( 3) =  .10139413E+01
Z( 4) =  .99376532E+00
```

Fig. 6 [continued]. Results of the discrete optimization.

PART III: CENTERING, TOLERANCING AND TUNING

A Nonlinear Programming Approach to Optimal Design Centering, Tolerancing, and Tuning

JOHN W. BANDLER, SENIOR MEMBER, IEEE, PETER C. LIU, MEMBER, IEEE, AND HERMAN TROMP

Abstract—A theory of optimal worst-case design embodying centering, tolerancing, and tuning is presented. Some simplified problems and special cases are discussed. Projections and slack variables are used to explain some of the concepts. The worst-case tolerance assignment and design centering problem falls out as a special case. Practical implementation requires a reasonable and relevant number of parameters and constraints to be identified to make the problem tractable. Two circuits, a simple LC low-pass filter and a realistic high-pass filter, are studied under a variety of different problem situations to illustrate both the benefits to be derived from our approach and the difficulties encountered in its implementation.

I. Introduction

COMPONENT TOLERANCE ASSIGNMENT is now considered to be an integral part of the design process [1]–[7]. The optimal worst-case tolerance problem with variable nominal point has benefitted in terms of increased tolerances [5]–[7]. Tuning [7], [8], on the other hand, does not seem to have been given its proper place in the design process. This work, therefore, brings in tuning of one or more components basically to further increase tolerances to reduce cost or to make unrealistically toleranced solutions more attractive. The mathematical formulation of an approach which embodies centering, tolerancing, and tuning in a unified manner is presented. Simplified problems and appropriate geometric interpretations are discussed. The worst-case purely toleranced problem and purely tuned problem fall out as special cases, as is to be expected. Numerical examples involving simple functions and a realistic as well as a simple circuit, illustrate the concepts.

II. Fundamental Concepts and Definitions

A *design* consists of design data of the *nominal point* $\boldsymbol{\phi}^0$, the *tolerance vector* $\boldsymbol{\varepsilon}$ and the *tuning vector* $\boldsymbol{t}$ where, for k parameters,

$$\boldsymbol{\phi}^0 \triangleq \begin{bmatrix} \phi_1^{\,0} \\ \phi_2^{\,0} \\ \vdots \\ \phi_k^{\,0} \end{bmatrix}, \quad \boldsymbol{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_k \end{bmatrix}, \text{ and } \boldsymbol{t} \triangleq \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{bmatrix}. \tag{1}$$

We assume that the parameters can be varied continuously and chosen independently. Extra conditions such as discretization and imposed parameter bounds may be treated as constraints [6]. Some of the parameters can be set to zero or held constant.

An *outcome* $\{\boldsymbol{\phi}^0,\boldsymbol{\varepsilon},\boldsymbol{\mu}\}$ of a *design* $\{\boldsymbol{\phi}^0,\boldsymbol{\varepsilon},\boldsymbol{t}\}$ implies a point

$$\boldsymbol{\phi} = \boldsymbol{\phi}^0 + \boldsymbol{E}\boldsymbol{\mu} \tag{2}$$

where

$$\boldsymbol{E} \triangleq \begin{bmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \ddots & \\ & & & \varepsilon_k \end{bmatrix} \tag{3}$$

and $\boldsymbol{\mu} \in R_\mu$. R_μ is a set of multipliers determined from realistic situations of the tolerance spread. For example,

$$R_\mu \triangleq \{\boldsymbol{\mu} \mid -1 \le \mu_i \le -a_i \text{ or } a_i \le \mu_i \le 1,\ i \in I_\phi\} \tag{4}$$

where

$$0 \le a_i \le 1 \tag{5}$$

and

$$I_\phi \triangleq \{1,2,\cdots,k\}. \tag{6}$$

The most commonly used continuous range is obtained by setting a_i to zero. A commercial stock may have the better toleranced components taken out, thus $0 < a_i \le 1$. Unless otherwise stated, we consider

$$R_\mu \triangleq \{\boldsymbol{\mu} \mid -1 \le \mu_i \le 1,\ i \in I_\phi\}. \tag{7}$$

The *tolerance region* R_ε is a set of points described by (2) for all $\boldsymbol{\mu} \in R_\mu$. In the case of $-1 \le \mu_i \le 1$, $i \in I_\phi$,

$$R_\varepsilon \triangleq \{\boldsymbol{\phi} \mid \phi_i = \phi_i^{\,0} + \varepsilon_i\mu_i,\ -1 \le \mu_i \le 1,\ i \in I_\phi\} \tag{8}$$

which is a convex *regular polytope* of k dimensions with sides of length $2\varepsilon_i$, $i \in I_\phi$, and centered at $\boldsymbol{\phi}^0$. The extreme points of R_ε are obtained by setting $\mu_i = \pm 1$. Thus, the set of *vertices* may be defined as

$$R_v \triangleq \{\boldsymbol{\phi} \mid \phi_i = \phi_i^{\,0} + \varepsilon_i\mu_i,\ \mu_i \in \{-1,1\},\ i \in I_\phi\}. \tag{9}$$

The number of points in R_v is 2^k. Let each of these points be indexed by $\boldsymbol{\phi}^i$, $i \in I_v$, where

$$I_v \triangleq \{1,2,\cdots,2^k\}. \tag{10}$$

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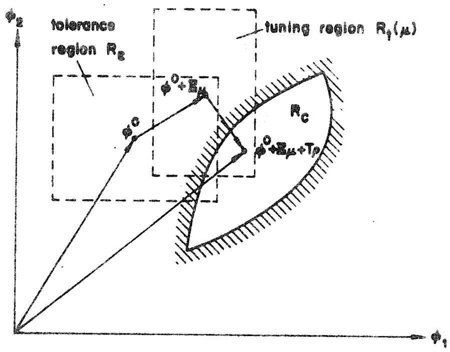

Fig. 1. Illustration of regions R_ε, R_t, and R_c. If $\mu = 0$ then R_t is centered at ϕ^0.

Thus

$$R_v = \{\phi^1, \phi^2, \cdots, \phi 2^k\}.$$

The *tuning region* $R_t(\mu)$ is defined as the set of points

$$\phi = \phi^0 + E\mu + T\rho \tag{11}$$

for all $\rho \in R_\rho$, where

$$T \triangleq \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_k \end{bmatrix}. \tag{12}$$

The components of ρ will be called slack variables since they do not directly contribute to the objective function. Some of the common examples of R_ρ are

$$R_\rho \triangleq \{\rho \mid -1 \le \rho_i \le 1, i \in I_\phi\} \tag{13}$$

or in the case of *one-way tuning* or *irreversible trimming*,

$$R_\rho = \{\rho \mid 0 \le \rho_i \le 1, i \in I_\phi\} \tag{14}$$

or

$$R_\rho = \{\rho \mid -1 \le \rho_i \le 0, i \in I_\phi\}. \tag{15}$$

Unless otherwise indicated, the case given by (13) is considered.

The *constraint region* R_c is given by

$$R_c \triangleq \{\phi \mid g_i(\phi) \ge 0, \quad \text{for all } i \in I_c\} \tag{16}$$

where

$$I_c \triangleq \{1,2,\cdots,m_c\} \tag{17}$$

is the index set for the performance specifications and parameter constraints. R_c is assumed to be not empty. Other conditions and assumptions will be imposed on R_c as the theory is developed further.

The definitions are illustrated in Fig. 1 by a two-dimensional example.

A *tunable constraint region* is denoted by $R_c(\psi)$, where ψ represents other independent variables. Fig. 2 depicts three different regions of an example of $R_c(\psi)$. Overlapping of these regions is allowable. The value of ψ may be continuous

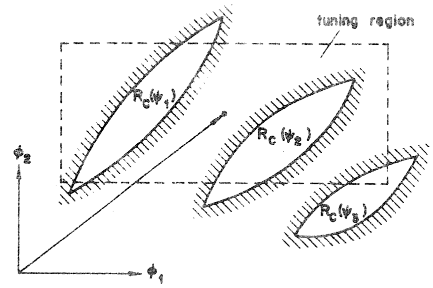

Fig. 2. Example of three different settings of tunable constraint regions.

or discrete. $R_c(\psi) = R_c$ in the ordinary sense if ψ is a constant.

III. The Original Problem P_0

The problem may be stated as follows: obtain a set of optimal design values $\{\phi^0, \varepsilon, t\}$ such that any outcome $\{\phi^0, \varepsilon, \mu\}$, $\mu \in R_\mu$, may be tuned into R_c for some $\rho \in R_\rho$.

It is formulated as the nonlinear programming problem:

$$P_0\text{: minimize } C(\phi^0, \varepsilon, t)$$

$$\text{subject to } \phi \in R_c$$

where

$$\phi = \phi^0 + E\mu + T\rho \tag{18}$$

and constraints $\phi^0, \varepsilon, t \ge 0$, for all $\mu \in R_\mu$ and some $\rho \in R_\rho$. C is an appropriate function chosen to represent a reasonable approximation to known component cost data.

Stated in an abstract sense, the *worst-case solution* of the problem must satisfy

$$R_t(\mu) \cap R_c \ne \varnothing \tag{19}$$

for all $\mu \in R_\mu$, where $\varnothing$ denotes a null set.

IV. The Reduced Problem P_1

The original problem P_0 of the preceding section can be reduced by separating the components into *effectively tuned* and *effectively toleranced* parameters. Let

$$I_\varepsilon \triangleq \{i \mid \varepsilon_i > t_i, i \in I_\phi\} \tag{20}$$

$$I_t \triangleq \{i \mid t_i \ge \varepsilon_i, i \in I_\phi\} \tag{21}$$

$$\varepsilon_i' \triangleq \varepsilon_i - t_i, i \in I_\varepsilon \tag{22}$$

and

$$t_i' \triangleq t_i - \varepsilon_i, i \in I_t. \tag{23}$$

It is obvious that I_t and I_ε are disjoint and $I_t \cup I_\varepsilon = I_\phi$.

Now, consider the problem

$$P_1\text{: minimize } C(\phi^0, \varepsilon, t)$$

$$\text{subject to } \phi \in R_c$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \varepsilon_i'\mu_i, & \text{for } i \in I_\varepsilon \\ t_i'\rho_i', & \text{for } i \in I_t \end{cases} \tag{24}$$

for all $-1 \leq \mu_i \leq 1$, $i \in I_\varepsilon$, and for some $-1 \leq \rho_i' \leq 1$, $i \in I_t$.

Theorem 1

A feasible solution to the *reduced problem* P_1 is a feasible solution to the original problem P_0.

Proof: Given $\boldsymbol{\phi}^0, \boldsymbol{\varepsilon}, \boldsymbol{t}$ we will show that

$$1)\ \varepsilon_i\mu_i + t_i\rho_i = \varepsilon_i'\mu_i,\ i \in I_\varepsilon \tag{25}$$

$$2)\ \varepsilon_i\mu_i + t_i\rho_i = t_i'\rho_i',\ i \in I_t \tag{26}$$

under the restrictions on μ_i, ρ_i, and ρ_i'.

1) Since ρ_i can be freely chosen from $-1 \leq \rho_i \leq 1$, we can let $\rho_i = -\mu_i$ giving

$$(\varepsilon_i - t_i)\mu_i = \varepsilon_i'\mu_i. \tag{27}$$

2) For any $-1 \leq \rho_i' \leq 1$ and all $-1 \leq \mu_i \leq 1$, we can choose

$$-1 \leq \rho_i = \frac{(t_i - \varepsilon_i)\rho_i' - \varepsilon_i\mu_i}{t_i} \leq 1, \qquad t_i \neq 0. \tag{28}$$

Thus any point with components represented by (24) of the reduced problem can be represented by (18) of the original problem.

Intuitively, this theorem states the fact that a feasible solution to a restrictive problem is also a feasible solution to an appropriate less restrictive problem. The variable transformation (22) and (23) may be considered as extraneous constraints to be satisfied.

Theorem 2

A feasible solution to the original problem P_0 implies a feasible solution to the reduced problem P_1 if R_c is one-dimensionally convex [3].

Proof: 1) We note, for $i \in I_\varepsilon$, that

$$\begin{aligned}\phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \leq \phi_i^0 - \varepsilon_i + t_i \leq \phi_i^0 + (\varepsilon_i - t_i)\mu_i \\ \leq \phi_i^0 + \varepsilon_i - t_i \leq \phi_i^0 + \varepsilon_i + t_i\rho_i(1)\end{aligned} \tag{29}$$

where $\rho_i(-1)$ corresponds to $\mu_i = -1$ and $\rho_i(1)$ corresponds to $\mu_i = 1$. If R_c is one-dimensionally convex, the following assumption

$$\begin{bmatrix} \vdots \\ \phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \\ \vdots \end{bmatrix}, \quad \begin{bmatrix} \vdots \\ \phi_i^0 + \varepsilon_i + t_i\rho_i(1) \\ \vdots \end{bmatrix} \in R_c \tag{30}$$

implies that

$$\begin{bmatrix} \vdots \\ \phi_i^0 + (\varepsilon_i - t_i)\mu_i \\ \vdots \end{bmatrix} \in R_c \tag{31}$$

where we consider changes in the ith component only and impose the required restrictions on μ_i and ρ_i.

2) On the other hand, for $i \in I_t$, given feasible $\rho_i(-1)$ and $\rho_i(1)$ such that

$$\phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \leq \phi_i^0 + \varepsilon_i + t_i\rho_i(1) \tag{32}$$

there exists a feasible ρ_i' such that

$$\begin{aligned}\phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \leq \phi_i^0 + (t_i - \varepsilon_i)\rho_i' \\ \leq \phi_i^0 + \varepsilon_i + t_i\rho_i(1).\end{aligned} \tag{33}$$

This is true for $t_i = \varepsilon_i$ and can be seen for $t_i > \varepsilon_i$ by rewriting this inequality as

$$\frac{-\varepsilon_i + t_i\rho_i(-1)}{t_i - \varepsilon_i} \leq \rho_i' \leq \frac{\varepsilon_i + t_i\rho_i(1)}{t_i - \varepsilon_i}. \tag{34}$$

Hence, if R_c is one-dimensionally convex, the assumption implies that

$$\begin{bmatrix} \vdots \\ \phi_i^0 + (t_i - \varepsilon_i)\rho_i' \\ \vdots \end{bmatrix} \in R_c. \tag{35}$$

Thus, a feasible solution to the original problem can be transformed to a feasible solution of the reduced problem P_1.

A Geometric Interpretation

Let us define a *projection matrix* $\boldsymbol{P}$ as a diagonal matrix such that

$$\boldsymbol{P} \triangleq \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_k \end{bmatrix} \tag{36}$$

where

$$p_i = \begin{cases} 0, & \text{for } i \in I_t \\ 1, & \text{for } i \in I_\varepsilon \end{cases}. \tag{37}$$

The projection of a point $\boldsymbol{\phi}$ may be denoted as $\boldsymbol{\phi}_p = \boldsymbol{P\phi}$. It may be noted that the projections of two points $\boldsymbol{\phi}^a, \boldsymbol{\phi}^{b(j)} = \boldsymbol{\phi}^a + \alpha\boldsymbol{e}_j$, where $\boldsymbol{e}_j$ is the jth unit vector, for $j \in I_t$, and some constant α, coincide. The projection concept and the introduction of slack variables provide a key to understanding the tuning concept.

Let

$$R_{\varepsilon t} \triangleq \{\boldsymbol{\phi} \mid \phi_i^0 - \varepsilon_i' \leq \phi_i \leq \phi_i^0 + \varepsilon_i',\ i \in I_\varepsilon\} \tag{38}$$

and

$$R_{t\varepsilon} \triangleq \{\boldsymbol{\phi} \mid \phi_i^0 - t_i' \leq \phi_i \leq \phi_i^0 + t_i',\ i \in I_t\} \tag{39}$$

denote the regions defined by the effectively toleranced and effectively tuned parameters. Then consider the following regions

$$R_{\varepsilon tp} \triangleq \{\boldsymbol{\phi}_p \mid \boldsymbol{\phi}_p = \boldsymbol{P\phi},\ \boldsymbol{\phi} \in R_{\varepsilon t}\} \tag{40}$$

$$R_{ct\varepsilon} \triangleq R_c \cap R_{t\varepsilon}, \tag{41}$$

and

$$R_{ct\varepsilon p} \triangleq \{\boldsymbol{\phi}_p \mid \boldsymbol{\phi}_p = \boldsymbol{P\phi},\ \boldsymbol{\phi} \in R_{ct\varepsilon}\}. \tag{42}$$

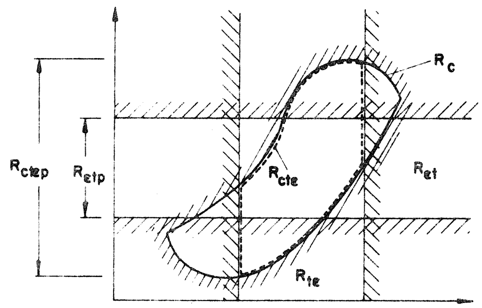

Fig. 3. Geometric interpretation of reduced problem P_1.

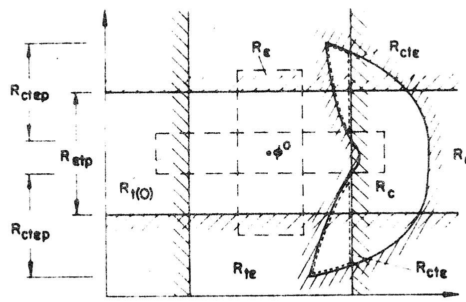

Fig. 4. Example of $R_{\varepsilon tp} \nsubseteq R_{ctep}$. $R_t(\mu)$ for $\mu = 0$ is indicated, for convenience.

Fig. 3 illustrates the definition of the regions. Any point whose components are given by (24) lies in the intersection of $R_{\varepsilon t}$ and $R_{t\varepsilon}$. Suppose the projection of $R_{ct\varepsilon}$ onto the subspace spanned by the effectively toleranced parameters includes the projection of that point. Then it may be tuned into $R_{ct\varepsilon}$ by adjusting the value of ρ_i', $i \in I_t$.

The reduced problem P_1 may be stated as: solve a pure tolerance problem (i.e., no tuning) in the subspace spanned by the toleranced variables with $R_{\varepsilon tp}$ as the tolerance region and $R_{ct\varepsilon p}$ as the constraint region. In other words, the regions defined by a feasible solution must satisfy the condition that

$$R_{\varepsilon tp} \subseteq R_{ct\varepsilon p}. \tag{43}$$

Fig. 4 illustrates a case where $R_{\varepsilon tp} \nsubseteq R_{ct\varepsilon p}$. An outcome, for example, at ϕ^0 cannot be tuned to R_c within the effective tuning range. However, there exists a solution to the original formulation by tuning both components. R_c is not one-dimensionally convex in this case.

Special Cases

Case 1: $I_\varepsilon = \varnothing$, the Pure Tuning Problem: In this case, $R_{\varepsilon t}$ is the entire space and P is a zero matrix. $R_{\varepsilon tp}$ is a single point at the origin. The problem has a solution if

$$R_{ct\varepsilon} \neq \varnothing. \tag{44}$$

Case 2: $I_t = \varnothing$, the Pure Tolerance Problem: In this case, $R_{t\varepsilon}$ is the entire space and P is a unit matrix:

$$R_{\varepsilon tp} = R_{\varepsilon t} \quad \text{and} \quad R_{ct\varepsilon p} = R_{ct\varepsilon} = R_c.$$

The problem has a solution if

$$R_{\varepsilon t} \subseteq R_c. \tag{45}$$

From a tolerance–tuning point of view, the first case is trivial theoretically. Except when there is only one single point R_c, the pure tuning problem is equivalent to an optimization of the nominal parameter values. On the other hand, the pure tolerance problem is very important from a practical point of view.

Extension of P_1 for Tunable Constraint Region

Three types of components can be identified when the constraint region is considered to be tunable. They are a) toleranced components, b) components tuned by the manufacturer, and c) components tunable by the customer. In this case,

$$\phi \in R_c(\psi)$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \varepsilon_i'\mu_i, & \text{for } i \in I_\varepsilon \\ t_i'\rho_i', & \text{for } i \in I_{tm} \\ t_i'\rho_i'(\psi), & \text{for } i \in I_{tc} \end{cases} \tag{46}$$

where I_{tm} identifies components b) and I_{tc} identifies components c).

Setting the ψ to a particular value will control the setting of ρ_i', $i \in I_{tc}$, such that ϕ will be in that particular constraint region $R_c(\psi)$.

V. The Reduced Problem P_2

It is impossible to test all the points in $R_{\varepsilon tp}$ to be in $R_{ct\varepsilon p}$. In order to make the problem tractable a number of simplifying assumptions could be made to obtain an acceptable solution to the problem with reasonable effort. To this end we replace the continuous range $-1 \leq \mu_i \leq 1$ by a discrete set $\mu_i \in \{-1,1\}$, $i \in I_\varepsilon$. Now, consider the problem

$$P_2\text{: minimize } C(\phi^0,\varepsilon,t)$$

$$\text{subject to } \phi \in R_c$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \varepsilon_i'\mu_i, & \text{for } i \in I_\varepsilon \\ t_i'\rho_i', & \text{for } i \in I_t \end{cases} \tag{47}$$

for all $\mu_i \in \{-1,1\}$, $i \in I_\varepsilon$, and some $-1 \leq \rho_i' \leq 1$, $i \in I_t$.

Let us define the set of *projected vertices* (or the vertices of the projected region) by

$$R_{vp} \triangleq \{\phi_p \mid \phi_p = P\phi, \phi \in R_v\}. \tag{48}$$

The condition may be now stated as

$$R_{vp} \subseteq R_{ct\varepsilon p}.$$

Theorem 3

A feasible solution to *reduced problem P_2* implies a feasible solution to reduced problem P_1 if $R_{ct\varepsilon p}$ is one-dimensionally convex.

This is a pure tolerance problem in the subspace spanned by the effectively toleranced parameters. For a proof in the tolerance parameter space, see Bandler [3].

VI. The Objective Functions

Several *objective functions* (or *cost functions*) have been proposed [1]–[5]. In practice, a suitable modeling problem would have to be solved to determine the cost-tolerance relationship. Here, it is assumed that the tolerances and tuning ranges (either absolute or relative) are the main variables and that the total cost of the design is the sum of the cost of the individual components.

The objective function should have the following properties

$$\begin{aligned} &C(\boldsymbol{\phi}^0,\boldsymbol{\varepsilon},\boldsymbol{t}) \rightarrow c, && \text{as } \boldsymbol{\varepsilon} \rightarrow \boldsymbol{\infty} \\ &C(\boldsymbol{\phi}^0,\boldsymbol{\varepsilon},\boldsymbol{t}) \rightarrow \infty, && \text{for any } \varepsilon_i \rightarrow 0 \\ &C(\boldsymbol{\phi}^0,\boldsymbol{\varepsilon},\boldsymbol{t}) \rightarrow C(\boldsymbol{\phi}^0,\boldsymbol{\varepsilon}), && \text{as } \boldsymbol{t} \rightarrow \boldsymbol{0} \\ &C(\boldsymbol{\phi}^0,\boldsymbol{\varepsilon},\boldsymbol{t}) \rightarrow \infty, && \text{for any } t_i \rightarrow \infty. \end{aligned} \tag{49}$$

Suitable objective functions will be, for example, of the form

$$C = \sum_{i=1}^{k} \frac{c_i}{x_i} + \sum_{i=1}^{k} c_i' y_i \tag{50}$$

where x_i and y_i denote the tolerances and tuning ranges, respectively. In the case of relative tolerances or relative tuning ranges $x_i = \varepsilon_i/\phi_i^0 \times 100$, $y_i = t_i/\phi_i^0 \times 100$. We may set the appropriate c_i' to zero if tuning is considered either free, or fixed or is not required. c_i may be set to zero if the corresponding tolerance is fixed.

VII. Mathematical Example

Consider the constraints

$$\phi_2 - \phi_1 - 2 \geq 0 \tag{51}$$

$$-\phi_2^2 + 16\phi_1 \geq 0. \tag{52}$$

A convex region R_c is defined by these constraints.

We will take R_μ as an infinite set of discrete points $\boldsymbol{\mu}(i)$, $i = 1,2,\cdots$, where $-1 \leq \mu_1(i) \leq 1$ and $-1 \leq \mu_2(i) \leq 1$. Thus a relevant problem may be formulated as follows. Minimize

$$C = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \tag{53}$$

with respect to ε_1, ε_2, ϕ_1^0, and ϕ_2^0, and subject to

$$\begin{aligned} g_1 &= \varepsilon_1 \geq 0 \\ g_2 &= \varepsilon_2 \geq 0 \\ g_3 &= \phi_1^0 \geq 0 \\ g_4 &= \phi_2^0 \geq 0 \end{aligned} \tag{54}$$

$$g_5(i) = (\phi_2^0 + \varepsilon_2\mu_2(i)) - (\phi_1^0 + \varepsilon_1\mu_1(i)) - 2 \geq 0, \quad i = 1,2,\cdots \tag{55}$$

$$g_6(i) = -(\phi_2^0 + \varepsilon_2\mu_2(i))^2 + 16(\phi_1^0 + \varepsilon_1\mu_1(i)) \geq 0, \quad i = 1,2,\cdots \tag{56}$$

where $-1 \leq \mu_1(i) \leq 1$ and $-1 \leq \mu_2(i) \leq 1$.

Optimality requires that

$$\begin{bmatrix} -\dfrac{1}{\varepsilon_1^2} \\ -\dfrac{1}{\varepsilon_2^2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \sum_i u_5(i) \begin{bmatrix} -\mu_1(i) \\ \mu_2(i) \\ -1 \\ 1 \end{bmatrix} + \sum_i u_6(i) \begin{bmatrix} 16\mu_1(i) \\ -2\mu_2(i)(\phi_2^0 + \varepsilon_2\mu_2(i)) \\ 16 \\ -2(\phi_2^0 + \varepsilon_2\mu_2(i)) \end{bmatrix} \tag{57}$$

$$u_1 g_1 = \cdots = u_4 g_4 = u_5(i) g_5(i) = u_6(i) g_6(i) = 0, \quad i = 1,2,\cdots \tag{58}$$

$$u_1,\cdots,u_4,u_5(i),u_6(i) \geq 0, \qquad i = 1,2,\cdots \tag{59}$$

where u denotes a multiplier. To solve the above equations, assume that ε_1, ε_2, ϕ_1^0, and ϕ_2^0 are not zero, therefore, set u_1, u_2, u_3, and u_4 to zero. Minimize $g_5(i)$ of (55) and $g_6(i)$ of (56) with respect to $\boldsymbol{\mu}(i)$. This leads, respectively, to

$$(\phi_2^0 - \varepsilon_2) - (\phi_1^0 + \varepsilon_1) - 2 \geq 0 \tag{60}$$

using $\boldsymbol{\mu}(i) = [1\ -1]^T$ and

$$-(\phi_2^0 + \varepsilon_2)^2 + 16(\phi_1^0 - \varepsilon_1) \geq 0 \tag{61}$$

using $\boldsymbol{\mu}(i) = [-1\ 1]^T$. The optimality conditions (57)–(59) are correspondingly reduced yielding the solution

$$\begin{aligned} \varepsilon_1 &= 0.5 \\ \varepsilon_2 &= 0.5 \\ \phi_1^0 &= 4.5 \\ \phi_2^0 &= 7.5. \end{aligned}$$

Consider next the problem of minimizing

$$C = \frac{1}{\varepsilon_2} \tag{62}$$

with respect to t_1', ε_2, ϕ_1^0, ϕ_2^0, and $\rho_1(i)$, and subject to

$$\begin{aligned} g_1 &= t_1' \geq 0 \\ g_2 &= \varepsilon_2 \geq 0 \\ g_3 &= \phi_1^0 \geq 0 \\ g_4 &= \phi_2^0 \geq 0 \end{aligned} \tag{63}$$

$$g_5 = 0.1 - \frac{t_1'}{\phi_1^0} \geq 0 \tag{64}$$

$$g_6(i) = (\phi_2^0 + \varepsilon_2\mu_2(i)) - (\phi_1^0 + t_1'\rho_1'(i)) - 2 \geq 0, \quad i = 1,2,\cdots \tag{65}$$

$$g_7(i) = -(\phi_2^0 + \varepsilon_2\mu_2(i))^2 + 16(\phi_1^0 + t_1'\rho_1'(i)) \geq 0, \quad i = 1,2,\cdots \tag{66}$$

$$g_8(i) = 1 - \rho_1'(i) \geq 0, \qquad i = 1,2,\cdots \tag{67}$$

$$g_9(i) = 1 + \rho_1'(i) \geq 0, \qquad i = 1,2,\cdots. \tag{68}$$

Here, ε_1 is considered fixed at 0.5 and there is a maximum effective tuning range of 10 percent. Hence, the first component does not contribute to the cost. The effective tuning range $t_1' = t_1 - 0.5$ is used as a variable.

The optimality conditions require that

$$\begin{bmatrix} 0 \\ -\dfrac{1}{\varepsilon_2^2} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ 0 \end{bmatrix} + u_5 \begin{bmatrix} -\dfrac{1}{\phi_1^0} \\ 0 \\ \dfrac{t_1'}{{\phi_1^0}^2} \\ 0 \\ 0 \end{bmatrix} + \sum_i u_6(i) \begin{bmatrix} -\rho_1'(i) \\ \mu_2(i) \\ -1 \\ 1 \\ -t_1' e_i \end{bmatrix}$$
$$+ \sum_i u_7(i) \begin{bmatrix} 16\rho_1'(i) \\ -2(\phi_2^0 + \varepsilon_2\mu_2(i))\mu_2(i) \\ 16 \\ -2(\phi_2^0 + \varepsilon_2\mu_2(i)) \\ 16 t_1' e_i \end{bmatrix}$$
$$+ \sum_i u_8(i) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -e_i \end{bmatrix} + \sum_i u_9(i) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e_i \end{bmatrix} \qquad (69)$$

$$u_1 g_1 = \cdots = u_5 g_5 = u_6(i) g_6(i) = \cdots = u_9(i) g_9(i) = 0, \qquad i = 1,2,\cdots \qquad (70)$$

$$u_1,\cdots,u_5,u_6(i),\cdots,u_9(i) \geq 0, \qquad i = 1,2,\cdots. \qquad (71)$$

Minimize $g_6(i)$ of (65) and $g_7(i)$ of (66) with respect to $\mu_2(i)$. We use $\mu_2(i) = -1$ in (65) and $\mu_2(i) = 1$ in (66) for this purpose. The corresponding $\rho_1'(i) = -1$ and $\rho_1'(i) = 1$, respectively, are obtained by maximizing $g_6(i)$ and $g_7(i)$ with respect to $\rho_1'(i)$. This yields the solution

$$\begin{aligned} t_1' &= 0.5432 \\ \varepsilon_2 &= 1.444 \\ \phi_1^0 &= 5.4321 \\ \phi_2^0 &= 8.3333. \end{aligned}$$

As expected, the inclusion of tunable elements can increase the tolerance on the components. The tolerance of the second parameter increases from $\varepsilon_2 = 0.5$ to $\varepsilon_2 = 1.444$ when the first component is allowed to have a maximum effective tuning range of 10 percent. This means that an actual absolute tuning of 1.0432 and a tolerance of 0.5 are designed for ϕ_1. The result can only be accomplished by allowing the nominal point to move. For example, the first component moved from 4.5 to 5.4321, a shift of 20 percent.

VIII. Frequency Domain Implementation

Data for a specific problem is contained in a data vector $\boldsymbol{a}^i$ which has the form

$$\boldsymbol{a}^i \triangleq \begin{bmatrix} r \\ \boldsymbol{\mu} \\ \psi \\ S \\ w \end{bmatrix}, \qquad i = 1,2,\cdots,m_a \qquad (72)$$

where ψ is an independent parameter denoting frequency or any number to identify a particular function for which the vertex $\boldsymbol{\phi}^r$ is chosen. $\boldsymbol{\mu}$ is the vector associated with $\boldsymbol{\phi}^r$, in particular,

$$r = 1 + \sum_{j=1}^{k} \left[\frac{\mu_j^r + 1}{2}\right] 2^{j-1}, \qquad \mu_j^r \in \{-1,1\}. \quad (73)$$

m_a is the total number of distinct vectors $\boldsymbol{a}^i$. S is a specification and w a weighting factor associated with each ψ. In our present work,

$$w = \begin{cases} +1, & \text{if } S \text{ is an upper specification} \\ -1, & \text{if } S \text{ is a lower specification.} \end{cases}$$

The performance constraints may now be formulated in the form of

$$g = w(S - F) \geq 0 \qquad (74)$$

with appropriate subscripts. F is the circuit response function evaluated at sample point ψ and point $\boldsymbol{\phi}$ which is given by

$$\boldsymbol{\phi} = \boldsymbol{P}\boldsymbol{\phi}^r + \sum_{j \in I_t} (\phi_j^0 + t_j'\rho_j'(r))\boldsymbol{e}_j. \qquad (75)$$

The projection matrix $\boldsymbol{P}$ and the index sets I_t and I_ε are fixed for a particular problem. They are determined before optimization takes place.

Let the n optimization variables be denoted by $\boldsymbol{x}$ including the variable nominal values, tolerances, tuning variables and all the appropriate slack variables $\rho_j'(r)$, $j \in I_t$, $r \in I_v$. Let m be the total number of constraints which include the performance specifications, slack variable bounds, parameter bounds, and any other extra constraints not considered above. In general, for linear network design in the frequency domain

$$n = k_0 + k_\varepsilon + k_t(1 + n_v) \qquad (76)$$

and

$$m = \left[\sum_{i=1}^{n^\psi} n_v(i)\right] + 2k_t n_v + \cdots \qquad (77)$$

where k_0, k_ε, and k_t are the number of variable nominal parameters, toleranced and tuned parameters, respectively; $n_v \leq 2^{k_\varepsilon}$ is the number of distinct vertices chosen; n^ψ is the number of frequency points considered; $n_v(i)$ is the number of vertices chosen at the ith frequency point and $2k_t n_v$ is the number of slack variable bounds.

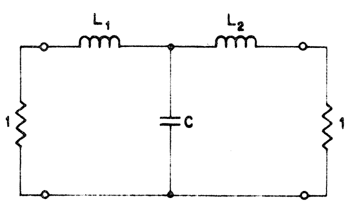

Fig. 5. Circuit for LC low-pass filter example.

TABLE I
SPECIFICATIONS FOR LC LOW-PASS FILTER

Frequency Range (rad/s)	Sample Points (rad/s)	Insertion Loss Specification (dB)	Type	Weight w
0 - 1	0.45, 0.50, 0.55, 1.0	1.5	upper (passband)	+1
2.5	2.5	25	lower (stopband)	-1

TABLE II
DATA FOR LOW-PASS FILTER

a^i	1	2	3	4	5
r	6	6	6	8	1
	+1	+1	+1	+1	-1
	-1	-1	-1	+1	-1
	+1	+1	+1	+1	-1
	0.45	0.50	0.55	1.0	2.5
S	1.5	1.5	1.5	1.5	25
w	1	1	1	1	-1

Low-Pass Filter

The LC low-pass filter shown in Fig. 5 is considered [5], [6]. Table I summarizes the specifications. The critical vertices used in the data vector a^i can be obtained from published vertex selection schemes [6]. These schemes utilize first partial derivative information at some local points or local regions to predict the worst vertices. Very often updating of a^i is required at suitable intervals. In this case, the numerical experience we have gained previously from the tolerance problems [5], [6] allows us to chose the minimal set of vertices. These are: ϕ^6 at $\psi = 0.45, 0.50, 0.55$ rad/s; ϕ^8 at $\psi = 1.0$ rad/s and ϕ^1 at $\psi = 2.5$ rad/s, where $\phi = [L_1 C L_2]^T$. Updating was not required in this example except when all the three components are toleranced and tuned simultaneously. Table II summarizes the data for the filter.

Several cases have been studied [9] but the results of the case L_1 tuned with C and L_2 toleranced will be presented. The objective function used is based on the relative tolerances of C and L_2 in the form

$$C = \frac{x_2}{x_5^2} + \frac{x_3}{x_6^2} \tag{78}$$

where, assuming $t_C = t_{L_2} = 0$, and some fixed value of ε_{L_1}, we take

$$\begin{aligned} x_1 &= \phi_1^0 = L_1^0 \\ x_2 &= \phi_2^0 = C^0 \\ x_3 &= \phi_3^0 = L_2^0 \\ x_4^2 &= t_1' = t_{L_1} - \varepsilon_{L_1} \\ x_5^2 &= \varepsilon_2 = \varepsilon_C \\ x_6^2 &= \varepsilon_3 = \varepsilon_{L_2}. \end{aligned}$$

The cost of element L_1 is assumed fixed. It, therefore, is not included in (78). The last three transformations are chosen to avoid changes of sign. There are three distinct projected vertices: ϕ_p^6, ϕ_p^8, and ϕ_p^1. The projection matrix in this case is

$$P = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}. \tag{79}$$

Therefore, the other variables may be identified as

$$\begin{aligned} x_7 &= \rho_1'(6) \\ x_8 &= \rho_1'(8) \\ x_9 &= \rho_1'(1). \end{aligned} \tag{80}$$

Substituting the numerical values from Table II into (75) we have the following:

$$a^1, a^2, a^3 \Rightarrow \phi = P\phi^6 + (\phi_1^0 + t_1'\rho_1'(6))e_1$$

$$= \begin{bmatrix} x_1 + x_4^2 x_7 \\ x_2 - x_5^2 \\ x_3 + x_6^2 \end{bmatrix} \tag{81}$$

$$a^4 \Rightarrow \phi = P\phi^8 + (\phi_1^0 + t_1'\rho_1'(8))e_1$$

$$= \begin{bmatrix} x_1 + x_4^2 x_8 \\ x_2 + x_5^2 \\ x_3 + x_6^2 \end{bmatrix} \tag{82}$$

$$a^5 \Rightarrow \phi = P\phi^1 + (\phi_1^0 + t_1'\rho_1'(1))e_1$$

$$= \begin{bmatrix} x_1 + x_4^2 x_9 \\ x_2 - x_5^2 \\ x_3 - x_6^2 \end{bmatrix}. \tag{83}$$

The performance specifications g_i, $i = 1,2,\cdots,5$, may now be formed. Additional constraints are given by

$$\left.\begin{aligned} g_{5+2i-1} &= 1 + x_{6+i} \\ g_{5+2i} &= 1 - x_{6+i} \end{aligned}\right\} \quad i = 1,2,3 \tag{84}$$
$$g_{12} = t_r - x_4^2/x_1.$$

The last constraint g_{12} is designed to limit the effective tuning range to t_r.

The resulting nonlinear programming problem (9 variables, 12 constraints) is solved by a least pth optimization algorithm due Charalambous [10] and the quasi-Newton method developed by Fletcher [11] and Gill and Murray [12]. The starting point corresponds to the optimally toleranced nominal point and arbitrary small tolerance and tuning values. Typically, a few hundred function evaluations with less than 30 s of CDC 6400 computing time is

TABLE III
L_1 TUNED, C AND L_2 TOLERANCED

Parameters	$t_r = 0.2$	$t_r = 0.1$	$t_r = 0.05$
L_1^0	2.0932	2.2442	2.1953
C^0	0.9360	0.9059	0.9062
L_2^0	1.7718	1.7569	1.7920
$100\ t_1'/L_1^0$	20.00 %	10.00 %	5.00 %
$100\ \varepsilon_2/C^0$	15.99 %	14.23 %	12.60 %
$100\ \varepsilon_3/L_2^0$	21.62 %	18.41 %	16.23 %
$\rho_1'(6)$		-1.0000	
$\rho_1'(8)$		-1.0000	
$\rho_1'(1)$		1.0000	
	$n = 9$	$m = 12$	

† For the optimally toleranced solution [5] $L_1{}^0 = L_2{}^0 = 1.9990$, $C^0 = 0.9056$, $100\varepsilon_1/L_1{}^0 = 100\varepsilon_3/L_2{}^0 = 9.89\%$, $100\varepsilon_2/C^0 = 7.60\%$.

required. Table III summarizes the results. Three different tuning ranges are used. The 5-percent tuning of L_1 increases the tolerances of the other two components by as much as 65 percent. A 10-percent positive and negative shift is obtained for $L_1{}^0$ and $L_2{}^0$, respectively. C^0 is shifted slightly. The slack variables assume values of -1, -1, and 1, indicating that the worst cases do occur at the vertices and, correspondingly, the tuning is set to extreme values.

Tuning of C presents a very interesting case. The symmetry property observed in the pure tolerance problem is preserved. Due to this symmetry, a 5-percent tuning range of C produces an increase of 90 percent in the tolerances of L_1 and L_2.

Suppose the designer has no prior knowledge of the choice of the tuning component. We consider an objective function of the form

$$C = \sum_{i=1}^{3} \left[\frac{\phi_i{}^0}{\varepsilon_i} + c \frac{t_i}{\phi_i{}^0} \right]. \tag{85}$$

One additional vertex ϕ^3 is considered in order to bound the solution during optimization. We omit details of the constraints, and summarize the final results in Table IV for different c. There are 21 variables and 36 constraints, hence, the computational effort has substantially increased over the previous case. The advantage gained in the general formulation is that the optimization will automatically choose the most appropriate component for tuning, which is C in the objective of (85).

The same designs can be obtained by the reduced formulation using C as a tuned and toleranced component and L_1 and L_2 as toleranced components.

High-Pass Filter

This problem was suggested by Pinel and Roberts [13]. The circuit diagram is shown in Fig. 6 and the basic specifications for the design are listed in Table V. The insertion loss relative to the loss at 990 Hz is to be constrained as indicated with resistances R_5 and R_7 related to $L_5{}^0$ and

TABLE IV
OPTIMAL TUNING

Parameters	$c = 10$	$c = 20$	$c = 50$
$L_1^0 = L_2^0$	1.8440	1.9221	2.0492
C^0	1.1730	1.0486	0.9069
$100\ \varepsilon_1/L_1^0 = 100\ \varepsilon_3/L_2^0$	31.62 %	23.84 %	16.15 %
$100\ \varepsilon_2/C^0$	31.62 %	22.36 %	14.14 %
$100\ t_1/L_1^0 = 100\ t_3/L_2^0$	2.54 %	0.00 %	0.00 %
$100\ t_2/C^0$	54.31 %	35.89 %	14.14 %
$\rho_1(6)$	-1.0000	-0.7165	0.9743
$\rho_2(6)$	0.1645	0.2466	1.0000
$\rho_3(6)$	-1.0000	-0.9992	-0.9846
$\rho_1(8)$	-1.0000	-1.0000	-0.8813
$\rho_2(8)$	-1.0000	-1.0000	-1.0000
$\rho_3(8)$	-1.0000	-1.0000	-0.9876
$\rho_1(1)$	1.0000	0.9887	0.9933
$\rho_2(1)$	1.0000	1.0000	1.0000
$\rho_3(1)$	1.0000	0.9989	0.9029
$\rho_1(3)$	1.0000	0.8433	-0.6051
$\rho_2(3)$	-0.1645	-0.1468	0.6434
$\rho_3(3)$	1.0000	0.8944	0.6441
$100\ \varepsilon_1'/L_1^0 = 100\ \varepsilon_3'/L_2^0$	29.08 %	23.84 %	14.14 %
$100\ t_2'/C^0$	22.69 %	13.53 %	0.00 %
	$n = 21$	$m = 36$	

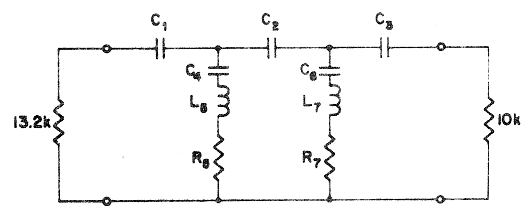

Fig. 6. High-pass filter.

TABLE V
SPECIFICATIONS FOR HIGH-PASS FILTER

Frequency Range (Hz)	Basic Sample Points (Hz)	Relative Insertion Loss (dB)	Weight w
170	170	45.	-1
360	360	49.	-1
440	440	42.	-1
630 - 680	630	4.	+1
680 - 1800	680 710 725 740	1.75	+1
630 - 1800	630 650 680 860 910 930 1050	-0.05	-1

Reference Frequency: 990 Hz

R_5, R_7 related to L_5^0 and L_7^0 through $Q = \dfrac{2\pi 990 L_5^0}{R_5} = \dfrac{2\pi\ 990 L_7^0}{R_7} = 1456$

L_7^0 with constant Q. The terminations are fixed, the designable parameters being C_1, C_2, C_3, C_4, L_5, C_6, and L_7.

The objective function throughout was taken as

$$C = \sum_{i=1}^{7} \frac{\phi_i^0}{\varepsilon_i} \tag{86}$$

where

$$\phi^0 = \begin{bmatrix} C_1^0 \\ C_2^0 \\ C_3^0 \\ C_4^0 \\ L_5^0 \\ C_6^0 \\ L_7^0 \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_{C_1} \\ \varepsilon_{C_2} \\ \varepsilon_{C_3} \\ \varepsilon_{C_4} \\ \varepsilon_{L_5} \\ \varepsilon_{C_6} \\ \varepsilon_{L_7} \end{bmatrix}.$$

The optimization package used here is DISOPT [14], which has been previously employed in worst-case tolerance problems [6]. The same quasi-Newton unconstrained minimization procedure as for the work described in the previous section is incorporated into DISOPT. The extrapolation feature [15] was chosen to accelerate convergence to the constrained optimum.

Verification of the designs to be described was carried out using all 2^7 vertices plus the nominal point at 170, 360, 440, 630–680, and 680–1800 Hz. Forty-two logarithmically spaced points were taken for the latter interval, and 8 for the former interval.

Table VI indicates the effort required to obtain the results of Table VII. Because of the complexity of the problems preliminary runs of the program were required before the final number of constraints were established. This information along with a realistic assessment of cost is given.

Case 1: No Tuning ($t = 0$)

Table VI summarizes the particular frequencies, specifications and the particular vertex number employed to obtain the final tolerances listed in Table VII. Table VII also lists the shifts in nominal parameter values with respect to those of an uncentered design [7], [13].

Case 2: 3 Percent Tuning for L_5

Results corresponding to the ones for Case 1 are tabulated in Tables VI and VII. Note that all the tolerances have increased. Fig. 7 shows the nominal response as well as the worst upper and lower outcomes based on all 2^7 vertices.

A more detailed verification of the results was made. Sixty logarithmically spaced points were taken from the critical region 630–680 Hz as well as 40 from 600–630 Hz. All the vertices were checked plus the nominal point, followed by 4000 Monte Carlo simulations uniformly distributed in the effective tolerance region. No violations were detected, and the upper and lower limits of response given by the vertices bounded the results from the Monte Carlo analysis except at 638.2 Hz, where the lowest relative loss obtained from the vertices was −0.0243 dB, whereas the Monte Carlo analysis yielded −0.0246 dB.

As a further check on the optimality of these results, L_5 was allowed to be both toleranced and tuned as distinct from being effectively toleranced from the point of view of optimization. The same vertices, an additional 25-ρ variables and 50 additional constraints on the ρ variables were used without any significant improvement in the results. The values of the ρ variables confirmed the assumption that L_5 should be effectively toleranced for 3 percent tuning.

Case 3: 3 Percent Tuning for L_5 *and* L_7

As indicated by Table VII a further improvement in all tolerances has been obtained.

TABLE VI
DATA FOR OPTIMIZATION OF HIGH-PASS FILTER

Frequency (Hz)	S (dB)	w	Vertex Number: Case 1 No Tuning	Case 2 L_5 Tuned	Case 3 L_5 and L_7 Tuned	Case 4 L_7 Tuned
170	45	-1	8	8	8	8
360	49	-1	48	48	48	48
440	42	-1	128	128	128	128
630	4	+1	1	1	1	1
630	-0.05	-1	60,100,104, 108,120,126	58,60,100, 104,108,120 126	60,108,120	60,87,95, 100,104,108, 120,126
637	-0.05	-1	-	-	-	87
640	-0.05	-1	-	58	108	52,58,60
643	-0.05	-1	-	-	-	85,93,117
650	-0.05	-1	nominal,12, 50,58,102	nominal,12, 34,42,50,58, 102,106,126	nominal,12,34, 42,44,58,106, 126	nominal,12, 36,42,50,58, 85,93,94, 102,106,126
658	-0.05	-1	-	-	42	58,69,85
665	-0.05	-1	-	-	34,42	34,58
670	-0.05	-1	-	-	-	2
680	1.75	+1	123	123	123	123
680	-0.05	-1	2,6	2,6	2,6	2,6
710	1.75	+1	43,83	43,83	43,83,123	43,83
725	1.75	+1	43,83	43,83	43,83	43,83
730	1.75	+1	-	-	43,83	43
740	1.75	+1	43,83	43,83	43,83	43,83
860	-0.05	-1	118,126	118,126	118,126	118,126
910	-0.05	-1	118,126	118,126	118,126	118,126
930	-0.05	-1	118,126	118,126	118,126	118,126
1040	-0.05	-1	-	-	-	3
1050	-0.05	-1	3	3	3	3
Number of Response Constraints			31	37	37	55
Total Number of Constraints m			45	51	51	69
Number of Variables n			14	14	14	14
Starting Point			Given by Pinel [13]	Optimum	for	Case1
Number of Trial Runs			3	1	2	1
Total Computing Effort (min)†			15	5	6	7
Computing Cost Including Trials†			\$94	\$31	\$37	\$44

† On a CDC 6400.

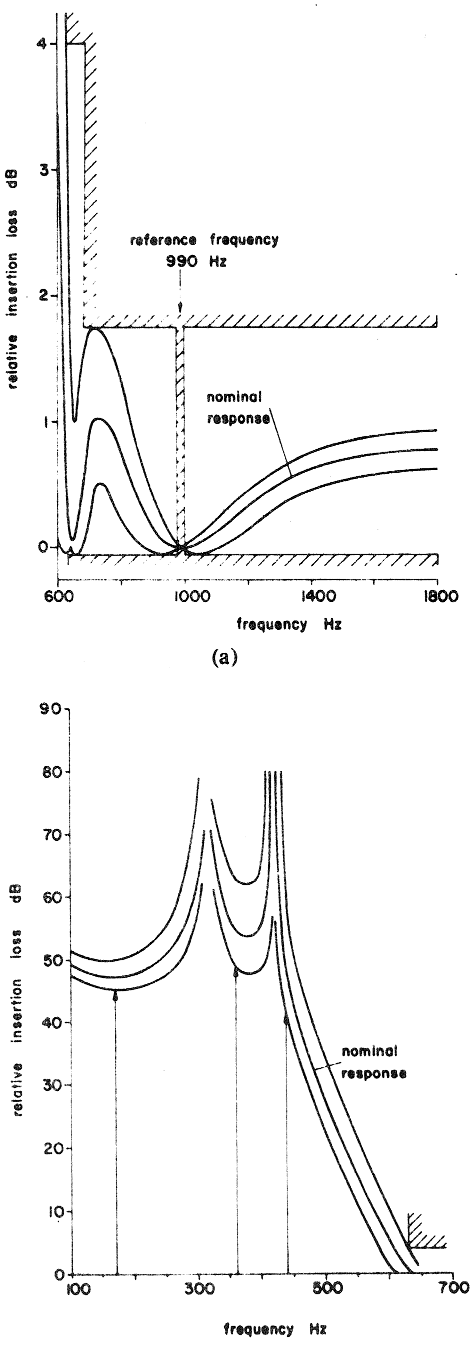

Fig. 7. (a) Passband details of optimized high-pass filter (Case 2). (b) Stopband details of optimized high-pass filter (Case 2).

Case 4: 3 Percent Tuning for L_7

The results for this problem (Table VII) are slightly worse than those for Case 2. A slight violation of the specification at 658 Hz was detected. We conclude that if only one inductor is to be tuned, L_5 should be chosen.

IX. Conclusions

A theory of optimal worst-case design embodying centering, tolerancing and tuning has been presented. The concept of a tunable constraint region that allows variable specifica-

TABLE VII
Results for High-Pass Filter

	Parameters	Case 1 No Tuning	Case 2 L_5 Tuned	Case 3 L_5 and L_7 Tuned	Case 4 L_7 Tuned
C_1	tolerance (%)	5.71	6.77	7.90	6.63
	nom. shift(%)	+18.1	+17.8	+18.3	+17.6
C_2	tolerance (%)	4.33	4.97	5.32	4.77
	nom. shift(%)	+16.2	+15.2	+14.4	+15.3
C_3	tolerance (%)	4.72	5.81	7.23	5.83
	nom. shift(%)	+16.6	+18.0	+18.8	+17.8
C_4	tolerance (%)	4.54	5.03	5.15	4.78
	nom. shift(%)	-3.8	-2.2	-1.2	[illegible]
L_5	tolerance (%)	3.29	3.95	4.44	3.82
	nom. shift(%)	-3.0	-3.0	-4.3	-4.1
C_6	tolerance (%)	6.32	7.05	7.27	6.66
	nom. shift(%)	-7.3	-5.1	-3.6	-6.0
L_7	tolerance (%)	3.64	4.34	5.04	4.32
	nom. shift(%)	-6.4	-7.9	-7.9	-6.3
	Cost	157	135	121	138*

* Violation of specifications. Relative loss = −0.052 dB at 658 Hz.

tions as set by the customer has also been incorporated. This may find application, for example, in tunable filters. The purely toleranced and purely tuned problems become special cases. Further simplification has been discussed in the light of one-dimensional convexity.

As expected, the inclusion of tunable elements can increase the tolerances on the components. The results seem to justify the reduction of the general tolerance–tuning problem into one containing effectively toleranced and effectively tuned components, where appropriate. If the separation of the components is not decided in advance, the general problem with the cost function reflecting both tolerances and tuning ranges is appropriate, since an optimization program requires an explicit number of variables and constraints in advance.

A component may be both tuned and toleranced simultaneously. Thus, one can represent the effects of an uncertainty of a tuned component if the tuning range is larger than the tolerance. On the other hand, if the tolerance is larger than the tuning range (see, for example, Table VII), it may be considered to be a toleranced component with some small tuning capacity. The tuning range may or may not appear in the objective function. The different weightings of tuning and tolerancing in the objective exhibit the flexibility of the formulation. With a very heavy weighting in the tuning, we will obtain a solution equivalent to a pure tolerance problem. Zero tuning is automatically indicated by the result of the formulation. Reducing the weighting will increase the tolerance as well as the tuning with a net effect of reducing the effective tolerance $\varepsilon_i' = \varepsilon_i - t_i$ until a crossover occurs from effective tolerance to effective tuning. Beyond that, the effective tuning value will continue to increase until a threshold value occurs. Below the threshold, the solution in terms of effective tuning and tolerance problem is unaffected. The tolerances of other components will continue to increase with decreasing weighting on the tuning.

A cost function tending to maximize tolerances and minimizing tuning has been implemented successfully in this context. For the high-pass filter the 3-percent tuning range on the inductors was considered free, thus tuning did not enter into the objective function. A reduced problem involving effective tolerances was found adequate since, as shown in Table VII, the tolerances exceed the tuning ranges. A good starting point for the tuning problem is a worst-case toleranced solution. The small tuning ranges in the high-pass filter problem meant that relatively small nominal shifts were obtained.

It may be added that, as far as the authors are aware, this seems to be the most general formulation to date dealing with the centering, tolerancing and tuning problems at the design stage.

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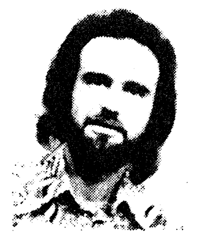
REFERENCES

[1] J. F. Pinel and K. A. Roberts, "Tolerance assignment in linear networks using nonlinear programming," *IEEE Trans. Circuit Theory*, vol. CT-19, Sept. 1972, pp. 475–479.
[2] J. F. Pinel, "Tolerance assignment and network alignment of linear networks in the frequency domain," in *IEEE Short Course on Computer Aided Network Design*, 73-SC-06, Mar. 1973, pp. 17–25.
[3] J. W. Bandler, "Optimization of design tolerances using nonlinear programming," *J. Optimization Theory and Applications*, vol. 14, July 1974, pp. 99–114.
[4] B. Karafin, "The general component tolerance assignment problem in electrical networks," Ph.D. dissertation, Univ. Pennsylvania, Philadelphia, 1974.
[5] J. W. Bandler and P. C. Liu, "Automated network design with optimal tolerances," *IEEE Trans. Circuits Syst.*, vol. CAS-21, Mar. 1974, pp. 219–222.
[6] J. W. Bandler, P. C. Liu, and J. H. K. Chen, "Worst case network tolerance optimization," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 630–641, Aug. 1975.
[7] J. F. Pinel, K. A. Roberts, and K. Singhal, "Tolerance assignment in network design," in *Proc. IEEE Int. Symp. Circuits and Systems*, pp. 317–320.
[8] J. F. Pinel, "Computer-aided network tuning," *IEEE Trans. Circuit Theory*, vol. CT-18, Jan. 1971, pp. 192–194.
[9] P. C. Liu, "A theory of optimal worst-case design embodying centering, tolerancing and tuning, with circuit applications," Ph.D. dissertation, McMaster Univ., Hamilton, Ont., Canada, 1975.
[10] C. Charalambous, "Nonlinear least *p*th optimization and nonlinear programming," Dep. Combinatorics and Optimization, Univ. Waterloo, Waterloo, Ont. Canada, Rep. CORR 74-3, Jan. 1974.
[11] R. Fletcher, "FORTRAN subroutines for minimization by quasi-Newton methods," Atomic Energy Research Establishment, Harwell, Berkshire, England, Rep. AERE-R7125, 1972.
[12] P. E. Gill and W. Murray, "Quasi-Newton methods for unconstrained optimization," *J. Inst. Math. Appl.*, vol. 9, 1972, pp. 91–108.
[13] J. F. Pinel, Bell-Northern Research, Ottawa, Ont., Canada, private communication, Aug. 30, 1974.
[14] J. W. Bandler and J. H. K. Chen, "DISOPT—a general program for continuous and discrete nonlinear programming problems," *Int. J. Syst. Sci.*, vol. 6, pp. 665–680. Documented listing available through Faculty of Engineering, McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-29, Mar. 1974.
[15] J. W. Bandler and W. Y. Chu, "Computational merits of extrapolation in least *p*th approximation and nonlinear programming," *Proc. 12th Allerton Conf. Circuit and System Theory*, pp. 912–921.



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Integrated Approach to Microwave Design

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Abstract—**A new, integrated approach to microwave design is presented involving concepts such as optimal design centering, optimal design tolerancing, optimal design tuning, parasitic effects, uncertainties in models and reference planes, and mismatched terminations. The approach is of the worst case type, and previously published design schemes fall out as particular cases of the ideas presented. The mathematical and computational complexity as well as the benefits realized by our approach is illustrated by transformer examples, including a realistic stripline circuit.**

I. Introduction

The use of nonlinear programming techniques for the design of microwave circuits has been well established. Applications hitherto reported by the authors, for example, fall into two categories. 1) The improvement of a response in the presence of parasitics [1], [2], in which case the function to be minimized is of the error function type and the constraints, if any, are normally imposed on the design parameters. 2) Design centering and tolerance assignment to yield a minimum cost circuit that satisfies certain specifications, usually imposed on the frequency response, for all possible values of the actual parameters [3]. The function to be minimized is of the cost function type and the constraints are due to the specifications. Tuning elements may be introduced to further increase possible unrealistic tolerances and thus decrease the cost or make a circuit meet specifications [4].

No consideration, however, of optimal tolerancing or tuning of microwave circuits has been reported where parasitic effects were taken into account. A major complication is introduced here, since the models available for common parasitic elements normally include uncertainties on the value of the model parameters. These uncertainties are due to the fact that the model is usually only approximate and that approximations have to be made in the implementation of existing model formulas. A typical example of the latter is the relationship between the characteristic impedance and width of a symmetric stripline, where the formula involves elliptic integrals.

The model uncertainties can well be of the same order of magnitude as the tolerances on the physical network parameters so that a realistic design, including tolerances, can only be found when allowance is made for them. In the approach adopted, an attempt is made to deal with the model uncertainties in the same way as with the other tolerances. This involves, however, a complication in the formulation of the problem. The physical tolerances affect the physical parameters, whereas the model parameter uncertainties affect a set of intermediate parameters (which will be called the model parameters) in the calculation of the response.

In the present paper we consider design of microwave circuits with the following concepts treated as an integral part of the design process: optimal design centering, optimal design tolerancing, optimal design tuning, parasitic effects, uncertainties in the circuit modeling, and mismatches at the source and the load.

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II. Theory

The Tolerance-Tuning Problem

In this section we introduce some of the notation and briefly review the parameters involved in the tolerance-tuning problem.

We consider first a vector of nominal design parameters ϕ^0 and a corresponding vector containing the manufacturing tolerances ε. Thus, for k variables,

$$\phi^0 \triangleq \begin{bmatrix} \phi_1^{\,0} \\ \phi_2^{\,0} \\ \vdots \\ \phi_k^{\,0} \end{bmatrix} \qquad \varepsilon \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_k \end{bmatrix}. \tag{1}$$

A possible outcome of a design is then

$$\phi = \phi^0 + E\mu_\varepsilon \tag{2}$$

where

$$\mu_\varepsilon \triangleq \begin{bmatrix} \mu_{\varepsilon_1} \\ \mu_{\varepsilon_2} \\ \vdots \\ \mu_{\varepsilon_k} \end{bmatrix} \tag{3}$$

and

$$E \triangleq \begin{bmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \ddots & \\ & & & \varepsilon_k \end{bmatrix}. \tag{4}$$

The vector μ_ε determines the actual outcome and can, for example, be bounded by

$$-1 \le \mu_{\varepsilon_i} \le 1, \qquad i = 1,2,\cdots,k. \tag{5}$$

It is assumed that the designer has no control over μ_ε. This leads to the concept of the tolerance region R_ε, namely, the set of points ϕ of (2) subject to, for example, (5). An untuned design implies ϕ as given by (2). Consider a vector t containing tuning variables corresponding to (1). Thus

$$t \triangleq \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{bmatrix}. \tag{6}$$

A design outcome with tuning implies

$$\phi = \phi^0 + E\mu_\varepsilon + T\mu_t \tag{7}$$

where

$$\mu_t \triangleq \begin{bmatrix} \mu_{t_1} \\ \mu_{t_2} \\ \vdots \\ \mu_{t_k} \end{bmatrix} \tag{8}$$

and

$$T \triangleq \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_k \end{bmatrix}. \tag{9}$$

The vector μ_t determines the setting of the tuning elements and we consider, for convenience,

$$-1 \le \mu_{t_i} \le 1, \qquad i = 1,2,\cdots,k. \tag{10}$$

Hence, we have a tuning region R_t centered at $\phi^0 + E\mu_\varepsilon$ for each outcome μ_ε.

The worst case tolerance-tuning problem is to obtain an optimal set $\{\phi^0,\varepsilon,t\}$ such that all possible outcomes (controlled by μ_ε) can be tuned so as to satisfy the design specifications (by adjusting μ_t) if tuning is available. If tuning is not available all outcomes must satisfy the design specifications. A detailed discussion has been presented [4].

Model Uncertainties

Taking ϕ as the vector of physical design parameters which have to be determined and appear in the cost function, we may consider an n-dimensional vector p containing the model parameters, e.g., the parameters appearing in an electrical equivalent circuit. In general, $n \neq k$. We have an associated vector of nominal model parameters p^0 and a vector of model uncertainties δ, where

$$p^0 \triangleq \begin{bmatrix} p_1^{\,0} \\ p_2^{\,0} \\ \vdots \\ p_n^{\,0} \end{bmatrix} \qquad \delta \triangleq \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix}. \tag{11}$$

A possible model can then be described by

$$p = p^0 + \Delta\mu_\delta \tag{12}$$

where

$$\mu_\delta \triangleq \begin{bmatrix} \mu_{\delta_1} \\ \mu_{\delta_2} \\ \vdots \\ \mu_{\delta_n} \end{bmatrix} \tag{13}$$

and

$$\Delta \triangleq \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \ddots & \\ & & & \delta_n \end{bmatrix}. \tag{14}$$

Thus μ_δ determines the particular model under consideration. We will assume

$$-1 \le \mu_{\delta_i} \le 1, \qquad i = 1,2,\cdots,n \tag{15}$$

and also the functional dependence on ϕ implied by

$$p = p^0(\phi) + \Delta(\phi)\mu_\delta. \tag{16}$$

Given a tolerance region in the ϕ space it would be hard, in general, to envisage its effect in the p space, even if $\delta = 0$. The selection of worst case p is complicated by the modeling uncertainties. Especially when $n < k$ more than one $\{\mu_\varepsilon,\mu_\delta\}$ may give the same worst case p. In selecting candidates we will assume, intuitively, that the following is sufficient:

$$\mu_{\varepsilon_i},\mu_{\delta_j} = \pm 1, \qquad i = 1,2,\cdots,k, \quad j = 1,2,\cdots,n. \tag{17}$$

Mismatch Considerations

We consider environmental influences in the form of mismatches at the source and load. The situation is depicted in Fig. 1. The discussion is directed towards handling terminations with prescribed maximum reflection-coefficient amplitudes and arbitrary reference planes, the

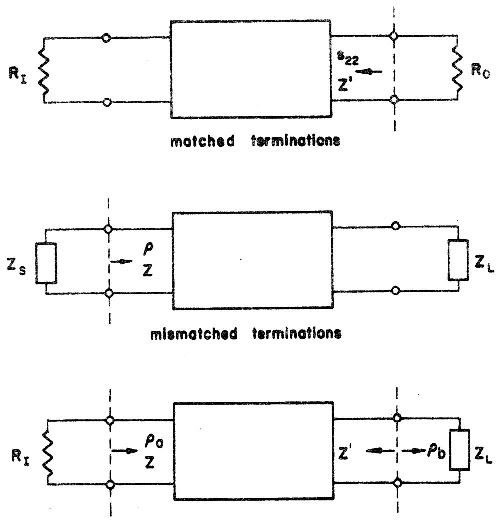

Fig. 1. Two-port circuit viewed with respect to three sets of terminations for defining impedances Z and Z' and reflection coefficients ρ, ρ_a, ρ_b, and s_{22}.

mismatches at different frequencies being, pessimistically, taken as independent.

Fig. 1(a) shows the ideal situation of matched resistive terminations R_I and R_O. Assume that the actual complex terminations as seen by the circuit are Z_S and Z_L, as shown in Fig. 1(b). Then the reflection coefficient

$$\rho_S = \frac{Z_S - R_I}{Z_S + R_I} \tag{18}$$

at the source, and

$$\rho_L = \frac{Z_L - R_O}{Z_L + R_O} \tag{19}$$

at the load. The actual reflection coefficient ρ at the source is given by

$$\rho = \frac{Z - Z_S^*}{Z + Z_S} \tag{20}$$

using the notation of Fig. 1(b). The asterisk denotes the complex conjugate.

Consider the situation depicted in Fig. 1(c). We have, for a matched source and mismatched load, the input impedance Z with the reflection coefficients

$$\rho_a = \frac{Z - R_I}{Z + R_I} \tag{21}$$

and

$$\rho_b = \frac{Z_L - Z'^*}{Z_L + Z'} \tag{22}$$

where Z' is the impedance at the output when the input is matched. Associated with the latter situation is the parameter s_{22} given by [Fig. 1(a)]

$$s_{22} = \frac{Z' - R_O}{Z' + R_O}. \tag{23}$$

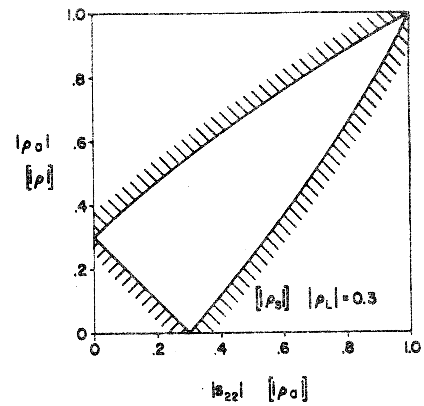

Fig. 2. Feasible region of reflection coefficients given that $|\rho_S| = |\rho_L| = 0.3$.

From (18), (20), and (21) we can obtain ρ in terms of ρ_S and ρ_a. Similarly, from (19), (22), and (23) we can obtain ρ_b in terms of s_{22} and ρ_L. Using Carlin and Giordano [5] we may readily derive the following expressions. For all possible phases

$$\frac{||\rho_a| - |\rho_S||}{1 - |\rho_a||\rho_S|} \le |\rho| \le \frac{|\rho_a| + |\rho_S|}{1 + |\rho_a||\rho_S|} \tag{24}$$

where, assuming a lossless circuit, $|\rho_a| = |\rho_b|$ and

$$\frac{||\rho_L| - |s_{22}||}{1 - |\rho_L||s_{22}|} \le |\rho_b| \le \frac{|\rho_L| + |s_{22}|}{1 + |\rho_L||s_{22}|}. \tag{25}$$

A particular example showing the extreme values of $|\rho_a|$ and $|\rho|$ is shown in Fig. 2.

Explicit upper and lower bounds on $|\rho|$ may be derived. Simplest is the upper bound, given for all possible phases of ρ_S and ρ_L and constant amplitude by

$$\max |\rho| = \frac{K_p + |s_{22}|}{1 + K_p|s_{22}|} \tag{26}$$

where

$$K_p = \frac{|\rho_L| + |\rho_S|}{1 + |\rho_L||\rho_S|}. \tag{27}$$

Let

$$K_q = \frac{|\rho_L| - |\rho_S|}{1 - |\rho_L||\rho_S|} \tag{28}$$

and

$$K_r = -K_q. \tag{29}$$

Assuming all possible phases of ρ_S and ρ_L, but constant amplitude as before, we obtain the following lower bounds.

$$\min |\rho| = \begin{cases} \dfrac{|s_{22}| - K_p}{1 - K_p|s_{22}|}, & \text{if } K_p < |s_{22}| \\ \dfrac{K_q - |s_{22}|}{1 - K_q|s_{22}|}, & \text{if } K_p > |s_{22}|,\ |\rho_L| > |\rho_S|,\ K_q > |s_{22}| \\ \dfrac{K_r - |s_{22}|}{1 - K_r|s_{22}|}, & \text{if } K_p > |s_{22}|,\ |\rho_L| < |\rho_S|,\ K_r > |s_{22}| \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

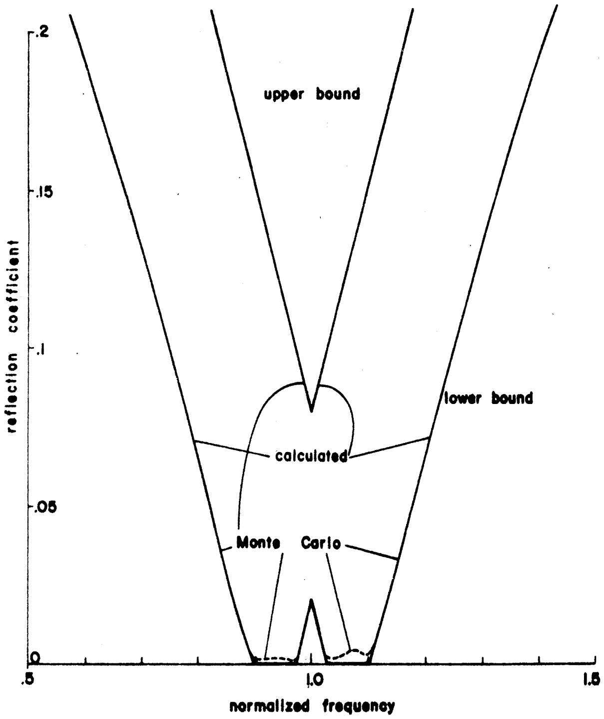

Fig. 3. Upper and lower bounds on reflection coefficient calculated from (26) and (30) and checked by a Monte Carlo analysis (1000 points) for an ideal one-section transformer from 50 to 20 Ω with $|\rho_S| = 0.05$ and $|\rho_L| = 0.03$.

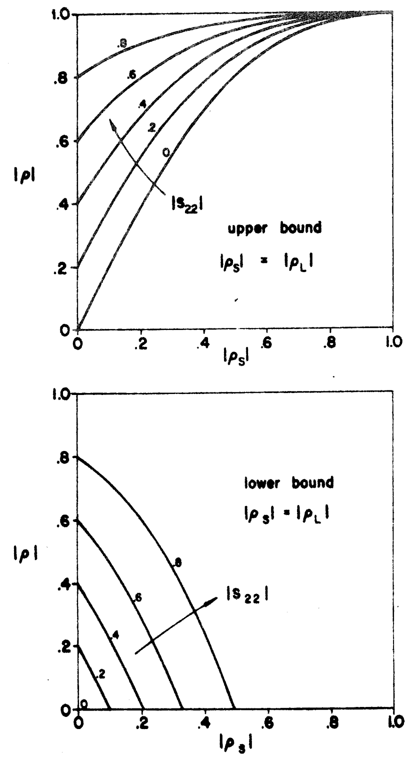

Fig. 4. Upper and lower bounds on $|\rho|$ for $|\rho_S| = |\rho_L|$.

Fig. 3 shows a comparison of these relations with the results of a Monte Carlo analysis with 1000 uniformly distributed values for the phases of ρ_S and ρ_L on $[0,2\pi]$ for a particular example of an ideal one-section transformer from 50 to 20 Ω with $|\rho_S| = 0.05$ and $|\rho_L| = 0.03$.

Assume now all possible amplitudes up to $|\rho_S|$ and $|\rho_L|$ in addition to all possible phases. The upper bound remains the same as (26) but the lower bound becomes

$$\min |\rho| = \begin{cases} \dfrac{|s_{22}| - K_p}{1 - K_p|s_{22}|}, & \text{if } K_p < |s_{22}| \\ 0, & \text{if } K_p \geq |s_{22}|. \end{cases} \quad (31)$$

An illustration for $|\rho_S| = |\rho_L|$ is shown in Fig. 4. We note that under this restriction, the results are not affected by whether all possible amplitudes are considered or not.

Design Specifications

Let all the performance specifications and constraints be expressed in the form

$$g_i \geq 0 \quad (32)$$

where g_i is, in general, an ith nonlinear function of $\boldsymbol{p}(\boldsymbol{\phi})$. Thus we may consider mismatches by an expression of the form

$$g_i = g_i^0(\boldsymbol{p}) + \mu_{\rho_i}(\boldsymbol{p},\rho_{S_i},\rho_{L_i}) \quad (33)$$

where subscript i may denote a sample point and where ρ_S represents the source mismatch and ρ_L the load mismatch. The function μ_{ρ_i} has the effect of shifting the constraint.

Given mismatches, model uncertainties, and so on, obviously influence the nominal design parameters and manufacturing tolerances. An objective, for example, is to find an optimal set $\{\boldsymbol{\phi}^0,\boldsymbol{\varepsilon},\boldsymbol{t}\}$ such that all possible outcomes (controlled by $\boldsymbol{\mu}_\varepsilon$), all possible models (controlled by $\boldsymbol{\mu}_\delta$), and all possible mismatches (controlled by $\boldsymbol{\mu}_\rho$) are accommodated in satisfying the design specifications.

III. Examples

To illustrate some of the ideas presented, we consider two simple circuits. The first includes tuning, the second considers possible model uncertainties, parasitic effects, and mismatched terminations.

Two-Section Transformer

An upper specified reflection coefficient of 0.55 for a two-section lossless transmission-line transformer with quarter-wave-length sections and an impedance ratio of 10:1 was considered at 11 uniformly spaced frequencies on 100-percent relative bandwidth.

Table I shows some results of minimizing certain objective (cost) functions of relative tolerances and tuning ranges. The functions are chosen to penalize small tolerances and large tuning ranges. The design parameters are the normalized characteristic impedances of the two sections, namely, Z_1 and Z_2. The problem has already been considered from the purely tolerance point of view [3]. The parameter ε_i' is the effective tolerance [4] of the ith parameter, i.e.,

$$\varepsilon_i' \triangleq \varepsilon_i - t_i \quad \text{for} \quad \varepsilon_i > t_i. \quad (34)$$

TABLE I
TWO-SECTION 10:1 QUARTER-WAVE TRANSFORMER DESIGN CENTERING, TOLERANCING, AND TUNING

Cost Function*	C_1	C_1	C_1	C_2	C_3	C_4	C_5
Z_1^0	2.1487	2.0340	2.2754	2.5025	1.8748	2.1487	2.1487
Z_2^0	4.7307	4.5355	4.9467	5.3337	4.2642	4.7307	4.7307
$\epsilon_1/Z_1^0 \times 100\%$	12.74	17.83	17.60	25.08	31.62	31.62	12.74
$\epsilon_2/Z_2^0 \times 100\%$	12.74	17.60	17.83	31.62	25.08	31.62	12.74
$t_1/Z_1^0 \times 100\%$	-	10.00	-	-	31.62	18.88	0.00
$t_2/Z_2^0 \times 100\%$	-	-	10.00	31.62	-	18.88	0.00
$\epsilon_1'/Z_1^0 \times 100\%$	-	7.83	-	-	0.00	12.74	12.74
$\epsilon_2'/Z_2^0 \times 100\%$	-	-	7.83	0.00	-	12.74	12.74

$*C_1 = Z_1^0/\epsilon_1 + Z_2^0/\epsilon_2$
$C_2 = Z_1^0/\epsilon_1 + Z_2^0/\epsilon_2 + 10(t_2/Z_2^0)$
$C_3 = Z_1^0/\epsilon_1 + Z_2^0/\epsilon_2 + 10(t_1/Z_1^0)$
$C_4 = Z_1^0/\epsilon_1 + Z_2^0/\epsilon_2 + 10(t_1/Z_1^0 + t_2/Z_2^0)$
$C_5 = Z_1^0/\epsilon_1 + Z_2^0/\epsilon_2 + 500(t_1/Z_1^0 + t_2/Z_2^0)$

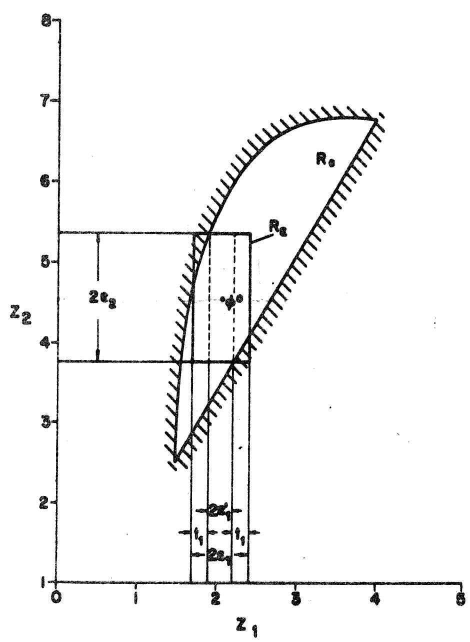

Fig. 5. Optimal solution corresponding to Column 3 of Table I. R_c is the constraint region, i.e., the region for which $|\rho| \leq 0.55$.

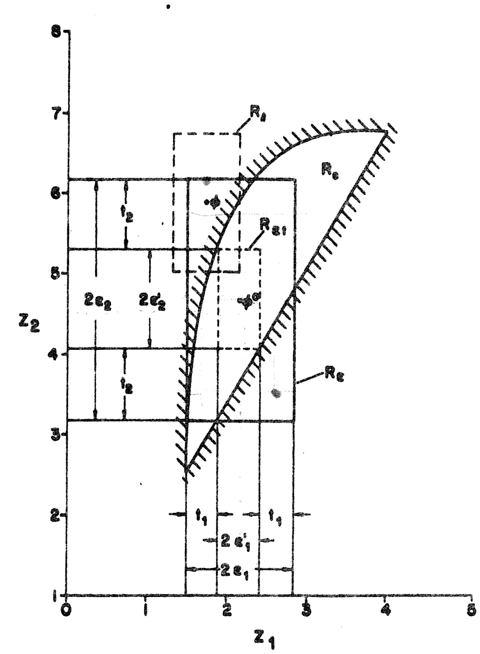

Fig. 6. Optimal solution corresponding to Column 7 of Table I. $R_{\epsilon t}$ is the *effective* tolerance region.

A number of interesting, but not unexpected, features may be noted. Column 2 of Table I shows results for no tuning [3]. Columns 3 and 4 show results when Z_1 and Z_2 are tunable, respectively, by 10 percent. Note that the nominal points move and the tolerances increase. Fig. 5 illustrates the optimal solution corresponding to Column 3. The remaining results indicate solutions when the tuning ranges are variables and included in the objective functions. Observe that the results in the final two columns are essentially the same as those in Column 2. The last column shows how the tuning ranges are automatically set to 0 when they are heavily weighted in the cost function, i.e., they are assumed to be expensive. Fig. 6 corresponds to the situation of Column 7.

Tuning of any component enhances all the tolerances, as expected. Furthermore, if tuning is expensive, it is rejected by the general formulation, which is useful if the designer has a number of possible alternative tunable components and is not sure which components should be effectively tuned ($t_i \geq \epsilon_i$) and which should be effectively toleranced.

One-Section Stripline Transformer

A more realistic example of a one-section transformer on stripline from 50 to 20 Ω is now considered. The physical

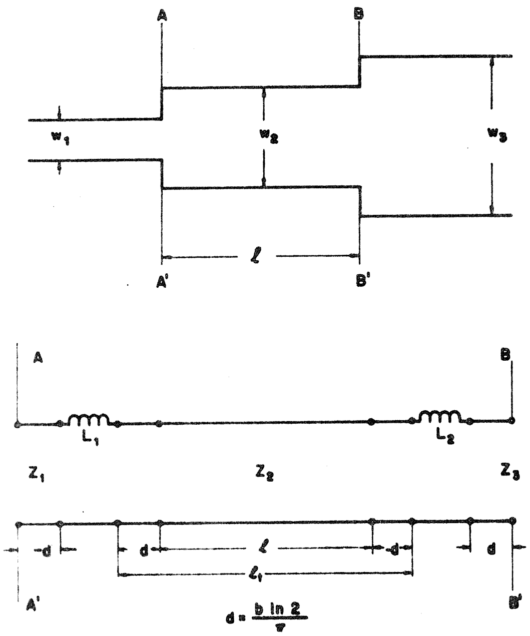

Fig. 7. Stripline transformer and equivalent circuit.

circuit and its equivalent are depicted in Fig. 7. The specifications are listed in Table II. Also shown are source and load mismatches to be accounted for as well as fixed tolerances on certain fixed nominal parameters and assumed uncertainties in model parameters.

Thirteen physical parameters implying 2^{13} extreme points are

$$\phi = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ l \\ \sqrt{\varepsilon_{r1}} \\ \sqrt{\varepsilon_{r2}} \\ \sqrt{\varepsilon_{r3}} \\ b_1 \\ b_2 \\ b_3 \\ t_{s1} \\ t_{s2} \\ t_{s3} \end{bmatrix} \begin{matrix} \left.\right\} \text{variable nominal and variable tolerances} \\ \left.\right\} \text{fixed nominal and fixed tolerances} \end{matrix} \tag{35}$$

where w denotes strip width, l the length of the middle section, ε_r the dielectric constant, t_s the strip thickness, and b the substrate thickness. Tolerances on ε_r, b, and t_s were imposed independently for the three lines allowing independent outcomes. Nominal values for corresponding parameters were the same throughout.

Six model parameters implying 2^6 extreme points are

$$p = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ L_1 \\ L_2 \\ l_t \end{bmatrix} \tag{36}$$

where D denotes effective linewidth, L the junction parasitic inductance, and l_t the effective section length.

TABLE II
ONE-SECTION STRIPLINE TRANSFORMER

Center Frequency	5 GHz
Frequency Band	4.5 - 5.5 GHz
Reflection Coefficient Specification	0.25 (upper)
Source Impedance	50 Ω (nominal)
Load Impedance	20 Ω (nominal)
Source Mismatch (Maximum)	0.025 (reflection coeff.)
Load Mismatch (Maximum)	0.025 (reflection coeff.)
ε_r	2.54 ± 1%
b	6.35 mm ± 1%
t_s	0.051 mm ± 5%
Uncertainty on L_1, L_2	3%
D_1, D_2, D_3	1%
ℓ_t	1 mm

The formula for D_i used is [6]

$$D_i = w_i + \frac{2b_i}{\pi} \ln 2 + \frac{t_{si}}{\pi}\left[1 - \ln \frac{2t_{si}}{b_i}\right], \qquad i = 1,2,3. \tag{37}$$

The formula is claimed to be good for $w_i/b_i > 0.5$. A 1-percent uncertainty was rather arbitrarily chosen for D_i. The characteristic impedance Z_i is then found as

$$Z_i = \frac{30\pi(b_i - t_{si})}{D_i\sqrt{\varepsilon_{ri}}}. \tag{38}$$

The values of L_i were calculated as [7]

$$L_i = \frac{30\bar{b}_i}{c} K_i, \qquad i = 1,2 \tag{39}$$

where c is the velocity of light *in vacuo* and

$$K_i = \ln\left[\left(\frac{1-\alpha_i^2}{4\alpha_i}\right)\left(\frac{1+\alpha_i}{1-\alpha_i}\right)^{\{[\alpha_i+(1/\alpha_i)]/2\}}\right] + \frac{2}{A_i}$$

$$\alpha_i = \frac{D_i}{D_{i+1}} < 1$$

$$A_i = \left(\frac{1+\alpha_i}{1-\alpha_i}\right)^{2\alpha_i} \frac{1+S_i}{1-S_i} - \frac{1+3\alpha_i^2}{1-\alpha_i^2}$$

$$S_i = \sqrt{1 - \frac{D_{i+1}^2}{\bar{\lambda}_{gi}^2}}$$

$$\bar{\lambda}_{gi} = \frac{c}{f\sqrt{\bar{\varepsilon}_{ri}}}$$

$$\bar{b}_i = 0.5(b_i + b_{i+1})$$

$$\sqrt{\bar{\varepsilon}_{ri}} = 0.5(\sqrt{\varepsilon_{ri}} + \sqrt{\varepsilon_{r(i+1)}}).$$

Mean values across the junctions of adjacent sections of $\sqrt{\varepsilon_r}$ and b are taken since actual values in our model can be different across junctions. Data for estimating the

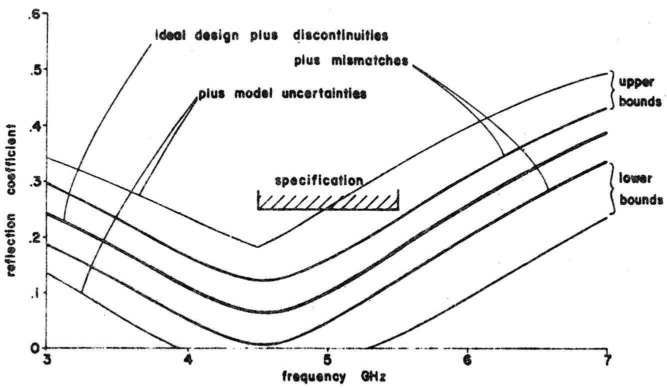

Fig. 8. Worst case analyses for the stripline transformer. Note that physical parameter tolerances are not included.

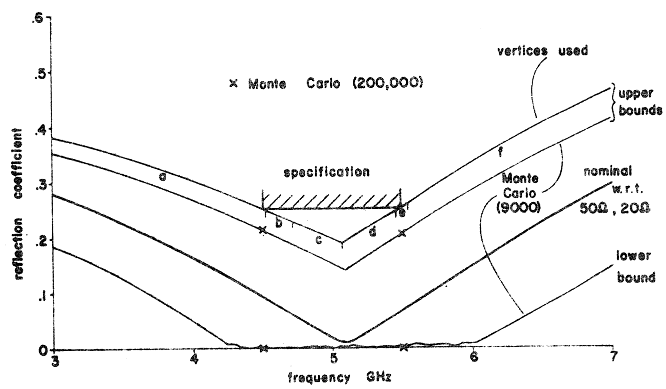

Fig. 9. Final results for the stripline transformer. The letters $a,b,\cdots,f$ indicate different vertices (designs) determining the worst case in different frequency bands.

uncertainties on L_t are available [6], [7]. Other approximations have, however, been introduced due to the tolerancing. A 3-percent uncertainty on L_t was adopted.

The length l_t is nominally the same as l. Experimental results [6] indicate possibly large inaccuracies in d (see Fig. 7) and that it depends at least on α, so that it is actually different for the two junctions. A rather pessimistic estimated error of 1 mm on l_t was chosen.

Maximum mismatch reflection coefficients of 0.025 were chosen for the source and load. Note that these values are assumed with respect to 50 and 20 Ω, respectively. The relevant formulas developed in Section II cannot be applied directly, since Z_1 and Z_3, which are affected by tolerances, must be considered for normalization. We take, most pessimistically,

$$|\rho_S| = \frac{0.025 + |\rho_1|}{1 + 0.025|\rho_1|} \tag{40}$$

where

$$\rho_1 = \frac{50 - Z_1}{50 + Z_1}$$

and

$$|\rho_L| = \frac{0.025 + |\rho_3|}{1 + 0.025|\rho_3|} \tag{41}$$

where

$$\rho_3 = \frac{20 - Z_3}{20 + Z_3}.$$

Fig. 8 summarizes some of the results obtained from worst case analyses. Depicted are curves of the ideal design with discontinuity (parasitic) effects taken into account; upper and lower bounds on the response with source and load mismatches also added; finally, upper and lower responses with model uncertainties further deteriorating the situation.

A worst case study was made to select a reasonable number of constraints from the possible $2^{19} = 2^{13} \times 2^6$, since 2^{19} would have required about 5000 s of CDC 6400 computing time per frequency point. The vertex selection procedure for the 13 physical parameters follows Bandler *et al.* [3]. From each of the selected vertices the worst values of the modeling parameters are chosen. Only the band edges are used during optimization. After each optimization the selection procedure is repeated, new constraints being added, if necessary.

TABLE III
RESULTS FOR ONE-SECTION STRIPLINE TRANSFORMER

Cost Function	$\frac{1}{100}\left(\frac{w_1^0}{\varepsilon_{w_1}} + \frac{w_2^0}{\varepsilon_{w_2}} + \frac{w_3^0}{\varepsilon_{w_3}} + \frac{\ell^0}{\varepsilon_\ell}\right)$		
Sample Points	4.5, 5.5		GHz
Number of Variables	8		
State of Solution	Intermediate	Final	
Number of Final Constraints	18	21	
Number of Optimizations	7	9	
CDC 6400 Time	2	4	min
Minimal Cost	4.82	4.93	
w_1^0	4.660	4.642	mm
w_2^0	8.968	8.910	mm
w_3^0	15.463	15.442	mm
ℓ^0	8.494	8.437	mm
$\varepsilon_{w_1}/w_1^0 \times 100$	0.94	0.92	%
$\varepsilon_{w_2}/w_2^0 \times 100$	1.20	1.13	%
$\varepsilon_{w_3}/w_3^0 \times 100$	0.74	0.70	%
$\varepsilon_\ell/\ell^0 \times 100$	0.64	0.65	%

Results on centering and tolerancing using DISOPT [8] are shown in Table III. The final number of constraints used is 21 after 9 optimizations required to identify the final constraints. Less than 4 min on the CDC 6400 was altogether required. (An intermediate, less accurate, solution is obtained using 18 constraints after 7 optimizations requiring 2 min on the CDC 6400.) To verify that the solution meets the specification, the constraint selection procedure was repeated at 21 points in the band.

Fig. 9 presents final results for this example. The reason for the discrepancy between the worst cases when vertices are used and when the Monte Carlo analysis is used is that the Monte Carlo analysis does not employ the pessimistic approximations of (40) and (41).

IV. Conclusions

The concepts we have described and the results obtained are promising. Our approach is the most direct way of currently obtaining minimum cost designs under practical situations, at least in the worst case sense. It is felt that this work is a significant advance in the art of computer-aided design, since the approach permits the inclusion of all realistic degrees of freedom of a design and all physical phenomena that influence the subsequent performance.

The approach automatically creates a tradeoff between physical tolerances (implying the cost of the network), model parameter uncertainties (implying our knowledge of the network), the quality of the terminations, and, eventually, the cost of tuning. Our approach to mismatches permits input and output connecting lines of arbitrary length—an important step towards modular design.

The conventional computer-aided design process, which seeks a single nominal design or its extension which attempts to find a design center influenced by sensitivities (see, for example, Rauscher and Epprecht [9]), would normally be a preliminary investigation to find a starting point for the work we have in mind.

References

[1] J. W. Bandler, "Computer optimization of inhomogeneous waveguide transformers," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 563–571, Aug. 1969.

[2] H. Tromp and G. Hoffman, "Computer optimization of 3 dB microstrip hybrids taking into account dispersive, coupling and junction effects," *Proc. 1973 European Microwave Conf.* (Brussels, Belgium, Sept. 1973), Paper A.12.4.

[3] J. W. Bandler, P. C. Liu, and J. H. K. Chen, "Worst case network tolerance optimization," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 630-641, Aug. 1975.

[4] J. W. Bandler, P. C. Liu, and H. Tromp, "A nonlinear programming approach to optimal design centering, tolerancing and tuning," *IEEE Trans. Circuits and Systems*, vol. CAS-23, pp. 155–165, March 1976.

[5] H. J. Carlin and A. B. Giordano, *Network Theory*. Englewood Cliffs, N.J.: Prentice-Hall, 1964, pp. 331–334.

[6] H. M. Altschuler and A. A. Oliner, "Discontinuities in the center conductor of symmetric strip transmission line," *IRE Trans. Microwave Theory Tech.*, vol. MTT-8, pp. 328–339, May 1960.

[7] V. Nalbandian and W. Steenaart, "Discontinuities in symmetric striplines due to impedance steps and their compensations," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 573–578, Sept. 1972.

[8] J. W. Bandler and J. H. K. Chen, "DISOPT—A general program for continuous and discrete nonlinear programming problems," *Int. J. Systems Science*, vol. 6, pp. 665–680, 1975.

[9] C. Rauscher and G. Epprecht, "Simplified approach to sensitivity optimization by use of scattering parameters," in *Proc. 1974 European Microwave Conf.* (Montreux, Switzerland, Sept. 1974), pp. 394–398.

PART IV: MISCELLANEOUS TOPICS IN DESIGN

IEEE Proc. ISCAS/76

EFFICIENT, AUTOMATED DESIGN CENTERING AND TOLERANCING

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We present an approach to optimal assignment of component tolerances with centering and, eventually, tuning. The main objective is to automate the process and improve computational efficiency. The development of selection schemes for critical vertices of the tolerance region is discussed. As the process proceeds vertices can be added or purged automatically. The exploitation of symmetry is considered.

Introduction

The historical development of optimal centering and tolerancing is indicated by a number of relevant papers[1-7]. The authors have already demonstrated the benefits of allowing the nominal design to be variable while optimizing the tolerances[6,7] and the way in which tuning can be brought in[8,9]. The principal drawback with the previous work is the interaction necessary by the designer to reduce the computational effort.

A typical problem solving sequence is shown in Fig.1. In this paper, we shall mainly deal with the efficient automatic solution of problem 3, but most of the methods and results will also be useful in problems 2 and 4, which have essentially the same structure.

Tolerance Problem Implementation

The methods to be described are illustrated by the results for the 5-section transmission-line stepped impedance lowpass filter, shown in Fig.2. Relevant element data and design specifications are shown in Table I. The nominal characteristic impedances are fixed and have a uniform fixed relative tolerance. ℓ_i, i=1,...,5, are the line lengths, normalized w.r.t. the quarter wavelength at the passband edge. They have variable nominal values and a uniform variable absolute tolerance. The parasitic capacitors C_i, i=1,...,6, were calculated from formulas given by Marcuvitz[10] for a step in the inner conductor of a coaxial line at the edge of the passband and with an inner diameter of the outer conductor of 1 inch.

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The parameter vector is given by

$$\underset{\sim}{\phi} = [\ell_1\ \ell_2\ \ell_3\ \ell_4\ \ell_5\ Z_1\ Z_2\ Z_3\ Z_4\ Z_5]^T. \qquad (1)$$

In this case there are 2^{10} = 1024 vertices of the tolerance region[6,7] given, in general, by

$$R_\varepsilon \triangleq \{\underset{\sim}{\phi} \mid \phi_i = \phi_i^0 + \varepsilon_i\mu_i,\ -1 \le \mu_i \le 1\ ,\ i \in I_\phi\}, \qquad (2)$$

where superscript 0 distinguishes the nominal parameter values, ε_i the tolerance on the ith component and, for k components,

$$I_\phi \triangleq \{1,2,\ldots,k\}. \qquad (3)$$

The problem is to optimally locate R_ε in the constraint region R_c given by

$$R_c \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{g}(\underset{\sim}{\phi}) \ge \underset{\sim}{0}\}, \qquad (4)$$

where $\underset{\sim}{g}$ is a vector of constraint functions obtained from the design specifications.

In principle, the network response should be calculated for all possible values of the parameter vector; or at least at the 2^k vertices of the tolerance region, when a one-dimensional convexity condition is satisfied[5]. In order to reduce the problem, we need a vertex selection method which gives an accurate prediction of the vertices which are critical in the optimization.

The tolerance optimization problem can then be solved, the vertices being updated if necessary, usually at the end of each of a sequence of optimizations. The tolerancing-tuning problem can be solved by a similar approach. The minimax approximation problem with fixed tolerances has also the same basic structure. The selected vertices are used for the error functions, and enter the objective function.

A vertex selection method which the authors had found acceptable involves changing each parameter one at a time from the nominal point and examining the partial derivatives of the response or constraint functions[7]. A disadvantage, in general, is that sometimes one vertex per frequency point is selected, which is often insufficient to reasonably constrain the parameters if the starting point is far from the solution. On the other hand, too many can be selected, most of which are not critical. In our example, even at the optimum the one-at-a-time scheme misses some critical constraints.

Results for the lowpass filter, summarized in Table II, were obtained by the one-at-a-time vertex selection method, followed by manual selection and testing of all vertices after each optimization. The minimax optimization with fixed tolerances was started with the parameter values obtained from the nominal minimax approximation. Two runs were needed using MINOPT[11]. Fig.3 shows results in the passband. Analogous results for $\varepsilon_\ell = 0.002$ show that the specifications cannot be satisfied. The passband ripple of the nominal response of the nominal minimax approximation was 0.00123 dB. The worst deviation of the minimax approximation with fixed tolerances is 0.0125 dB if $\varepsilon_\ell = 0.001$, and 0.02007 dB when $\varepsilon_\ell = 0.002$.

Tolerance optimization was started from the parameter values resulting from minimax approximation with $\varepsilon_\ell = 0.001$. Six runs of DISOPT[12] were needed, with intermediate testing of vertices. The final run took 60 sec on a CDC 6400 using 228 function evaluations. The number of variables is 4 and the number of final constraints 49. Fig.4 shows the results in the passband.

Symmetry was also exploited. There is a double mirror-symmetry in the parameter vector ϕ. The insertion loss $L(\phi)$ is given by

$$L(\phi) = L(S\phi), \qquad (5)$$

where

$$S = \begin{bmatrix} \begin{matrix} 0&0&0&0&1\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0&0&0&0&1\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0 \end{matrix} \end{bmatrix}. \qquad (6)$$

The number of vertices essentially different (in response value) under symmetry conditions is reduced from 1024 to 544.

The optimum ϕ^0 should lie in the plane of symmetry. If the nominal point is forced to be symmetric during optimization, the numbers of parameters and constraints are reduced. When symmetry is not exploited, care should be taken to ensure that of each pair of symmetric vertices, both are used as constraints or error functions. The savings from symmetry are clear from Table III.

Efficient Vertex Selection

Let $g(\phi)$ be a constraint function. The worst vertex is given by

$$\min_{\mu} g(\phi^0 + E\mu) \qquad (7)$$

subject to

$$1 - \mu_i \geq 0,\ 1 + \mu_i \geq 0,\ i \in I_\phi, \qquad (8)$$

where E is a diagonal matrix with ε_i as the ith element. Let $\check{\mu}$ be the minimum of (7). The Kuhn-Tucker conditions are at $\check{\mu}$

$$\nabla_\mu g(\phi^0 + E\check{\mu}) = \sum_{i=1}^{k} u_i^- \nabla_\mu (1 - \mu_i) + \sum_{i=1}^{k} u_i^+ \nabla_\mu (1 + \mu_i)$$

$$u_i^- (1 - \check{\mu}_i) = 0,\ u_i^+ (1 + \check{\mu}_i) = 0, \qquad (9)$$

$$u_i^- \geq 0,\quad u_i^+ \geq 0,\ i \in I_\phi$$

where ∇_μ is the first partial derivative operator w.r.t. μ. This leads at $\check{\mu}$ to

$$\frac{\partial g}{\partial \mu_i} = \varepsilon_i \frac{\partial g}{\partial \phi_i} = u_i^+ - u_i^-,\ i \in I_\phi. \qquad (10)$$

If the tolerance region is one-dimensionally convex the worst point should be a vertex, and any component $\check{\mu}_i$ of $\check{\mu}$ has two possible values. If $\check{\mu}_i = 1$, then $u_i^+ = 0$ and

$$\varepsilon_i \frac{\partial g}{\partial \phi_i} = -u_i^- \leq 0. \qquad (11)$$

If $\check{\mu}_i = -1$, then $u_i^- = 0$ and

$$\varepsilon_i \frac{\partial g}{\partial \phi_i} = u_i^+ \geq 0. \qquad (12)$$

Equations (11) and (12) yield

$$\check{\mu} = -\mathrm{sgn}\,[\nabla_\phi\, g(\phi^0 + E\check{\mu})], \qquad (13)$$

where ∇_ϕ is the first partial derivative operator w.r.t. ϕ.

Equation (13) can be solved iteratively for $\check{\mu}$. If $\mu^{(j)}$ is the jth approximation for $\check{\mu}$, then

$$\mu^{(j)} = -\mathrm{sgn}[\nabla_\phi\, g(\phi^0 + E\mu^{(j-1)})]. \qquad (14)$$

A suitable starting point would be $\mu^{(0)} = 0$.

More than one vertex may satisfy (13), only one being the worst vertex, generally. The first iteration, starting at the nominal point, does generally not lead to the worst vertex, e.g., when there is symmetry and the gradient at the nominal point is in the plane of symmetry. The first iteration leads to a symmetrical vertex, hence additional testing is necessary.

Two schemes were used, both starting with an iterative search from the nominal point. They have the structure of a branching process.

Testing Scheme 1 Whenever a vertex satisfies (13) all adjacent vertices (i.e.,differing in only one component of μ) are also tested.

Testing Scheme 2 Whenever a vertex satisfies (13) e iterative search is restarted from all adjacent rtices.

Parameters are called related if they have to be changed simultaneously when going from one vertex satisfying (13) to another one (same frequency).

Here, this happens with Z_i and ℓ_i. Testing scheme 1 may not work where there are related parameters. Testing scheme 2 will work, but generally less reliably. We reinterpret the term "adjacent": adjacent vertices differ only in the μ_i's corresponding to a set of related parameters. See Table IV.

The tolerance optimization problem was solved with the new vertex selection scheme using testing scheme 2, assuming related parameters. The solution was now found in 40 sec of CPU time, and 157 function evaluations in one optimization. The number of constraints was 33. Previously, 6 runs were needed of about 60 sec each, plus 2 to 3 minutes for vertex selection and intermediate testing.

Efficient Automatic Tolerancing

For the test problem we started with $\ell_1^0 = \ell_2^0 = \ell_3^0 = \ell_4^0 = \ell_5^0 = 0.1$, $\varepsilon_\ell = 0.001$ and used DISOPT[12]. We used vertex testing scheme 2 and related parameters. Optimization consists of 2 steps: feasibility check interrupted when all constrain- are satisfied and the main optimization process.

To get proper convergence, the amendment of limited vertex insertion is needed. A newly selected vertex is only added to the constraint table if it is worse than the vertices already selected for that particular frequency. With this amendment the solution of the test problem is found in 86 sec. We consider a scheme with approximate initial centering. We do the feasibility check only as long as new violated constraints are found by the vertex selection. Only two optimizations and 71 sec were needed.

Purging Schemes We would like to reduce the constraints further, especially if we go on to the tolerancing-tuning problem, where the number of slack variables increases with the number of constraints. The following basic purging scheme was used. The user specifies a purging percentage P. Let g_u and g_ℓ be the highest and the lowest constraint values, respectively, taken over the whole constraint table (all frequencies). The purging limit is then defined as

$$g_c = g_u - .01\,P\,(g_u - g_\ell). \qquad (15)$$

2nd Amendment (minimal band requirements) The user specifies frequency bands and the minimum number of constraints needed per band to limit the solution. We used 1 as the minimum number of constraints in ranges 0 - 0.55, 0.60 - 0.96, 0.97 - 1.0 and .5 - 10.0 GHz. Amendment 1 just restricts g_u.

Process 1 with 10% purging yielded the solution in 49 sec (213 sec on a Siemens 4004). The 2nd amendment was not used, while the 1st amendment eliminated the sample point at 10 GHz. With P = 30% the running time is 45 sec (189 sec on a Siemens 4004). The second amendment was used in the first purging step, to keep one constraint in the stopband.

We consider adaptive minimal band requirements, as realized in purging process 2: if any vertex purged before the previous optimization reappears in the vertex selection with a negative constraint value we discard the last optimization. We ensure that the vertex will not be purged next and increase the minimum number of constraints for that band by one. A vertex unjustifiably purged is not counted towards the minimum band requirements. Thus, one vertex more has to be found.

Process 2 for P=90% yielded the solution in 38 sec (149 sec on a Siemens 4004). It is slightly more efficient than scheme 1 with 30% purging. Another optimization is needed. It is, however, expected to be more efficient when we introduce tuning.

Conclusions

An approach to automating the optimal assignment of component tolerances along with centering has been presented. A new vertex selection method has been proposed with the aim of avoiding intermediate testing of all vertices. The information required of the designer relates to the physical properties of the problem.

References

[1] J.F.Pinel and K.A.Roberts,"Tolerance assignment in linear networks using nonlinear programming", IEEE Trans. Circuit Theory, vol. CT-19, Sep.1972, pp.475-479.

[2] J.F.Pinel,"Tolerance assignment and network alignment of linear networks in the frequency domain", IEEE Short Course on Computer Aided Network Design, 73-SC-06,Mar.1973, pp. 17-25.

[3] J.F.Pinel,K.A.Roberts and K.Singhal,"Tolerance assignment in network design",Proc.IEEE Int.Symp. Circuits Syst.(Newton,Mass.,Apr.1975),pp.317-320.

[4] B.Karafin,"The general component tolerance assignment problem in electrical networks",Ph.D.Thesis, Univ. Pennsylvania,1974.

[5] J.W.Bandler,"Optimization of design tolerances using nonlinear programming",J.Optimiz.Theory Applic.,vol.14,July 1974, pp.99-114.

[6] J.W.Bandler and P.C.Liu,"Automated network design with optimal tolerances",IEEE Trans.Circuits Syst.,vol.CAS-21,Mar.1974,pp.219-222.

[7] J.W.Bandler,P.C.Liu and J.H.K.Chen,"Worst case network tolerance optimization",IEEE Trans.Microwave Theory Tech.,vol.MTT-23,Aug.1975,pp.630-641.

[8] J.W.Bandler,P.C.Liu and H.Tromp,"A nonlinear pro-

gramming approach to optimal design centering, tolerancing, and tuning", IEEE Trans. Circuits Syst.,vol. CAS-23, Mar. 1976.

[9] P.C.Liu, "A theory of optimal worst-case design embodying centering, tolerancing and tuning, with circuit applications", Ph.D. Thesis, McMaster Univ., 1975.

[10] N.Marcuvitz, Waveguide Handbook, M.I.T. Rad. Lab. Ser., vol.10, New York:McGraw-Hill, 1951.

[11] J.W.Bandler,C.Charalambous and J.H.K. Chen, "MINOPT - an optimization program based on recent minimax results", Proc.Midwest Symp.Circuits Syst. (Montreal,Canada, Aug.1975), pp.520-524.

[12] J.W.Bandler and J.H.K.Chen, "DISOPT - a general programme for continuous and discrete non-linear programming problems", Int.J.Systems Science, vol.6, 1975, pp.665-680.

TABLE I LOWPASS FILTER

Passband (0-1 GHz)	0.02 dB
Stopband (2.5 - 10 GHz)	25.0 dB
$Z_1^0 = Z_3^0 = Z_5^0$	0.2 Ω
$Z_2^0 = Z_4^0$	5.0 Ω
ε_{Z_i}, i= 1,...,5	2 %
$C_1 = C_6$	15.6 pF
$C_2 = C_3 = C_4 = C_5$	28.5 pF
Parameter constraints	$\ell_1^0=\ell_5^0$, $\ell_2^0=\ell_4^0$ $\varepsilon_{\ell_1}=\varepsilon_{\ell_2}=\varepsilon_{\ell_3}=\varepsilon_{\ell_4}=\varepsilon_{\ell_5}=\varepsilon_\ell$
Cost function	$1/\varepsilon_\ell$

TABLE II RESULTS FOR LOWPASS FILTER

Problem	$\ell_1^0=\ell_5^0$	$\ell_2^0=\ell_4^0$	ℓ_3^0	ε_ℓ
Nominal minimax	.0343	.1440	.1207	-
Fixed tolerances	.0402	.1433	.1252	.001
Tolerance optimiz.	.0423	.1426	.1274	.00196

TABLE III LAST RUN IN MINIMAX (ε_ℓ=0.001)

	No symmetry	Symmetry
No.of variables	5	3
No.of constraints	32	20
CPU time (sec)	36.0	18.6
No.of function evaluations	252	185

TABLE IV VERTEX SELECTION AT OPTIMUM TOLERANCES

	No.of failures	CPU time (sec)	No.of vertices
One-at-a-time	3	12.3	1196
Testing scheme 1†	2	1.65	32
Testing scheme 2†	0	2.07	36
Testing scheme 2††	1	3.09	26

† Assuming related parameters

†† Not assuming related parameters

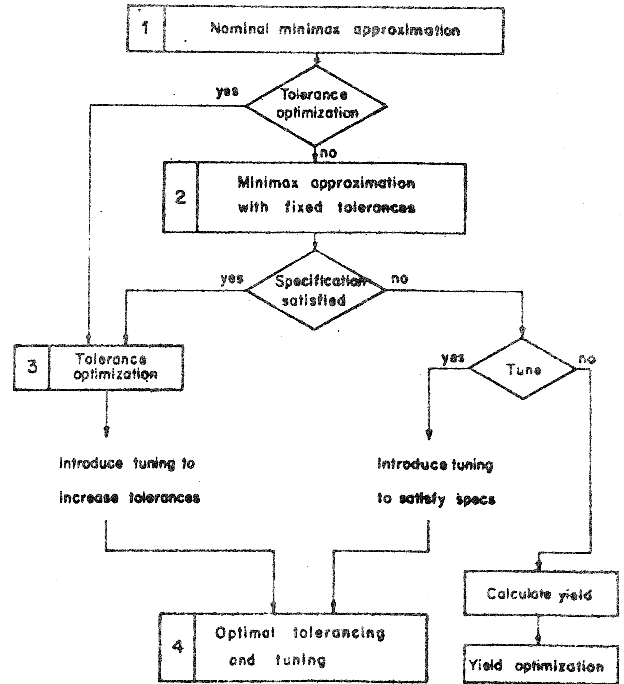

Fig.1 Typical problem solving sequence.

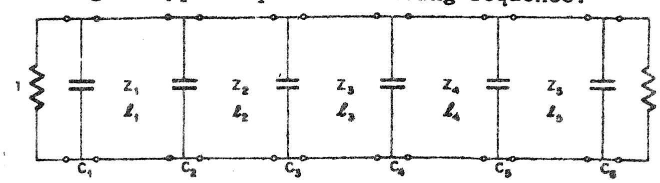

Fig.2 The lowpass filter.

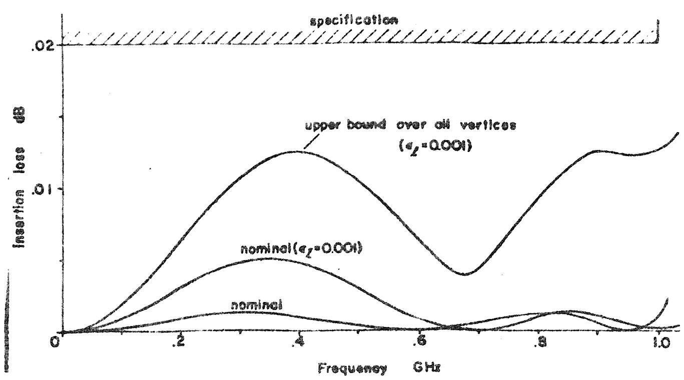

Fig.3 Minimax approximation (passband).

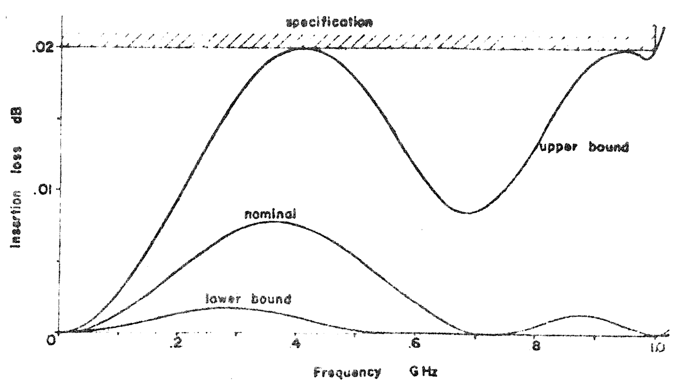

Fig.4 Optimally toleranced filter (passband).

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NONLINEAR OPTIMIZATION OF ENGINEERING DESIGN WITH EMPHASIS ON CENTERING, TOLERANCING AND TUNING

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ABSTRACT

This paper addresses the engineering problem of optimal design from the standpoint of minimizing cost of production subject to acceptable performance in the worst case under as many unknowns and nonideal outcomes that can be reasonably accommodated in the design process in an integrated fashion. Optimal design centering, optimal assignment of component tolerances and optimal tuning (including tuning by both the manufacturer and by the customer) in the face of uncertainties in the model and external factors affecting the performance are considered. It is explained how even for a relatively small number of components a very large number of constraints and variables may have to be considered.

Following the introduction a general statement of the requirements of the worst-case approach to the problem is made. A number of observations on important points concerning the size of the problem and its effective solution are made. A brief review of theoretical and computational work carried out by the author and his colleagues is presented.

INTRODUCTION

Optimal centering of engineering designs taking into account or optimizing the assignment of manufacturing tolerances is the subject of this review. Post-production tuning by the manufacturer attempting to correct for the effects of these tolerances is integrally involved in the presentation. Furthermore, the general approach accommodates tuning carried out by the customer both to correct for long term drift of the component values and to facilitate tunability in the sense of meeting a variety of possible performance specifications.

Even for a small number of designable components the solution process may involve very large numbers of possible constraints and variables. Indeed, the

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general problem the author has in mind involves an infinite number of variables and an infinite number of constraints. Thus, the subject appears relevant to the study of large engineering systems.

The reason for the size of the problem is clear: for a given design to be manufactured any of an infinite possible number of outcomes can occur, each outcome, in general, having to be independently tunable. Thus, even with guaranteed bounds on the tolerances, a very large number of possible situations must be simulated.

The presentation also considers the immunization of the design against the effects of uncertainties in the model parameters used in the simulation and against certain nonideal environmental effects causing possible deviation from ideal performance.

This paper considers worst-case design, i.e., each outcome after any necessary tuning must meet all design specifications under all anticipated conditions. This approach can often be justified as an end in itself. It may be a preliminary exercise to statistical design. We consider independent variables. Correlations may, for example, be accounted for by imposing known constraint data which reduces the number of independent variables.

THE PROBLEM

The Physical Variables

Consider a vector of k nominal design parameters

$$\underset{\sim}{\phi}^0 \triangleq \begin{bmatrix} \phi_1^0 \\ \phi_2^0 \\ \cdot \\ \cdot \\ \cdot \\ \phi_k^0 \end{bmatrix}, \tag{1}$$

a vector of k associated manufacturing tolerances

$$\underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_k \end{bmatrix} \tag{2}$$

and two corresponding vectors of, in general, k postmanufacturing tuning variables

$$\underset{\sim}{t}_m \triangleq \begin{bmatrix} t_{m1} \\ t_{m2} \\ \cdot \\ \cdot \\ \cdot \\ t_{mk} \end{bmatrix}, \quad \underset{\sim}{t}_c \triangleq \begin{bmatrix} t_{c1} \\ t_{c2} \\ \cdot \\ \cdot \\ \cdot \\ t_{ck} \end{bmatrix}. \tag{3}$$

The variables $\underset{\sim}{\phi}^0$, $\underset{\sim}{\varepsilon}$, $\underset{\sim}{t}_m$ and $\underset{\sim}{t}_c$ constitute a possible physical description of the design. The subscripts m and c distinguish, respectively, manufacturer and customer tuning.

The point $\underset{\sim}{\phi}$ denotes actual parameter values. The ith component is given by

$$\phi_i = \phi_i^0 + \varepsilon_i \, \mu_{\varepsilon i} + t_{mi} \, \mu_{tmi} + t_{ci} \, \mu_{tci} , \tag{4}$$

where $\mu_{\varepsilon i}$ determines the outcome due to (uncontrollable) manufacturing tolerances and μ_{tmi} and μ_{tci} indicate the setting of the (controllable) tuning variables. Thus,

$$\underset{\sim}{\mu}_\varepsilon \triangleq \begin{bmatrix} \mu_{\varepsilon 1} \\ \mu_{\varepsilon 2} \\ \cdot \\ \cdot \\ \cdot \\ \mu_{\varepsilon k} \end{bmatrix}, \quad \underset{\sim}{\mu}_{tm} \triangleq \begin{bmatrix} \mu_{tm1} \\ \mu_{tm2} \\ \cdot \\ \cdot \\ \cdot \\ \mu_{tmk} \end{bmatrix}, \quad \underset{\sim}{\mu}_{tc} \triangleq \begin{bmatrix} \mu_{tc1} \\ \mu_{tc2} \\ \cdot \\ \cdot \\ \cdot \\ \mu_{tck} \end{bmatrix} \tag{5}$$

identify the particular outcome and appropriate tuning, whereas $\underset{\sim}{\phi}^0$, $\underset{\sim}{\varepsilon}$, $\underset{\sim}{t}_m$ and $\underset{\sim}{t}_c$ are design parameters.

Performance Constraints and Deviations

The values of ϕ sufficient to give an acceptable design depend on other uncertainties influencing its performance. Examples are uncertainties in the model parameters obtained from the physical parameters, and non-ideal environmental effects altering the performance. Let $\underset{\sim}{g}(\underset{\sim}{\psi})$ denote a set of nonlinear functions such that

$$\underset{\sim}{g}(\underset{\sim}{\psi}) \geq \underset{\sim}{0} \tag{6}$$

represents an acceptable situation for a particular setting of $\underset{\sim}{\psi}$, another set of independent variables. Then $g^0(\psi)$ will be used to identify the nominal performance of the design under ideal environmental effects. The actual performance is given by

$$g_i = g_i^0 \, (\underset{\sim}{p}, \underset{\sim}{\psi}) + \mu_{gi} \, (\underset{\sim}{p}, \underset{\sim}{q}, \underset{\sim}{\psi}), \; i = 1, 2, \ldots, m(\underset{\sim}{\psi}), \tag{7}$$

where

$$\underset{\sim}{p} \triangleq \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{bmatrix} \tag{8}$$

represents the n-element model parameter vector, $\underset{\sim}{q}$ the external parameters and

$$\underset{\sim}{\mu}_g \triangleq \begin{bmatrix} \mu_{g1} \\ \mu_{g2} \\ \cdot \\ \cdot \\ \cdot \\ \mu_{gm} \end{bmatrix} \tag{9}$$

the deviation from ideal performance.

The Model Uncertainties

The ith element of the parameter vector of a possible model is

$$p_i = p_i^0 (\underset{\sim}{\phi}) + \delta_i(\underset{\sim}{\phi}) \mu_{\delta i} , \tag{10}$$

where

$$\underset{\sim}{\delta} \triangleq \begin{bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \delta_n \end{bmatrix} \tag{11}$$

determines the model uncertainties and

$$\underset{\sim}{\mu}_\delta \triangleq \begin{bmatrix} \mu_{\delta 1} \\ \mu_{\delta 2} \\ \cdot \\ \cdot \\ \cdot \\ \mu_{\delta n} \end{bmatrix} \tag{12}$$

the model under consideration.

A Common Worst-Case Assumption

Let

$$M^{\ell} \triangleq \{\underset{\sim}{\mu} \mid -1 \leq \mu_i \leq 1,\ i = 1, 2, \ldots, \ell\} \quad . \tag{13}$$

Often we take, without loss of generality,

$$\underset{\sim}{\mu}_{\varepsilon},\ \underset{\sim}{\mu}_{tm},\ \underset{\sim}{\mu}_{tc} \in M^{k},\ \underset{\sim}{\mu}_{g} \in M^{m},\ \underset{\sim}{\mu}_{\delta} \in M^{n} \tag{14}$$

and, in an effort to make the problem tractable, candidates for worst case are selected from the vertices of M^{ℓ}, namely, from

$$M_v^{\ell} \triangleq \{\underset{\sim}{\mu} \mid \mu_i \in \{-1, 1\},\ i = 1, 2, \ldots, \ell\} \ . \tag{15}$$

The Worst-Case Problem

The worst-case engineering design problem can now be stated as

$$\text{minimize} \quad C(\underset{\sim}{\phi}^{0},\ \underset{\sim}{\varepsilon},\ \underset{\sim}{t}_{m},\ \underset{\sim}{t}_{c},\ \underset{\sim}{q}) \quad , \tag{16}$$

where C is an appropriate, generally nonlinear cost function subject to

$$\underset{\sim}{\phi} \in R_c\ (\underset{\sim}{\psi}) \tag{17}$$

for all permissible $\underset{\sim}{\mu}_{\varepsilon}$ and $\underset{\sim}{\psi}$ and some permissible $\underset{\sim}{\mu}_{tm}$ and $\underset{\sim}{\mu}_{tc}(\underset{\sim}{\psi})$. The constraint region $R_c(\underset{\sim}{\psi})$ is given by

$$R_c(\underset{\sim}{\psi}) \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{g}(\underset{\sim}{\phi},\ \underset{\sim}{\psi}) \geq \underset{\sim}{0} \text{ for all permissible } \underset{\sim}{\mu}_{g},\ \underset{\sim}{\mu}_{\delta}\} \ . \tag{18}$$

Discussion

For each outcome considered critical independent tuning must be simulated, hence it is very important to accurately distinguish those constraints essential to determining the solution. Otherwise, variables indifferent to the optimization process will be generated along with the redundant constraints. Experience with such situations indicates that, computationally, an ill-conditioned, potentially time-consuming formulation is thereby created.

There seems to be no conceptual difference between tuning carried out by the manufacturer (at the time of manufacture or repair) and that exercised by the customer (during the lifetime of the product). The differences in designs fulfilling essentially the same purpose are in the mathematically superficial ones of exact function, cost and convenience of operation. Tuning, for example, designed to permit a product to satisfy a variety of specifications according to the setting of the tuning variable(s) only involves more constraints to be considered at the design stage than tuning provided for correcting the effects of component tolerances or drift.

The precise cost function used at the design stage will depend on whatever data is available for the problem in hand. Intuitively, large tolerances and little or no tuning by the manufacturer tend to reduce cost. Having to provide for convenient tuning by the customer will most often tend to increase costs, unless this flexibility permits larger manufacturing tolerances. Through having, for example, to provide for tighter tolerances, model uncertainties and the requirement of satisfactory operation under adverse external conditions will tend to increase costs. An example of a cost function widely used in circuit design is

$$\sum_{i=1}^{k} c_i \frac{\phi_i^0}{\varepsilon_i} , \tag{19}$$

where c_i are constant weights.

REVIEW

A number of works relevant in the area of circuits and systems are listed (1)-(21). They may be grouped according to the type of material they contain. Most are based on intuitive or pragmatic approaches to the general problem (1)-(6). A number attempt to address the problem from a theoretical point of view (7)-(14). The remaining ones are oriented towards computer implementations, aiming at efficient, general-purpose algorithms (15)-(21).

The conventional assumption that the worst case occurs at a vertex of the tolerance region has been examined by Bandler (7). A linearity assumption of the constraints with respect to the variables for small tolerances is, of course, sufficient. A one-dimensional convexity assumption of the constraint region with respect to each variable independently is, however, also sufficient (7). The implications of certain cost functions in conjunction with constraint regions which intersect with certain subspaces are discussed from the point of view of reducing the number of variables.

The benefits of allowing the nominal point to move in terms of substantially increased tolerances over, for example, what can be expected from a conventional minimax solution without considering tolerances have been illustrated by Bandler and Liu (5). The description of a rather general computer package called TOLOPT has been published by Bandler et al (15). It incorporates many features including several algorithms for constrained minimization, fairly efficient vertex selection techniques and the option of assigning optimal parameter values chosen from a given discrete set. A comprehensive user's guide (16), (17) and the complete program listing (17) are available from the present author.

The most general statement of the centering-tolerancing-tuning problem to the author's knowledge is that by Bandler, Liu and Tromp (13). The concepts and implications of the tolerance and tuning regions are discussed. A brief illustration of the idea of a tunable constraint region is given. Theorems relating to reducing the general problem along with geometrical interpretations are presented. Examples of circuit design illustrating effective tolerancing and tuning, cost functions and optimal tuning are found in that paper, which is based on two conference papers (11), (12).

The contribution by Bandler and Liu on the implications of biquadratic functions in the tolerance problem (10) appears to be an important forerunner of the kind of analytical preparation that could be involved in studying particular designs or classes of designs. In fact, the authors elaborate on the one-dimensional convexity assumption from the point of view of determining conditions under which the worst case will occur at the extreme points of an interval for those circuits and systems whose frequency response is biquadratic (bilinear complex) in each component.

The extension of the problem to include model uncertainties and mismatch terminations has been applied to a microwave circuit (14). Here, a complete study of the example was carried out so that the design procedure accounted for parasitic effects and uncertainties in model formulas, as well as completely arbitrary mismatched passive terminations and reference planes subject to given maximum reflection coefficients.

Recently, an efficient approach to the optimal assignment of component tolerances along with centering has been presented (20). The development of selection schemes include the purging as well as the addition of vertices of the tolerance region during the optimization process. The vertex selection method used is based on the iterative solution of necessary conditions for the worst vertex derived from the Kuhn-Tucker conditions.

A different approach is that of Bandler and Abdel-Malek (18), (21). Multidimensional polynomial approximations of the constraint functions are made using a minimal number of evaluations of the actual functions within an interpolation region. Updating of the approximations is periodically carried out during the optimization process, attention being focussed on interpolation regions containing active vertices of the tolerance region. The structure of the approximations permits efficient function and gradient evaluation even when all vertices are used. The approach facilitates rapid and accurate determination of solutions including yield estimation and optimization, even with relatively inefficiently written analysis programs. Partial derivatives, for example, of the performance with respect to the design parameters do not have to be provided.

CONCLUSIONS

Practical implementation of the ideas presented will be difficult, in general, in the light of the current state-of-the art. Unless the designer has a great deal of insight into the problem and its mathematical formulation he should expect a significant investment in computer effort. Future work, it is felt, will have the greatest impact if it is directed towards the development of algorithms for efficiently handling the variables and the constraints involved. Data enabling the formulation of meaningful cost functions will also put this subject on a more practical footing.

Finally, to help the engineer appreciate the relevance of the centering-tolerancing-tuning problem in the area of circuit and system design and to indicate some additional related achievements to date, further references are appended (22), (23).

REFERENCES

(1) Butler, E.M., Realistic design using large-change sensitivities and performance contours, IEEE Trans. Circuit Theory CT-18, 58 (1971).

(2) Karafin, B.J., The optimum assignment of component tolerances for electrical networks, Bell Syst. Tech. J. 50, 1225 (1971).

(3) Pinel, J.F., and Roberts, K.A., Tolerance assignment in linear networks using nonlinear programming, IEEE Trans. Circuit Theory CT-19, 475 (1972).

(4) Pinel, J.F., Tolerance assignment and network alignment of linear networks in the frequency domain, IEEE Short Course on Computer Aided Network Design 73-SC-06, 17 (1973).

(5) Bandler, J.W., and Liu, P.C., Automated network design with optimal tolerances, IEEE Trans. Circuits Syst. CAS-21, 219 (1974).

(6) Pinel, J.F., Roberts, K.A., and Singhal, K., Tolerance assignment in network design, Proc. IEEE Int. Symp. Circuits and Systems (Newton, Mass., 1975), 317.

(7) Bandler, J.W., Optimization of design tolerances using nonlinear programming, Proc. Princeton Conf. Information Sciences and Systems (Princeton, N.J., Feb. 1972), 655. Also, J. Optimization Theory and Applications 14, 99 (1974).

(8) Karafin, B., The General Component Tolerance Assignment Problem in Electrical Networks, Ph.D. Thesis, University of Pennsylvania, Philadelphia, 1974.

(9) Liu, P.C., A Theory of Optimal Worst-Case Design Embodying Centering, Tolerancing and Tuning with Circuit Applications, Ph.D. Thesis, McMaster University, Hamilton, Canada, 1975.

(10) Bandler, J.W., and Liu, P.C., Some implications of biquadratic functions in the tolerance problem, IEEE Trans. Circuits Syst. CAS-22, 385 (1975).

(11) Bandler, J.W., and Liu, P.C., The tolerance-tuning problem: a nonlinear programming approach, Proc. 12th Allerton Conf. Circuit and System Theory (Urbana, Ill., 1974), 922.

(12) Bandler, J.W., Liu, P.C., and Tromp, H., Practical design centering, tolerancing and tuning, Proc. IEEE Int. Symp. Circuits and Systems (Newton, Mass., 1975), 206.

(13) Bandler, J.W., Liu, P.C., and Tromp H., A nonlinear programming approach to optimal design centering, tolerancing and tuning, IEEE Trans. Circuits Syst. CAS-23, 155 (1976).

(14) Bandler, J.W., Liu, P.C., and Tromp, H., Integrated approach to microwave design, IEEE Trans. Microwave Theory Tech. MTT-24 (1976).

(15) Bandler, J.W., Liu, P.C., and Chen, J.H.K., Worst case network tolerance optimization, IEEE Trans. Microwave Theory Tech. MTT-23, 630 (1975).

(16) Bandler, J.W., Chen, J.H.K., Dalsgaard, P., and Liu, P.C., TOLOPT - a program for optimal, continuous or discrete, design centering and tolerancing, Proc. 13th Allerton Conf. Circuit and System Theory (Urbana, Ill., 1975), 633.

(17) Bandler, J.W., Chen, J.H.K., Dalsgaard, P., and Liu, P.C., TOLOPT - a program for optimal, continuous or discrete, design centering and tolerancing Part I: user's guide, Part II: Fortran listing, Report SOC-105,

Faculty of Engineering, McMaster University, Hamilton, Canada, 1975.

(18) Bandler, J.W., and Abdel-Malek, H.L., Optimal centering and tolerancing using available analysis programs, Report SOC-118, Faculty of Engineering, McMaster University, Hamilton, Canada, 1976.

(19) Director, S.W., and Hachtel, G.D., The simplical approximation approach to design centering and tolerance assignment, Proc. IEEE Int. Symp. Circuits and Systems (Munich, Germany, 1976), 706.

(20) Bandler, J.W., Liu, P.C., and Tromp, H., Efficient, automated design centering and tolerancing, Proc. IEEE Int. Symp. Circuits and Systems (Munich, Germany, 1976), 710.

(21) Bandler, J.W., Abdel-Malek, H.L., Johns, P.B., and Rizk, M.R.M., Optimal design via modeling and approximation, Proc. IEEE Int. Symp. Circuits and Systems (Munich, Germany, 1976), 767.

(22) Szentirmai, G., Ed., Computer Aided Filter Design, IEEE Press, New York, 1973.

(23) Bandler, J.W., An introduction to simulation and optimization, IEEE Int. Microwave Symp. Digest (Cherry Hill, N.J., 1976).

Teaching Optimal Design

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Abstract—Experiences and views related to teaching optimal design to electrical engineering undergraduates as well as course content are discussed in the context of numerical methods of analysis and design. A number of documented user-oriented computer programs extensively used by students in modeling and optimization of circuits and systems are referenced and are available from the author. Two of them, namely CANOP2 and MINOPT, are briefly described.

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I. Introduction

THIS paper presents some experiences and views on teaching engineering design via numerical optimization techniques [1], [2] to electrical engineering undergraduates. The context of the material is computer aided circuit and system analysis and design taught through appropriate courses.

Numerical analysis has long been a respectable subject for electrical engineers. The advent of the modern, high-speed, large memory digital computer has, within the framework of analysis-based engineering courses, been used mainly to solve larger computational problems than before without a corresponding advance in design philosophy. The transition: slide rule to computer, graph paper to CRT display, has still not, in the author's opinion, had a significant impact on the educator's outlook in optimal design.

This paper indicates current possibilities and limitations in both the application of optimization techniques as tools in design as well as in the undergraduate classroom. Some available programs are described or referenced [3]-[9].

II. Course Content

Background Material

In order to develop meaningful procedures of optimization and optimal design using computer aids, it is assumed that the electrical engineering undergraduate has already been exposed to the following topics: matrix analysis of linear systems, steady-state and transient analysis using a digital computer, sparse matrix techniques, sensitivity and tolerance analysis, computer solution of electromagnetic fields using iterative techniques, and nonlinear dc circuit analysis. It is also assumed that the student is already acquainted with least-squares approximation, minimization by steepest descent and simple (nongradient) direct search methods. As a guiding light (for the instructor, at least) Calahan's book [10] is still recommended. Chua and Lin [11] and Director [12] also cover much of the needed background.

Procedure

One aim of the course is to explain the underlying concepts of efficient iterative methods of solving constrained optimization problems, to indicate their limitations as well as their potential. Another is to present a variety of ways of formulating engineering design problems as optimization problems. Finally, all the ideas are brought together in hands-on experience using any of several packaged batch or interactive optimization programs in the context of a design project individually tailored to the student's interests and progress.

Tests and assignments are employed mainly to determine whether basic concepts have been grasped. The students build up towards the final project which dominates their time towards the end. The project report usually represents over 50 percent of the grade. There is no final examination.

Optimization and Optimal Design

Fig. 1 illustrates the most essential concepts involved in optimization such as the objective function U of several variables $\boldsymbol{\phi}$, the Taylor expansion assuming differentiability, the gradient vector operator ∇ and the Newton-type iteration (jth step) which seeks a point satisfying (for a quadratic model) the necessary condition for optimality of a zero gradient vector. It is felt that the parallel development of the basic process of solving a system of nonlinear equations $\boldsymbol{f}(\boldsymbol{\phi}) = \boldsymbol{0}$ along with the one-dimensional examples aid the student's understanding.

Fig. 2 summarizes typical engineering design situations treated in the course [1], [2]. Fig. 2(a) depicts upper and lower specifications on a response function of an independent variable ψ (e.g., frequency or time) implying a constraint region in the $\boldsymbol{\phi}$ space. Fig. 2(b) represents a Chebyshev or minimax approximation problem involving three extreme error functions e_1, e_2, and e_3 with the corresponding $\boldsymbol{\phi}$-space representation of $\max_i |e_i|$. Depicted is the phenomenon of discontinuous derivatives occuring when the max function shifts from one error function to another. The situation of many circuits with independent design parameter values lying within a tolerance region of a nominal design [13] is depicted in Fig. 2(c). Here a whole production line of designs may be involved.

Features of the nonlinear programming problem and nonlinear minimax approximation problem [2], which are central to optimal design, are sketched in parallel in Fig. 3. The ob-

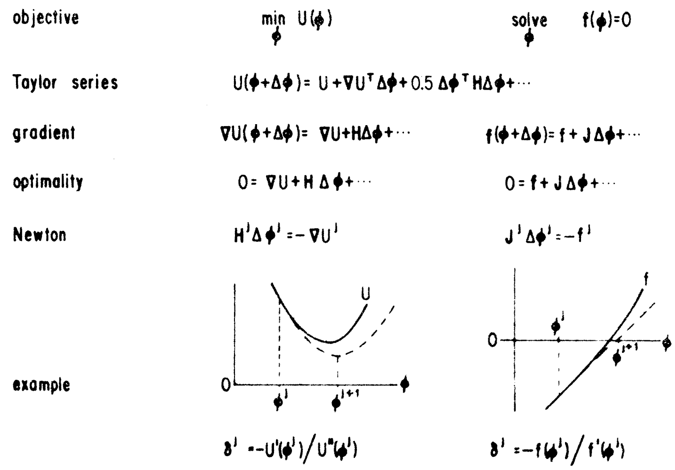

Fig. 1. Contrast of essential concepts in minimization and solution of nonlinear equations, with one-dimensional illustrations.

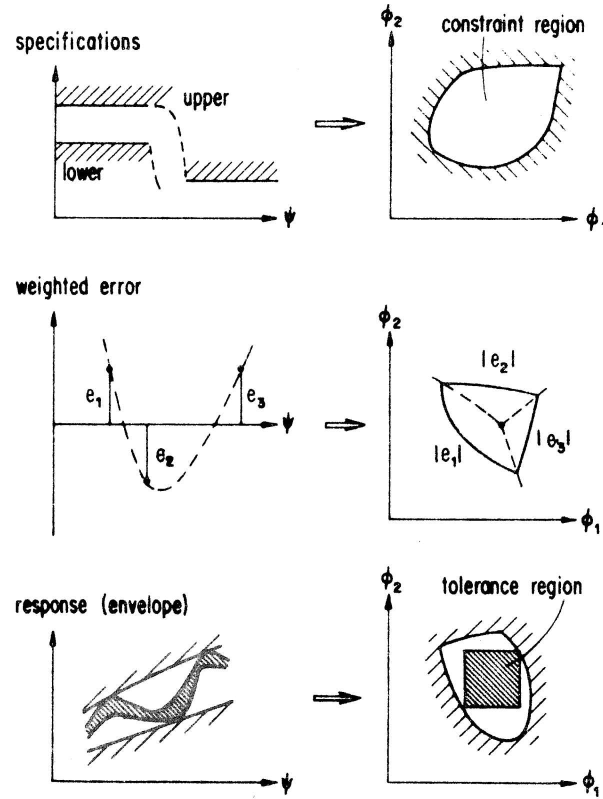

Fig. 2. Typical engineering design situations. (a) Upper and lower specifications with corresponding constraint region. (b) Error function with contour of the maximum. (c) Toleranced design satisfying the specifications.

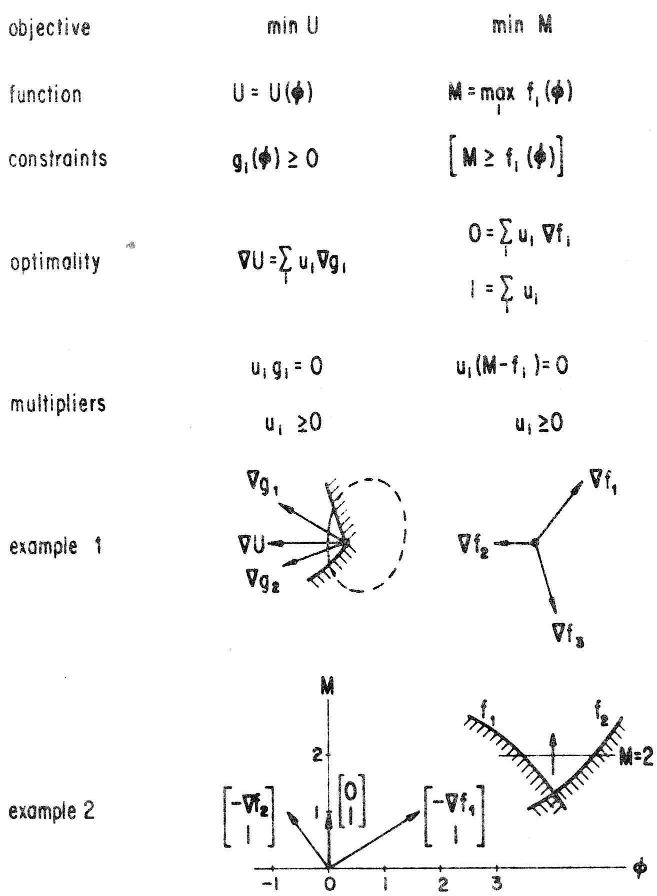

Fig. 3. Contrast of essential concepts in minimization subject to constraints and minimax approximation, with two-dimensional and one-dimensional illustrations.

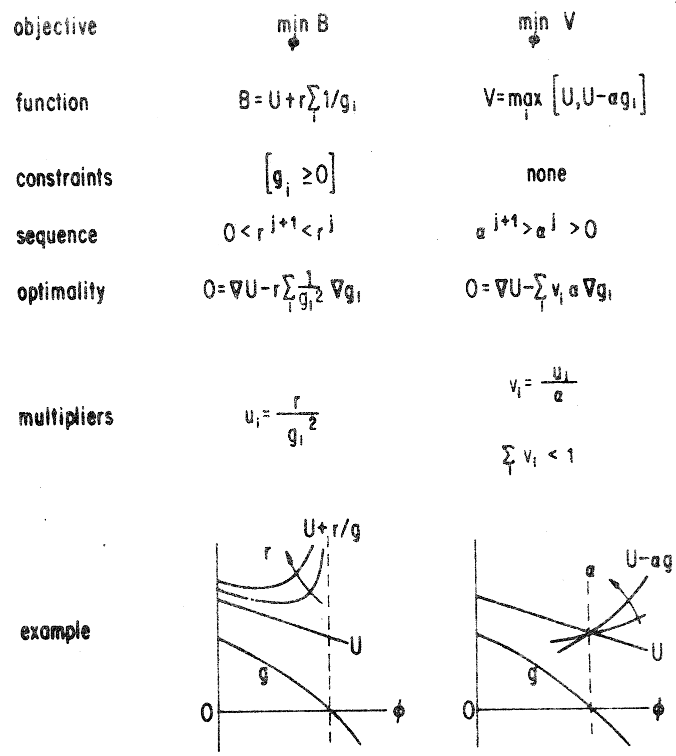

Fig. 4. Contrast of common barrier function approach and exact penalty function (minimax) approach to constrained minimization. One-dimensional illustrations show the effect of changing the parameters r and α.

jective functions are scalars. The constraints are explicit in the first problem and implied in the second. Necessary conditions for optimality involving nonnegative multipliers u_i with corresponding diagrams are contrasted. The understanding of algorithms and the interpretation of solutions are crucially related to the optimality conditions and hence the author dwells on them with a variety of illustrations of different cases. One case is the unifying example 2 of Fig. 3. Here we may interpret optimality either in the nonlinear programming or minimax senses.

Fig. 4 presents two rather general approaches to solving constrained optimization problems. The first is the widely used Fiacco-McCormick barrier function method involving a sequence of unconstrained solutions converging to the desired solution from the interior of the constraint region [14]. The second is an example of an exact penalty function method where a sufficiently large value of α will make the unconstrained minimax solution the desired one [15]. In the latter case constraints do not have to be satisfied during optimization, but in the former case they do.

Fig. 5 illustrates the basic approach to generalized least pth optimization when the maximum M of a set of functions is either positive or negative [16], [17]. In the former case of Fig. 5(a) we see the normalization of the functions, the retention only of positively going functions followed by the formation of a scalar least pth objective equivalent to a penalty function. In the latter case of Fig. 5(b) the normalization of all the functions also changes their sign. All functions are retained in a barrier type objective function. Minimization of the least pth objective in an engineering design problem, therefore, tends to pull a response towards a specification if the specification is violated and increase the margin by which the specification is satisfied if subsequently possible.

One of the most exciting areas developed in the course is that of optimal centering, tolerancing, and tuning [13]. It is a difficult problem to formulate efficiently in general and is still under intensive research. The results are extremely worthwhile in practice. Sufficiently straightforward examples such as resistive voltage divider circuits can be found to enable the students to program and solve meaningfully posed design problems.

Fig. 6 illustrates in one dimension the features of optimal worst case design. All possible production outcomes define a tolerance region, usually characterized by the nominal point ϕ^0, tolerance ϵ, and parameter μ such that

$$\phi = \phi^0 + \epsilon\mu, \quad -1 \leqslant \mu \leqslant 1$$

is an outcome. A tuned outcome is given by

$$\phi = \phi^0 + \epsilon\mu + t\rho$$

$$-1 \leqslant \mu \leqslant 1, \quad -1 \leqslant \rho \leqslant 1$$

where t represents the range and ρ the setting of the control. Depending on whether the tolerance exceeds the tuning as in Fig. 6(a) or the tuning exceeds the tolerance as in Fig. 6(b) we obtain, respectively, an effective tolerance region which must be entirely contained in the constraint region for 100 percent

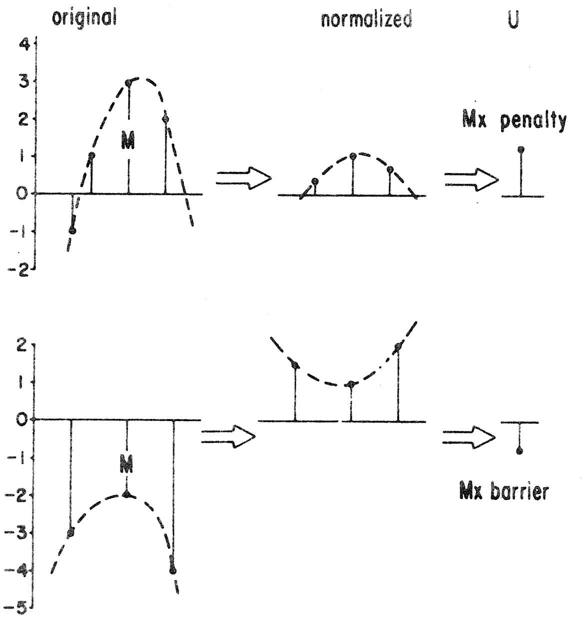

Fig. 5. Illustration of generalized least pth optimization [16], [17].

yield or an effective tuning region only one point of which need be in the constraint region. Generalizing the basic statement of the centering-tolerancing-tuning problem to many dimensions is relatively simple, but may result in a vast nonlinear programming problem. The conventional assumption that the worst case can be predicted by linearizing the constraints at the nominal point is, within the scope of the course, probably as far as one can go in developing a computationally feasible formulation.

III. Program Packages

Here two of the packages available to students will be given detailed attention.

Interactive Cascaded Network Optimization Package [4], [18]

The package called CANOP2 will analyze and optimize cascaded, linear, time-invariant networks in the frequency domain. It is based on CANOPT [19]. It plots responses and enforces equality on the variable parameters, if desired.

The program is organized in such a way that future additions or deletions of performance specifications, constraints, optimization methods, and circuit elements are readily implemented. Presently, the network is assumed to be a cascade of two-port building blocks terminated in a unit normalized, frequency-independent resistance at the source and a user-specified frequency-independent resistance at the load.

A variety of two-port lumped and distributed elements such as resistors, inductors, capacitors, lossless transmission lines, lossless short-circuited and open-circuited transmission-line stubs, series and parallel LC and RLC resonant circuits, and microwave allpass C- and D-sections can be handled. Upper and lower bounds on all relevant parameters can be specified by the user. A generalized least pth objective function or sequence of least pth objective functions incorporating simultaneously input reflection coefficient, insertion loss, relative group delay, and parameter constraints (if any) are automatically created. Constraints are treated by the objective function in essentially the same way as the performance specifications [19]. To distinguish between the various responses or constraint functions a scheme for interval translation and introduction of artificial points has been developed. The Fletcher method of minimizing unconstrained functions of many variables [20] is available to the user. The package incorporates the adjoint network method of sensitivity evaluation [2].

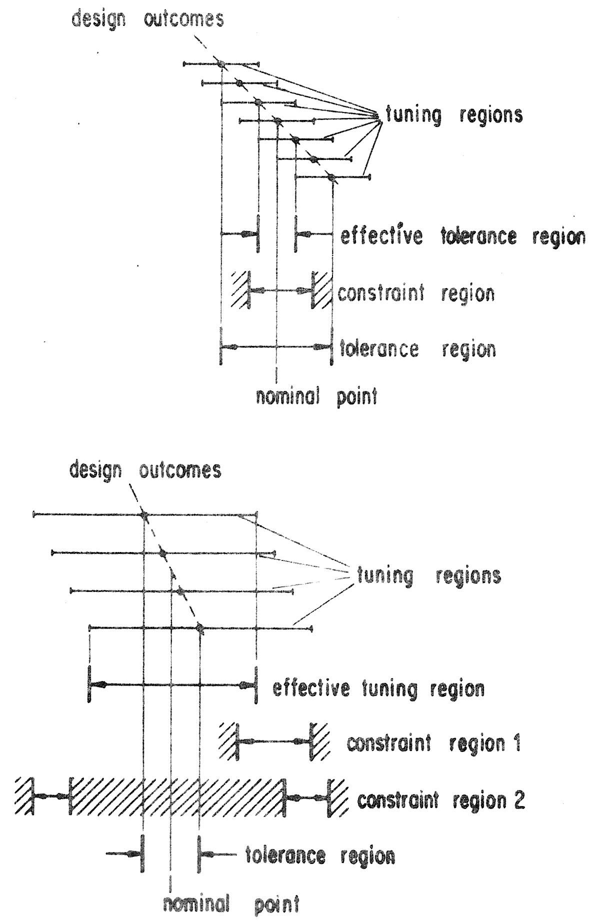

Fig. 6. One-dimensional illustration of the concepts involved in centering, tolerancing and tuning. (a) Tolerance exceeds tuning. (b) Tuning exceeds tolerance.

If equality (symmetry) of some parameters can be predicted, symmetry may be forced throughout the optimization. Results may be automatically presented numerically and graphically and analysis of different responses may be performed at the user's discretion and a new optimization may be requested at different frequencies. A summary of the features and options available is given in Table I.

The package written in Fortran IV was originally developed for batch processing on a CDC 6400 computer and has now been largely extended for use on Intercom. The user may interact at many points with the program to change parameters, frequency range, types and options and to request plots. The interactive user enters his data in free format, and is not required to learn any special language. He responds to simple questions in a straightforward manner.

A Sequential Least pth Optimization Program [5], [21]

MINOPT is a package of subroutines for solving design problems in which the objective is to best satisfy a given set of de-

TABLE I
SUMMARY OF FEATURES, OPTIONS, AND PARAMETERS REQUIRED

Features	Type	Options	Parameters
Objective Functions	Least pth	$1<p<\infty$	Value of p for each of a specified number of optimizations Artificial margin Difference in objective functions for termination
Performance Specifications	Upper (+1.)	Reflection coefficient(1)	Normalization frequency Number of points and constraints / Number of bands or intervals
and	Lower (-1.)	Insertion loss (2)	For each: Specification/constraint Weighting factor Type
Parameter Constraints	Single (0.)	Group delay (3) Parameter value (0)	Option Frequency (sample point) or parameter / Lower and upper frequencies (band edges) Number of subintervals
Analysis		Analysis only (0)	Option Specified or default values for: Number of iterations allowed
Optimization	Gradient	Fletcher optimization method (1)	Estimate of lower bound on objective function Test quantities for termination
Circuit Elements	Cascaded Two-port	Typical plus C- and D-sections	Number of elements Sequence of code numbers Parameter values Indicator for fixed, variable or equal (symmetrical) parameters Load resistance Parameters for C- and D-sections
Graph	Frequency response	Given response Other response Any range Automatic scaling Specified scaling	As many plots as desired Option Frequency (sample point) / Lower and upper frequencies (band edges)

sign specifications or constraints in the least pth or minimax sense. It assumes the availability of first partial derivatives of the functions concerned with respect to the design parameters. Essentially, a single least pth approximation can be done, or a sequence of least pth approximations with finite constant p can be carried out to produce highly accurate minimax solutions, if desired. An estimated lower bound on the minimax solution is employed by the algorithm. A feature to successively drop functions likely to be inactive at the solution is incorporated. The program is efficient and well-suited to conducting feasibility checks.

MINOPT is written in Fortran IV and has been tested on a CDC 6400 computer. The following is a brief description of the subroutines called by MINOPT.

USER Subprogram provided by the user to calculate the functions and first partial derivatives.

LPOBJ Formulates the least pth objective.

GDCHK Checks the derivatives at the starting point by numerical perturbation.

OUTPUT Outputs the optimum solution or the current estimate of the solution.

VA09A Fletcher minimization program.

Consider finding a second-order model of a fourth-order system, when the input to the system is an impulse, in the minimax sense. The transfer function of the system is

$$G(s) = \frac{(s+4)}{(s+1)(s^2+4s+8)(s+5)}$$

and of the model is

$$H(s) = \frac{\phi_3}{(s+\phi_1)^2+\phi_2^2}.$$

The problem is therefore equivalent to making the function

$$F(\boldsymbol{\phi}, t) = \frac{\phi_3}{\phi_2} \exp(-\phi_1 t) \sin \phi_2 t$$

best approximate

$$S(t) = \frac{3}{20} \exp(-t) + \frac{1}{52} \exp(-5t) - \frac{\exp(-2t)}{65} \cdot (3 \sin 2t + 11 \cos 2t)$$

in the minimax sense.

The problem was discretized in the time interval 0–10 s and the function to be minimized is

$$\max_{i \in I} |e_i(\boldsymbol{\phi})|, \qquad I = \{1, 2, \cdots, 51\}$$

where

$$e_i(\boldsymbol{\phi}) = F(\boldsymbol{\phi}, t_i) - S(t_i).$$

A printout of the results is shown in Fig. 7. There are 4 optimizations and 119 function evaluations required. It is interesting to observe the successive reduction in number of error functions actually calculated, so that the computing effort is far less than implied by the number 119.

IV. DISCUSSION

Documented, user-oriented computer programs extensively used by students in modeling and optimization of circuits and systems are available from the author. The examples solved by the students range over analog and digital circuits and systems, active and passive circuit design, low frequency and microwave design, frequency domain and time domain approximation, etc.

The efficient evaluation and utilization of sensitivities is central to the course. Not only nominal design may be achieved iteratively followed, for example, by tolerance *analysis*, but optimally centered, toleranced, and tuned designs are possible with reasonable additional effort. Sufficiently straightforward examples can be found to demonstrate conclusively the value of the general approach as well as the justification in exposing these ideas to the undergraduate student.

The solution of differential equations, the solution of nonlinear equations and the minimization of nonlinear functions of many variables are crucial to engineering modeling, analysis, and design. Once students have mastered the concepts of steepest descent and the basic Newton-type iteration, solution methods for these problems do not require much additional stretch of the imagination.

The author's preferred method of student evaluation in a design-oriented course is an individual project followed by a report. This allows students to move at different paces, permits experiences with different problems to be shared among them and encourages depth and breadth from the more motivated student. The main difficulty in managing this approach

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ARTIFICIAL MARGIN FOR THE NEXT OPTIMIZATION = 4.00000000E-03
NUMBER OF FUNCTIONS FOR THE NEXT OPTIMIZATION = 51
OPTIMIZATION 1
ITER    FUNCT      OBJECTIVE        VARIABLE          GRADIENT
  0       9      6.394211E-01     1.000000E+00     -7.787849E-01
                                  1.000000E+00     -3.780300E-01
                                  1.000000E+00      7.898472E-01

 20      36      7.778212E-03     8.520020E-01      6.395533E-06
                                  8.935317E-01      1.384626E-05
                                  1.422568E-01     -8.362492E-05

 22      38      7.778211E-03     8.520350E-01     -9.730915E-08
                                  8.935018E-01     -2.855661E-08
                                  1.422609E-01      6.064313E-07

IEXIT = 1
NORMAL EXIT
CURRENT MAXIMUM FUNCTION VALUE = 1.05144148E-02
ARTIFICIAL MARGIN FOR THE NEXT OPTIMIZATION = 7.27711352E-03
NUMBER OF FUNCTIONS FOR THE NEXT OPTIMIZATION = 13
OPTIMIZATION 2
ITER    FUNCT      OBJECTIVE        VARIABLE          GRADIENT
 35      55      1.161221E-03     7.061282E-01     -1.504854E-08
                                  9.479483E-01     -3.011574E-08
                                  1.251141E-01      1.234177E-08

IEXIT = 1
NORMAL EXIT
CURRENT MAXIMUM FUNCTION VALUE = 8.24480216E-03
AFTIFICIAL MARGIN FOR THE NEXT OPTIMIZATION = 7.93591219E-03
NUMBER OF FUNCTIONS FOR THE NEXT OPTIMIZATION = 6
OPTIMIZATION 3
ITER    FUNCT      OBJECTIVE        VARIABLE          GRADIENT
 40      68      5.436247E-05     6.876561E-01     -4.268539E-03
                                  9.525845E-01     -2.717345E-04
                                  1.231909E-01      1.732489E-01

 49      81      1.915435E-05     6.847436E-01     -3.683231E-07
                                  9.540264E-01      1.349722E-07
                                  1.228994E-01      1.685067E-06

IEXIT = 1
NORMAL EXIT
CURRENT MAXIMUM FUNCTION VALUE = 7.95178792E-03
ARTIFICIAL MARGIN FOR THE NEXT OPTIMIZATION = 7.94705799E-03
NUMBER OF FUNCTIONS FOR THE NEXT OPTIMIZATION = 4
OPTIMIZATION 4
ITER    FUNCT      OBJECTIVE        VARIABLE          GRADIENT
 60     111      3.631441E-09     6.844180E-01     -1.622166E-02
                                  9.540929E-01     -1.916376E-02
                                  1.228643E-01      1.199815E-01

 64     118      1.629539E-09     6.844178E-01     -1.428569E-02
                                  9.540931E-01     -3.729895E-03
                                  1.228642E-01      1.650471E-01

IEXIT = 1
NORMAL EXIT
CURRENT MAXIMUM FUNCTION VALUE = 7.94705954E-03
ARTIFICIAL MARGIN FOR THE NEXT OPTIMIZATION = 7.94705910E-03

                    FOLLOWING IS THE OPTIMUM SOLUTION
                    OBJECTIVE FUNCTION U = 1.62953865E-09
         X( 1) = 6.84417768E-01 GU(1) =-1.42856936E-02
         X( 2) = 9.54093084E-01 GU(2) =-3.72989486E-03
         X( 3) = 1.22864249E-01 GU(3) = 1.65047054E-01

         NUMBER OF FUNCTION EVALUATIONS = 119*

         *This total includes the number of function
         evaluations required for gradient checking,
         minimization and the determination of the
         artificial margin and index set.
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Fig. 7. Results for the system modeling example with $p = 2$. Starting point $\phi = [1\ 1\ 1]^T$.

has been in convincing the students that they have insufficient time to complete projects large enough to satisfy their own ambitions.

It is stressed that the efficient utilization of algorithms for design requires a reorientation in the thinking of the engineer who may or may not be well versed in simulation. Putting a simulation program into a simple loop, whether the designer is in that loop or not, severely limits his horizons. Nevertheless, a large number of simulation programs exist which do not provide for efficient means of changing design parameters as needed by design, for example, efficient first or large-change sensitivity evaluation is not provided for. The graduating engineer will likely meet many such programs. An approach to exploit such programs appropriately and with minimal effort involves multidimensional low-order approximations [22]. These approximations are also useful in modeling of experimental data and surface fitting in general.

The author believes that the student's time is at a premium and that the material presented to him should be sufficiently fundamental to be of value during advances in hardware and software. It seems, therefore, preferable to avoid the use of large, general purpose simulation programs if: 1) they require considerable investment of the student's time in mastering inessential details involved in their use at the expense of the theory of basic iterative methods or 2) they detract from the process of acquiring the expertise of setting up and running design problems *automatically*.

Probably the most difficult and time-consuming topics for the student to master are those of defining a design problem to the computer in terms of appropriate objectives, choice of nonredundant design variables, selection of essential performance and other constraints, proper scaling to facilitate rapid convergence and, above all, the correct interpretation of false or (for any reason) undesirable solutions. In contrast, subjects such as sensitivity evaluation and minimization algorithms (in their basic form) are readily grasped.

V. Conclusions

At a time of increased specialization, the optimization approach to problem solving is particularly opportune. It provides the engineer with a tool, e.g., a modest package for minimizing functions subject to constraints, which is applicable with varying effectiveness or efficiency to such diverse problems as transistor and other device modeling, rational function approximation, curve fitting, nonlinear (and linear) circuit analysis, tolerance assignment, and post production tuning strategies. Branch and bound strategies, the essence of which are rather straightforward, permit discrete solutions to be forced, e.g., in digital filter design or in the optimal utilization of available components. The use of off-the-shelf components or the suitable restriction of design parameter values or the use of loosely toleranced elements obviously reduces the cost of production. This philosophy is stressed in the classroom, and the available programs can help in realizing these objectives.

Two volumes should finally be mentioned as providing excellent background and motivation for the instructor, namely the reprint volumes of Director [23] and Szentirmai [24].

References

[1] J. W. Bandler, "Optimization methods for computer-aided design," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 533–552, Aug. 1969; also in *Microwave Integrated Circuits*, J. Frey, Ed. New York: Artech House, 1975.

[2] —, "Computer-aided circuit optimization," in *Modern Filter Theory and Design*, G. C. Temes and S. K. Mitra, Eds. New York: Wiley-Interscience, 1973.

[3] J. H. K. Chen, "DISOPT–a general program for continuous and discrete non-linear programming problems," Faculty of Engineering, McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-29, Mar. 1974 (revised June 1975).

[4] J. W. Bandler and J. R. Popović, "CANOP2–interactive cascaded network optimization package," Faculty of Engineering, McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-69, Dec. 1974.

[5] J. W. Bandler, C. Charalambous, and J. H. K. Chen, "MINOPT–an optimization program based on recent minimax results," Faculty of Engineering, McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-70, Dec. 1974; also computer program description in *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, p. 543, Aug. 1976.

[6] W. Y. Chu, "Extrapolation in least *p*th approximation and nonlinear programming," Faculty of Engineering, McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-71, Dec. 1974.

[7] J. W. Bandler and W. Y. Chu, "FLOPT2–a program for least *p*th optimization with extrapolation to minimax solutions," Faculty of Engineering, McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-84, Apr. 1975; also computer program description in *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 543–544, Aug. 1976.

[8] J. W. Bandler, J. H. K. Chen, P. Dalsgaard, and P. C. Liu, "TOLOPT–a program for optimal, continuous or discrete, design centering and tolerancing," Faculty of Engineering, McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-105, Sept. 1975.

[9] J. W. Bandler and W. Y. Chu, "FLOPT3–an interactive program for least *p*th optimization with extrapolation to minimax solutions," Faculty of Engineering, McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-108, Oct. 1975.

[10] D. A. Calahan, *Computer Aided Network Design*. New York: McGraw-Hill, 1972 (revised ed.).

[11] L. O. Chua and P. M. Lin, *Computer-aided Analysis of Electronic Circuits*. Englewood Cliffs, NJ: Prentice-Hall, 1975.

[12] S. W. Director, *Circuit Theory–A Computational Approach*. New York: Wiley, 1975.

[13] J. W. Bandler, P. C. Liu, and H. Tromp, "A nonlinear programming approach to optimal design centering, tolerancing and tuning," *IEEE Trans. Circuits and Systems*, vol. CAS-23, pp. 155–165, Mar. 1976.

[14] A. V. Fiacco and G. P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*. New York: Wiley, 1968.

[15] J. W. Bandler and C. Charalambous, "Nonlinear programming using minimax techniques," *J. Optimiz. Theory Appl.*, vol. 13, pp. 607–619, June 1974.

[16] ——, "Theory of generalized least *p*th approximation," *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 287–289, May 1972; also in *Computer-Aided Filter Design*, G. Szentirmai, Ed. New York: IEEE Press, 1973.

[17] C. Charalambous, "A unified review of optimization," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, pp. 289–300, Mar. 1974.

[18] J. W. Bandler and J. R. Popović, "Interactive cascaded network optimization package," in *Proc. 18th Midwest Symp. Circuits and Systems*, pp. 525–529, Aug. 1975.

[19] J. W. Bandler, J. R. Popović, and V. K. Jha, "Cascaded network optimization program," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, pp. 300–308, Mar. 1974.

[20] R. Fletcher, "A new approach to variable metric algorithms," *Computer J.*, vol. 13, pp. 317–322, Aug. 1970.

[21] J. W. Bandler, C. Charalambous, and J. H. K. Chen, "MINOPT–an optimization program based on recent minimax results," in *Proc. 18th Midwest Symp. Circuits and Systems*, pp. 520–524, Aug. 1975.

[22] J. W. Bandler, H. L. Abdel-Malek, P. B. Johns, and M. R. M. Rizk, "Optimal design via modeling and approximation," *IEEE Int. Symp. Circuits and Systems*, pp. 767–770, Apr. 1976.

[23] S. W. Director, Ed., *Computer Aided Circuit Design: Simulation and Optimization*. Stroudsburg, PA: Dowden, Hutchinson, and Ross, 1973.

[24] G. Szentirmai, Ed., *Computer-Aided Filter Design*. New York: IEEE Press, 1973.

PART V: MODELLING, APPROXIMATION AND STATISTICAL DESIGN

OPTIMAL DESIGN VIA MODELING AND APPROXIMATION

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Recent ideas and results developed by the authors involving concepts of modeling and approximation are reviewed. The approaches taken include abstract ones as well as a physically meaningful one in the area of time domain circuit analysis involving transmission-line modeling of lumped circuits. Optimal centering and tolerancing is also considered.

Introduction

Recent ideas and results developed by the authors involving concepts of modeling and approximation are reviewed. Both an abstract approach to approximation of the response functions with respect to the design parameters as well as a physically meaningful approach to time domain circuit analysis are discussed. The optimal assignment of component tolerances and optimal centering is considered. Low-order multidimensional approximations of the functions concerned allow rapid and accurate yield estimation and optimization. Transmission-line modeling of lumped circuits is used for optimization in the time domain. The response evaluation is exact for the model and exact derivatives are easily obtained.

Bounding and Approximating R_c

The constraint region R_c is the set of points ϕ, the vector of design parameters, for which all performance specifications and design constraints are satisfied. Upper and lower bounds on the parameters ϕ_i for which points satisfying all the requirements can be found provide useful design information[1]. In a statistical analysis, for example, constraints can be stacked in the order of increasing computational effort, upper and lower bounds appearing at the top of the stack. If any is violated, further testing becomes unnecessary.

See Fig.1. In practice, 2k optimizations are not required, if the necessary conditions for optimality are taken advantage of in the k-space.

Fig.2 shows an alternative approach to eliminating regions unlikely to contain acceptable solutions. A generalized least pth objective $U(\phi)$ based on all the constraint functions is formed[2]. $M(\phi)$ is the corresponding maximum. A sequence ϕ is generated. If R_c is convex or one-dimensionally convex[3] we can eliminate from further consideration the regions shown in Fig.2.

We can employ interior and exterior approximations[4] to R_c as shown in Fig.3. A best exterior approximation may be found by deflation of a suitable region R_E and a best interior approximation by inflation of a suitable region R_I[4], keeping $R_I \subset R_c \subset R_E$. Thus, the original functions would be calculated only for $\phi \in R_E - R_I$.

The interior approximation could also be used in design centering[3]. The tolerance region for independent variables R_ε is an example of an interior approximation. The upper and lower bounds on R_c form an exterior approximation.

Optimization Utilizing Polynomial Approximations

Consider the approximation of the constraint functions using values at selected sets of points (base points)[5,6]. In conventional optimization: (1) The functions are approximated by quadratic polynomials using $0.5(k + 2)(k + 1)$ base points (number of coefficients), where k is the dimensionality. The base points lie in a neighbourhood of the starting point (current estimate of the solution) within a step $\pm\delta$ in each variable in the ϕ space. (2) Optimization is carried out with the approximate functions. The solution becomes the next starting point. (3) If the solution is close to the interpolation region, e.g., each parameter has not changed by more than 1.5δ the step size is reduced, e.g., by a factor of 0.25. (4) The procedure is repeated from (1) until an appropriate termination criterion is satisfied.

For solving centering and tolerancing problems consider the following. See Fig.4. (1) As previously with δ chosen greater than the starting or current values of the tolerances ε_i, $i = 1,2,\ldots,k$. (2) As previously but where the problem under consideration is the worst case tolerance problem. (3) If the nominal point ϕ^0 moves too far from the interpolation region (Fig.4), e.g., parameters change by more than 1.5δ, the procedure is repeated from (1). (4) If the nominal point has not moved too far δ is reduced. If δ is still greater than the tolerances approximation as in (1) is carried out. If δ is greater than only some of the toler-

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ances approximation is carried out separately for constraints corresponding to the active vertices of R_ε spaced by less than twice the step size around the center of the hyperface (Fig.4). When δ becomes less than all the tolerances each constraint is re-approximated around the appropriate active vertices. (5) δ is subsequently reduced only when all active vertices stay within the corresponding interpolation region. (6) The procedure is repeated as necessary until parameter changes satisfy an appropriate termination criterion when the step size is reduced.

The structure of the approximations permits efficient function and gradient evaluation even when all 2^k vertices are used. Yield can be estimated by enlarging the tolerance region and using the quadratic approximation to find the yield by Monte Carlo analysis.

Transmission-line Modeling

We can apply an approach called the transmission-line matrix (TLM) method extensively used for the solution of field problems[7,8] to the time domain analysis of lumped networks. Lumped components are represented by transmission-line elements with various terminations. The exact response of the model to an impulse can be found by a numerical procedure, with attendant advantages in the physical description of errors and stability for stiff systems.

Consider a ladder network of series inductors and shunt capacitors. Fig.5 depicts elements and continuized models for a cascaded transmission-line representation. Stubs, appropriately terminated, or a combination of stubs and interconnecting lines can also be employed.

In the cascade analysis (two-port junctions) an ideal delta function pulse is launched from the first junction. The pulse scatters on reaching the next junction, being partly reflected and partly transmitted. This scattering occurs at every junction, pulses racing to and fro between junctions. For simplicity, assume equal lengths and velocities of propagation for all the sections. If the velocity is 1 m/s, then the time h in seconds for a pulse to travel between sections is numerically equal to the length h in meters.

The TLM iteration process is

$$ {}_j\underset{\sim}{V}^r(\ell) = \underset{\sim}{S}(\ell)\ {}_j\underset{\sim}{V}^i(\ell) $$

$$ {}_{j+1}V_1^i(\ell) = {}_jV_2^r(\ell-1) $$

$$ {}_{j+1}V_2^i(\ell) = {}_jV_1^r(\ell+1) $$

where j denotes the iteration, ℓ the junction number, S the junction scattering matrix, i incident and r reflected pulses, and subscripts 1 and 2 distinguish the two junction ports. Simple programming and simple calculation of exact sensitivities w.r.t. design variables is possible.

Examples

Consider the worst-case tolerance optimization of impedances Z_1 and Z_2 of a 2-section quarterwave lossless, 10:1, 100% bandwidth, transmission-line transformer[2]. See Figs.6 and 7. The expected solutions[9] were obtained from ε_1=0.2 and ε_2=0.4 (11 sample points). About 7 sec (18 function evaluations (f.e.)) and 2.5 sec (12 f.e.) for the initial solutions of Figs.6 and 7, respectively, were required on a CDC 6400 using δ=0.4 with FLNLP2[10]. Setting δ=0.1 gave the final results shown in 9.5 sec (24 f.e.) and 3 sec (18 f.e.), respectively.

Minimizing $1/\varepsilon_1 + 1/\varepsilon_2$ (Fig.6) subject to a yield (uniform distribution) $Y \geq 90\%$ enlarged ε_i by about 50%. Yield and sensitivities were estimated from formulas by Tromp[11].

Symmetrical LC lowpass filter was optimized in the time domain. Fig.8 shows a "specified" impulse response for $L_1 = L_2 = 1.0$, $C = 2.0$. Taking 100 sample points,[1] using TLM analysis, least 4th approximation yielded the solution in 21 sec (24 f.e.) and 17 sec (19 f.e.) from starting points a and b, respectively, with a maximum error of about 3×10^{-17}. The specifications of Fig.9 were met with a minimax error of 0.0021992 after 37 sec (46 f.e.) using 33 sample points for optimization.

References

1. C.Charalambous, "Discrete optimization", Proc.Allerton Conf. Circuit and System Theory (Urbana, Ill., Oct.1973), pp. 840-847.
2. J.W.Bandler and C. Charalambous, "Practical least pth optimization of networks", IEEE Trans.Microwave Theory Tech., vol.MTT-20, Dec.1972, pp.834-840.
3. J.W.Bandler, P.C.Liu and H.Tromp, "A nonlinear programming approach to optimal design centering, tolerancing and tuning", IEEE Trans.Circuits Syst., vol.CAS-23, Mar.1976.
4. J.W.Bandler and P.C.Liu, "Optimal design with tolerancing and tuning", IEEE CANDE Workshop (Kennebunkport, Maine, Sep. 1974).
5. S.L.Sobolev, "On the interpolation of functions of n variables", (transl.), Sov.Math.Dokl., vol.2, 1961, pp.343-346.
6. H.C.Thacher, Jr., and W.E. Milne, "Interpolation in several variables", SIAM J., vol.8, 1960, pp.33-42.
7. P.B.Johns and R.L.Beurle, "Numerical solution of 2-dimensional scattering problems using a transmission-line matrix", Proc.IEE, vol.118, Sep.1971, pp.1203-1208.
8. P.B.Johns, "The solution of inhomogeneous waveguide problems using a transmission-line matrix", IEEE Trans.Microwave Theory Tech., vol.MTT-22, Mar.1974, pp.209-215.
9. J.W.Bandler, P.C.Liu and J.H.K.Chen, "Worst case network tolerance optimization", IEEE Trans.Microwave Theory Tech., vol.MTT-23, Aug.1975, pp.630-641.
10. W.Y.Chu, " Extrapolation in least pth approximation and nonlinear programming", McMaster Univ., Report SOC-71, Dec.1974.
11. H.Tromp, private communication, Apr.1975.

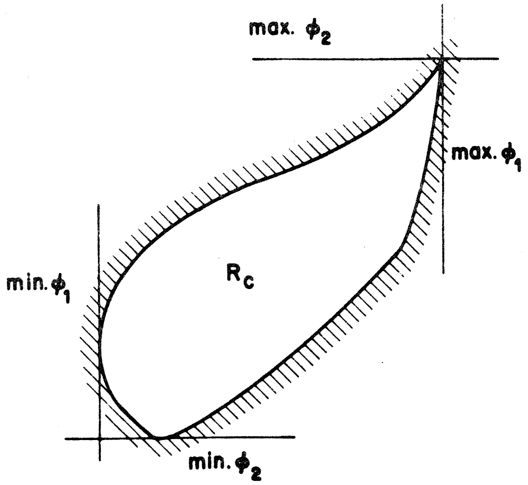

Fig.1 Bounding R_c.

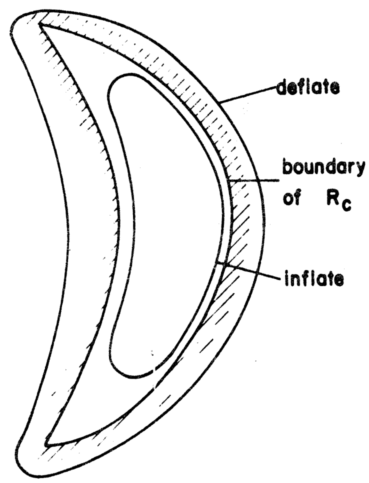

Fig.3 Interior and exterior approximations to R_c.

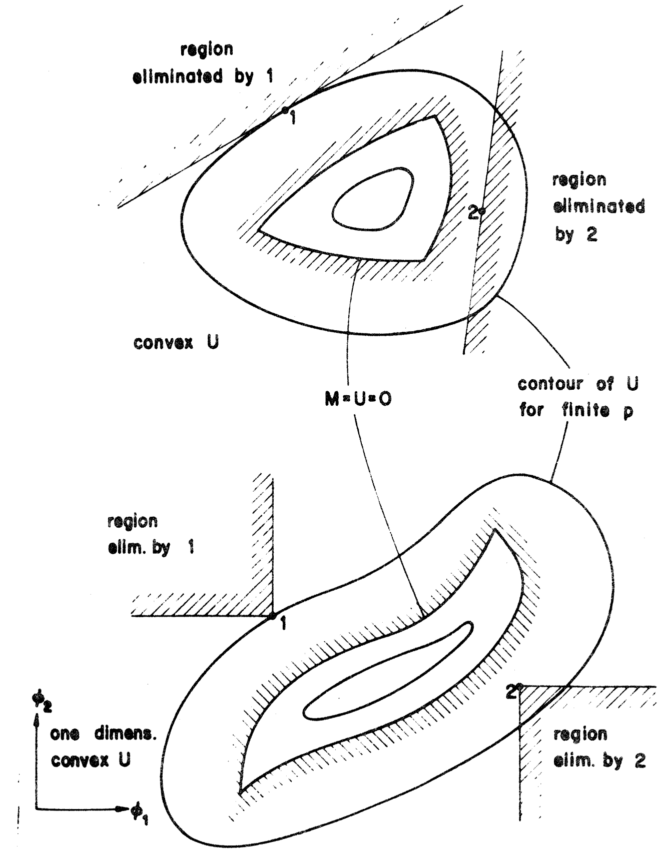

Fig.2 Elimination of regions under assumptions of convexity and one-dimensional convexity.
U > 0 → specification violated
U < 0 → specification satisfied

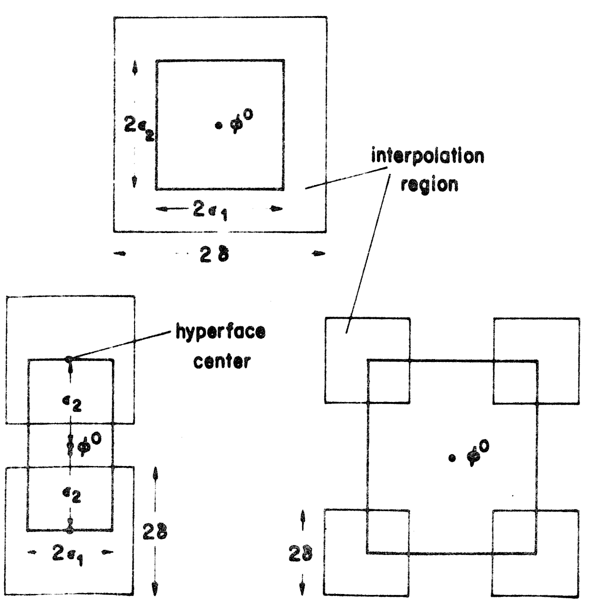

Fig.4 Tolerance regions and interpolation regions.

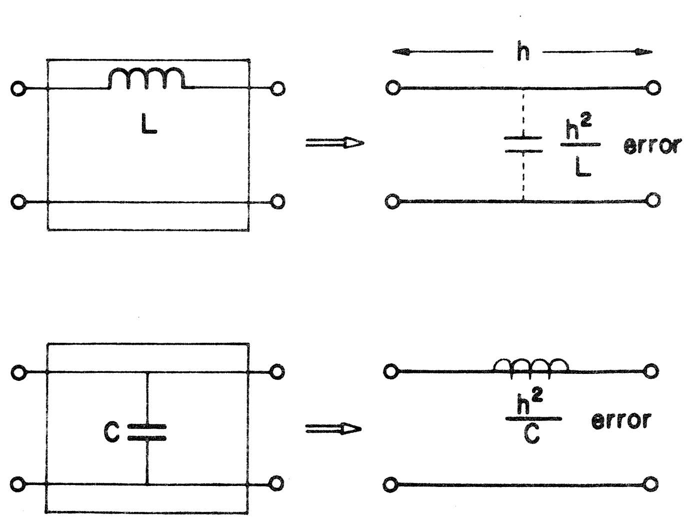

Fig.5 Lossless transmission-line models:an inductor and a capacitor become lines of characteristic impedance L/h and h/C,respectively.

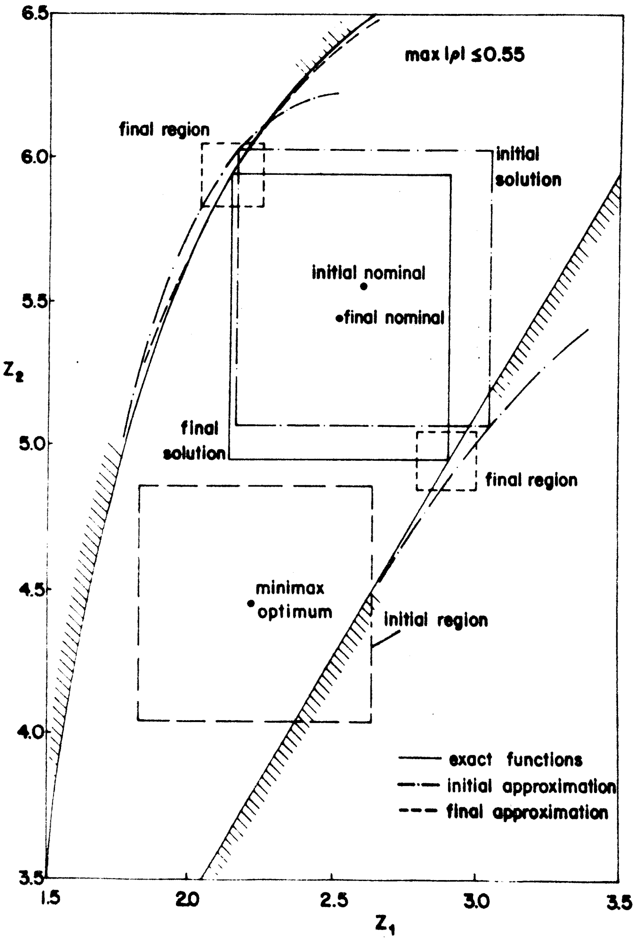

Fig.6 **Minimization** of $1/\varepsilon_1 + 1/\varepsilon_2$ for the two-section transformer.

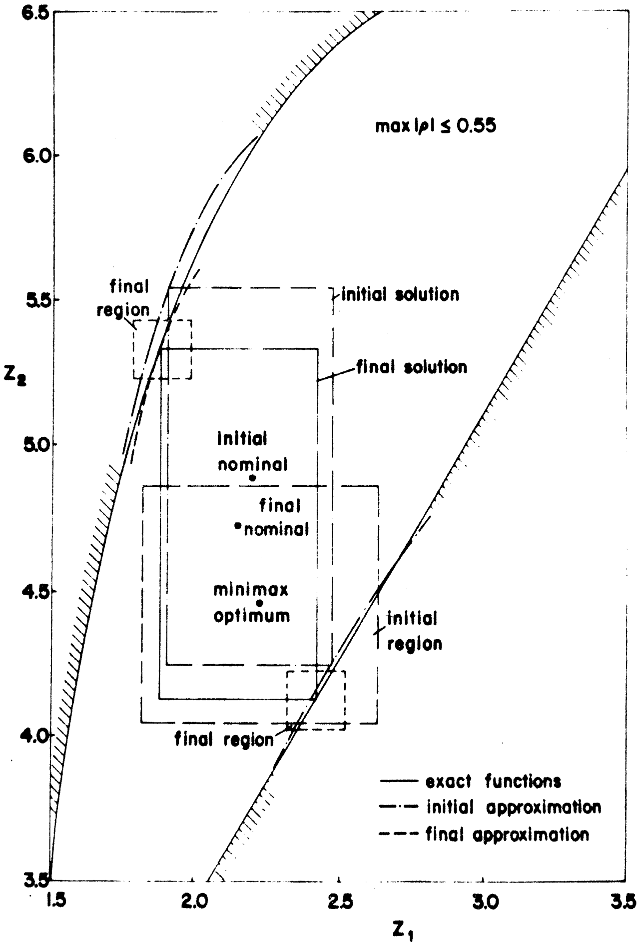

Fig.7 Minimization of $Z_1^0/\varepsilon_1 + Z_2^0/\varepsilon_2$ for the two-section transformer.

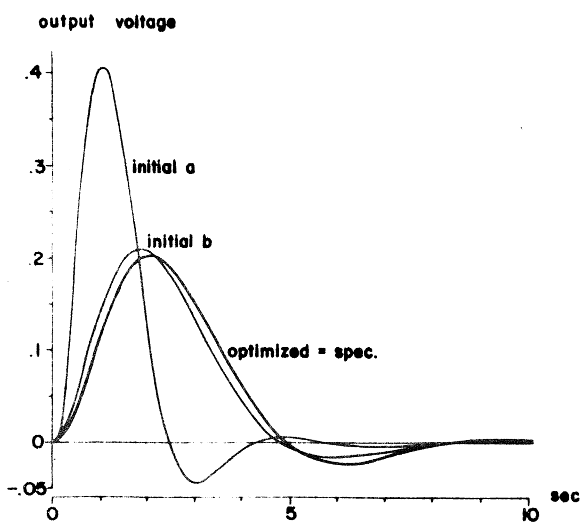

Fig. 8 Optimization using TLM analysis.Starting point a: $L_1 = L_2 = 0.5$, $C = 1.0$. Starting point b: $L_1 = L_2 = 0.8$, $C = 2.2$.

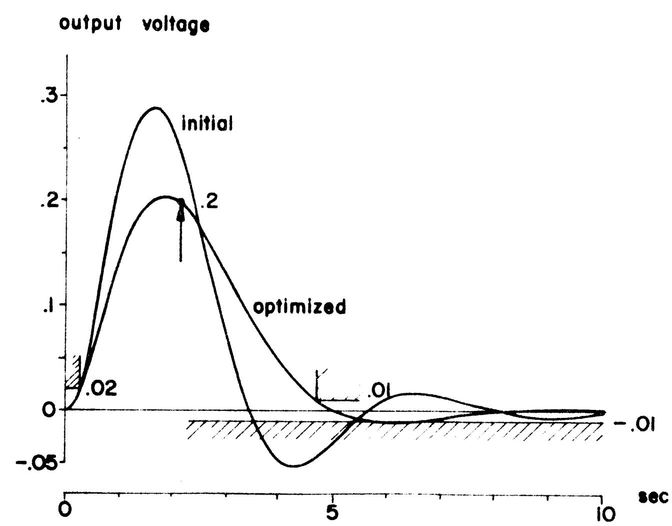

Fig.9 Optimization using TLM analysis.Starting point $L_1 = L_2 = C = 1.0$. Solution: $L_1 = L_2 = 0.76646$, $C = 2.3739$.

OPTIMAL CENTERING, TOLERANCING AND YIELD DETERMINATION VIA UPDATED APPROXIMATIONS AND CUTS

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Abstract

This paper presents a new approach to optimal design centering, the optimal assignment of parameter tolerances and the determination and optimization of production yield. Based upon multidimensional linear cuts of the tolerance orthotope and uniform distributions of outcomes between tolerance extremes in the orthotope, exact formulas for yield and yield sensitivities w.r.t. design parameters are derived. The formulas employ the intersections of the cuts with the orthotope edges, the cuts themselves being functions of the original design constraints. Our computational approach involves the approximation of all the constraints by low-order multidimensional polynomials. These approximations are continually updated during optimization. Inherent advantages of the approximations which we have exploited are that explicit sensitivities of the design performance are not required, available simulation programs can be used, inexpensive function and gradient evaluations can be made, inexpensive calculations at vertices of the tolerance orthotope are facilitated during optimization and, subsequently, inexpensive Monte Carlo verification is possible. Simple circuit examples illustrate worst-case design and design with yields of less than 100%. The examples also provide verification of the formulas and algorithms.

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I. INTRODUCTION

Optimal tolerance assignment is the process of associating the largest tolerances with design parameters to minimize cost. Design centering is the process of defining a set of nominal parameter values to maximize the tolerances or to maximize the yield for known but unavoidable statistical fluctuations. This paper integrates the concepts of design centering, the optimal assignment of parameter tolerances and the determination and optimization of production yield into an overall optimal design process.

Our computational approach should be viewed in the context of the following important work in this area: the nonlinear programming approach of Bandler et. al. [1,2] and by Pinel and Roberts [3], the branch and bound method of Karafin [4], the Monte Carlo approach of Elias [5] and the Director and Hachtel technique involving approximations of the feasible region [6]. It makes use of approximations of all the constraints by low-order multidimensional polynomials. These approximations are continually updated in critical regions identified during optimization and integrated with the nonlinear program which inscribes an orthotope in the constraint region by minimizing a suitable scalar objective function. This orthotope will actually be the optimum tolerance region for a worst-case design problem with independent variables. The readily differentiable approximations permit efficient gradient methods of minimization to be employed as well as inexpensive calculations at vertices of the tolerance orthotope, which tend to locate the critical regions. The yield problem commences when the orthotope is allowed to expand beyond the boundary of the constraint region. Attention is then directed to the critical regions which contribute to the yield calculation.

Section II describes the nature of the tolerance problem and discusses the implications of the conventional assumption of one-dimensional convexity [7,8]. Section III formally introduces the multidimensional polynomial. Our approach to choosing interpolation base points is given. The section includes an efficient algorithm for evaluating the approximations and their derivatives at different vertices in different well-chosen interpolation regions. Based upon multidimensional linear cuts of the tolerance orthotope and uniform distributions of outcomes between tolerance extremes in the orthotope, Section IV presents exact formulas for yield and yield sensitivities w.r.t. design parameters. The formulas employ the intersections of the cuts with the orthotope edges, the cuts themselves being functions of the original design constraints. Ways of treating linear and quadratic constraints (actual or approximate) are discussed linking this Section with the previous one. Section V details design algorithms embodying all the ideas of Sections II to IV. Phase 1 deals with optimization in the worst-case sense considering a single updated interpolation region. Phase 2 involves worst-case design considering two or more updated interpolation regions. Here, the approximations are improved w.r.t. expected yield so that the final algorithm dealing with yield less than 100% can make appropriate approximations to the boundary based on a single function of the least pth type [9] within each critical region.

Some illustrative examples are also included. A two-section quarter-wave transmission-line transformer is used to explain how a worst-case design is obtained and, further, is used for yield determination and optimization. A worst-case design and a well-centered design for yield less than 100% for a three-section lowpass LC filter as well as a check using Monte Carlo analysis are included. A practical example of a non-ideal two-section waveguide transformer is described. The worst-case design as well as yield determination for the enlarged tolerance region and a comparison between execution times for the Monte Carlo analysis applied to the actual constraints and the approximated constraints are given.

II. OPTIMAL CENTERING AND TOLERANCING

The tolerance assignment problem can be stated as: minimize some cost function

$$C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu})$$

subject, for example, to the constraint on yield

$$Y(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}) \geq Y_L \quad , \qquad (1)$$

where

$$\underset{\sim}{\phi}^0 \triangleq \begin{bmatrix} \phi_1^0 \\ \phi_2^0 \\ \vdots \\ \phi_k^0 \end{bmatrix} \geq \underset{\sim}{0} \;, \qquad \underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_k \end{bmatrix} \geq \underset{\sim}{0} \;, \qquad \underset{\sim}{\mu} \triangleq \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix} \qquad (2)$$

k is the number of designable parameters, $\underset{\sim}{\phi}^0$ is the nominal point, $\underset{\sim}{\varepsilon}$ is the tolerance vector, $\underset{\sim}{\mu}$ is a vector of random variables and Y_L is a yield specification.

R_c is the constraint region defined by m_c functions $g_i(\underset{\sim}{\phi})$ and given by

$$R_c \triangleq \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) \geq 0, \; i = 1, 2, \ldots, m_c\} \; . \qquad (3)$$

Thus, for the worst-case design [1,7], sometimes called the 100% yield, it is required that

$$R_{\varepsilon} \subset R_c \quad , \tag{4}$$

where R_{ε} is the tolerance region given by

$$R_{\varepsilon} \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu},\ -1 \leq \mu_i \leq 1,\ i = 1, 2, \ldots, k\}, \tag{5}$$

where $\underset{\sim}{E}$ is a k x k matrix with diagonal elements set to ε_i.

For a one-dimensionally convex region [7] it is sufficient that the set of all vertices R_v satisfy the following condition

$$R_v \subset R_c \ , \tag{6}$$

where R_v is defined by

$$R_v \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu},\ \mu_i \in \{-1, 1\},\ i = 1, 2, \ldots, k\}. \tag{7}$$

This leads to the following nonlinear programming problem for worst-case design: minimize some cost function

$$C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon})$$

subject to

$$g_i(\underset{\sim}{\phi}) \geq 0, \quad i = 1, 2, \ldots, m_c \text{ for all } \underset{\sim}{\phi} \in R_v \quad . \tag{8}$$

The active set of vertices at the solution is given by

$$R_{va} \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} \in R_v,\ g_i(\underset{\sim}{\phi}) = 0\} \ . \tag{9}$$

Unlike conventional optimization problems where a single point is of interest, tolerances and uncertainties create a region of interest. The solution is usually characterized by several critical points or regions so that more information about the constraint region is required. Under the foregoing assumptions it seems reasonable to assume that for a high but less than 100% yield the active vertices determined by a worst-case design will indicate regions where constraint violations are most likely. Accordingly, our interest must be directed to the active vertices as locations for centering reliable approximations to the boundary.

III. INTERPOLATION BY QUADRATIC POLYNOMIAL

An approximate representation of a function $f(\underset{\sim}{\phi})$ by using its values at a finite set of points is possible [10,11]. These points are called nodes or base points. Interpolation, in general, can be done by means of a linear combination of the set of all possible monomials.

The Quadratic Polynomial

In the case of a quadratic interpolating polynomial the number N of such monomials and hence the number of coefficients a_1, a_2, ..., a_N which have to be determined is given by

$$N = (k+1)(k+2)/2. \tag{10}$$

Using N base points $\underset{\sim}{\phi}^1$, $\underset{\sim}{\phi}^2$, ..., $\underset{\sim}{\phi}^N$ we must solve the linear system

$$\left(\begin{array}{ccc|ccc|ccc}
(\phi_1^1)^2 & (\phi_2^1)^2 \ldots & (\phi_k^1)^2 & \phi_1^1\,\phi_2^1 & \phi_1^1\,\phi_3^1 \ldots & \phi_{k-1}^1\,\phi_k^1 & \phi_1^1\,\phi_2^1 \ldots & \phi_k^1 & 1 \\
(\phi_1^2)^2 & (\phi_2^2)^2 \ldots & (\phi_k^2)^2 & \phi_1^2\,\phi_2^2 & \phi_1^2\,\phi_3^2 \ldots & \phi_{k-1}^2\,\phi_k^2 & \phi_1^2\,\phi_2^2 \ldots & \phi_k^2 & 1 \\
\vdots & & & & & & & & \\
(\phi_1^N)^2 & (\phi_2^N)^2 \ldots & (\phi_k^N)^2 & \phi_1^N\,\phi_2^N & \phi_1^N\,\phi_3^N \ldots & \phi_{k-1}^N\,\phi_k^N & \phi_1^N\,\phi_2^N \ldots & \phi_k^N & 1
\end{array}\right)
\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}
=
\begin{pmatrix} f(\underset{\sim}{\phi}^1) \\ f(\underset{\sim}{\phi}^2) \\ \vdots \\ f(\underset{\sim}{\phi}^N) \end{pmatrix} . \tag{11}$$

The solution of (11) exists when the set of base points is degree-2 independent [12]. This solution yields the interpolating polynomial

$$P(\underset{\sim}{\phi}) = a_1\phi_1^2 + a_2\phi_2^2 + \ldots + a_k\phi_k^2 + a_{k+1}\,\phi_1\,\phi_2 + a_{k+2}\,\phi_1\,\phi_3 + \ldots +$$

$$a_{N-k-1}\phi_{k-1}\phi_k + a_{N-k}\phi_1 + a_{N-k+1}\phi_2 + \ldots + a_{N-1}\phi_k + a_N . \tag{12}$$

Selection of the Base Points

Now, let $\bar{\phi}$ be the center of the interpolation region $\bar{R}$ and δ be a step vector defining the size of the region in the following manner. For any base point ϕ^n we have

$$\phi^n \in \bar{R} \triangleq \{\phi \mid | \phi_i - \bar{\phi}_i | \leq \delta_i \ , \ i = 1, 2, \ldots, k\} \ . \qquad (13)$$

The set of base points is given by

$$[\phi^1 \ \phi^2 \ \ldots \ \phi^N] = D \ [0 \ 1_k \ -1_k \ B] + [\bar{\phi} \ \ \bar{\phi} \ \ldots \ \bar{\phi}] \ , \qquad (14)$$

where D is a k x k matrix with diagonal elements set to δ_i, 0 is the zero vector of dimension k, 1_k is a k x k unit matrix, B is a k x $(\frac{k(k-1)}{2})$ matrix defined by

$$B = [\mu^1 \ \ \mu^2 \ \ldots \ \mu^L] \ , \qquad (15)$$

in which

$$L = \frac{k(k-1)}{2} \qquad (16)$$

and

$$-1 \leq \mu_i^j \leq 1, \ j = 1, 2, \ldots, L \ . \qquad (17)$$

See, for example, Fig. 1.

This choice of base points preserves one-dimensional convexity/concavity of the approximated function, since there are three base points along each axis (see Appendix).

Polynomial Evaluation at Vertices (Theory)

The method used for computing the polynomial and its gradients at the vertices as required by the tolerance problem, exploits simple properties of a quadratic approximation. The following two equations are used to obtain the polynomial value and its gradients at any vertex ϕ^r using values at another vertex ϕ^s.

$$P(\underset{\sim}{\phi}^r) = P(\underset{\sim}{\phi}^s) + (\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^s)^T \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^s) + \frac{1}{2}(\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^s)^T \underset{\sim}{H}(\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^s) \ , \tag{18}$$

$$\underset{\sim}{\nabla} P(\underset{\sim}{\phi}^r) = \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^s) + \underset{\sim}{H}(\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^s) \ , \tag{19}$$

where

$$\underset{\sim}{\nabla} \triangleq \begin{Bmatrix} \partial/\partial\phi_1 \\ \partial/\partial\phi_2 \\ \vdots \\ \partial/\partial\phi_k \end{Bmatrix} \tag{20}$$

and

$$\underset{\sim}{H} \triangleq \underset{\sim}{\nabla} \underset{\sim}{\nabla}^T P \tag{21}$$

is the Hessian matrix for the quadratic approximation.

Let $\underset{\sim}{\phi}^r$ and $\underset{\sim}{\phi}^s$ be related as follows

$$\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^s + 2\varepsilon_i \underset{\sim}{e}_i \ , \tag{22}$$

where $\underset{\sim}{e}_i$ is the unit vector in the ith direction.

Hence, we have

$$r = s + 2^{i-1} \tag{23}$$

according to the following vertex enumeration scheme:

$$r = 1 + \sum_{i=1}^{k} \frac{(\mu_i^r + 1)}{2} 2^{i-1} \quad , \qquad \mu_i^r \in \{-1, 1\} \ , \tag{24}$$

where

$$\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^0 + \underset{\sim}{E} \underset{\sim}{\mu}^r \ . \tag{25}$$

Then (18) and (19) reduce to

$$P(\underset{\sim}{\phi}^r) = P(\underset{\sim}{\phi}^s) + 2\varepsilon_i \nabla_i P(\underset{\sim}{\phi}^s) + 2\varepsilon_i^2 H_{ii} \ , \tag{26}$$

$$\underset{\sim}{\nabla} P(\underset{\sim}{\phi}^r) = \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^s) + 2\varepsilon_i \underset{\sim}{H}_i \ , \tag{27}$$

where ∇_i is the ith row of $\underset{\sim}{\nabla}$, H_{ii} is the ith diagonal element of $\underset{\sim}{H}$ and $\underset{\sim}{H}_i$ is the ith column of $\underset{\sim}{H}$.

If $\underset{\sim}{\phi}^r$ and $\underset{\sim}{\phi}^s$ fall into two different interpolation regions, which is the case if $\varepsilon_i > \delta_i$ (see Fig. 2), (26) and (27) cannot be used because of the different polynomials.

Now, let $\underset{\sim}{H}^{\ell}$, $\ell = 1, 2, \ldots, N_{in}$ denote the Hessian matrix for the different interpolation regions, where N_{in} is the number of interpolation regions.

Define the set I as

$$I \stackrel{\Delta}{=} \{i \mid \varepsilon_i \leq \delta_i\} \quad . \tag{28}$$

It is clear that if n_i is the number of elements of I, then

$$N_{in} = 2^{k-n_i} \quad . \tag{29}$$

<u>Polynomial Evaluation at Vertices (Algorithm)</u>

<u>Step 1</u> Compute $P^{\ell}(\underset{\sim}{\phi}^s)$ and $\underset{\sim}{\nabla}\, P^{\ell}(\underset{\sim}{\phi}^s)$ for all $s \in S$ where

$$S = \{s \mid s = 1 + \sum_{i=1}^{k} \frac{(\mu_i^s + 1)}{2}\, 2^{i-1},\ \mu_i^s = -1 \text{ if } i \in I,\ \mu_i^s \in \{-1,1\} \text{ if } i \notin I\}, \tag{30}$$

$$\ell = 1 + \sum_{i=1}^{k} \frac{(\mu_i^s + 1)}{2}\, 2^{\sum_{j=1}^{i} p_j - 1} \quad , \tag{31}$$

$$p_j = \begin{cases} 0 & \text{if } j \in I \\ 1 & \text{if } j \notin I \end{cases} \tag{32}$$

and where ℓ identifies an interpolation region. Set $J \leftarrow I$.

<u>Step 2</u> If J is empty stop.

<u>Step 3</u> Set $i \leftarrow i_1$ where $i_1 \in J$ and $i_1 \leq i$ for all $i \in J$.

<u>Step 4</u> Find $T = \varepsilon_i + \varepsilon_i$.

Step 5 Find the vectors $\underset{\sim}{G}_i^\ell = T\ \underset{\sim}{H}_i^\ell$ for all ℓ defined by (31).

Step 6 For all $s \in S$ and for all ℓ, calculate

$$P^\ell(\underset{\sim}{\phi}^r) = P^\ell(\underset{\sim}{\phi}^s) + T\ \nabla_i\ P^\ell(\underset{\sim}{\phi}^s) + \varepsilon_i\ G_{ii}^\ell\ , \tag{33}$$

$$\underset{\sim}{\nabla}\ P^\ell(\underset{\sim}{\phi}^r) = \underset{\sim}{\nabla}\ P^\ell(\underset{\sim}{\phi}^s) + \underset{\sim}{G}_i^\ell\ , \tag{34}$$

where r is defined by (23) and G_{ii}^ℓ is the ith element of $\underset{\sim}{G}_i^\ell$.

Step 7 Set

$$S \leftarrow S \cup \{r \mid r = s + 2^{i-1},\ s \in S\}\ , \tag{35}$$

$$J \leftarrow J - \{1, 2, \ldots, i\} \tag{36}$$

and go to Step 2.

This scheme is illustrated in Fig. 3 for different cases. The computational effort required for considering all vertices compared to that required for one vertex only is shown in Table I.

IV. YIELD ESTIMATION AND YIELD SENSITIVITIES

The yield in the case of uncorrelated uniformly distributed parameters is given by

$$Y = 1 - \frac{\sum\limits_{\ell} V^\ell}{2^k \prod\limits_{i=1}^{k} \varepsilon_i}\ , \tag{37}$$

where the V^ℓ are nonfeasible nonoverlapping hypervolumes in the tolerance region. Each is chosen according to an ℓth constraint. The assumption of no overlapping of nonfeasible regions defined by different constraints inside the orthotope is assured, for example, if

$$R_\varepsilon \cap R_i \underset{i \neq j}{\cap} R_j = \emptyset,\quad i, j = 1, 2, \ldots, m_c, \tag{38}$$

where

$$R_i \triangleq \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) < 0\} \tag{39}$$

and $\emptyset$ is the empty set.

In order to estimate V^{ℓ} our approach is first to find the intersections between the hypersurface $g_{\ell}(\underset{\sim}{\phi}) = 0$ and the orthotope edges. These intersections lead to an ℓth linear constraint approximation

$$\underset{\sim}{\phi}^{T} \underset{\sim}{q}^{\ell} - c^{\ell} \geq 0 \quad , \tag{40}$$

which is used to provide a multidimensional linear cut of the tolerance orthotope.

Hence, we can express V^{ℓ} as

$$V^{\ell} = \left\{ \frac{1}{k!} \prod_{j=1}^{k} \alpha_j^{\ell} \right\} \left\{ 1 + \sum_{\nu=1}^{k} (-1)^{\nu} \sum_{\beta=1}^{n_{\nu}^{\ell}} (\Gamma_{\beta}^{\ell})^{k} \right\} , \tag{41}$$

where

$$\Gamma_{\beta}^{\ell} = 1 - \sum_{1}^{\nu} 2\varepsilon_{i_{\beta}} / \alpha_{i_{\beta}}^{\ell} \tag{42}$$

and where n_{ν}^{ℓ} is the number of vertices differing in ν parameters from the nonfeasible reference vertex $\underset{\sim}{\phi}^{r} = \underset{\sim}{\phi}^{0} + \underset{\sim}{E}\, \underset{\sim}{\mu}^{r}$ and do not satisfy the ℓth constraint. The subscript i_{β} identifies those ν components of the $\underset{\sim}{\phi}$ vector which differ between the reference vertex and the violating vertex under consideration. We take

$$\mu_i^{r} = - \text{sign } (q_i^{\ell}) \;, \; i = 1, 2, \ldots, k. \tag{43}$$

$\sum_{\nu=1}^{k} n_{\nu}^{\ell}$ is the total number of vertices which do not satisfy the ℓth constraint. α_j^{ℓ} is the distance from the reference vertex to the point of intersection between the hyperplane $\underset{\sim}{\phi}^{T} \underset{\sim}{q}^{\ell} - c^{\ell} = 0$ and the orthotope edge along the jth direction (α_j^{ℓ} may be greater than $2\varepsilon_j$). Fig. 4 illustrates some cases for hypervolume calculation when $k = 3$.

The yield sensitivities can be expressed as

$$\frac{\partial Y}{\partial \phi_i^0} = - \frac{1}{2^k \prod_{j=1}^{k} \varepsilon_j} \sum_{\ell} \frac{\partial V^\ell}{\partial \phi_i^0} \qquad (44)$$

and

$$\frac{\partial Y}{\partial \varepsilon_i} = \left(\frac{1}{\varepsilon_i} \sum_{\ell} V^\ell - \sum_{\ell} \frac{\partial V^\ell}{\partial \varepsilon_i}\right) / \left(2^k \prod_{j=1}^{k} \varepsilon_j\right) \quad , \qquad (45)$$

where

$$\frac{\partial V^\ell}{\partial \phi_i^0} = \left\{ \frac{1}{k!} \left[\sum_{p=1}^{k} \frac{\partial \alpha_p^\ell}{\partial \phi_i^0} \prod_{\substack{j=1 \\ j \neq p}}^{k} \alpha_j^\ell \right] \right\} X$$

$$+ Z \left\{ k \sum_{\nu=1}^{k} (-1)^\nu \sum_{\beta=1}^{n_\nu^\ell} (\Gamma_\beta^\ell)^{k-1} \left[\sum_{1}^{\nu} \frac{\partial \alpha_{i_\beta}^\ell}{\partial \phi_i^0} \frac{2\varepsilon_{i_\beta}}{(\alpha_{i_\beta}^\ell)^2} \right] \right\} \qquad (46)$$

and where

$$X = 1 + \sum_{\nu=1}^{k} (-1)^\nu \sum_{\beta=1}^{n_\nu^\ell} (\Gamma_\beta^\ell)^k \quad , \qquad (47)$$

$$Z = \frac{1}{k!} \prod_{j=1}^{k} \alpha_j^\ell \quad , \qquad (48)$$

$$\frac{\partial V^\ell}{\partial \varepsilon_i} = \mu_i^r \frac{\partial V^\ell}{\partial \phi_i^0} - Z \left\{ k \sum_{\nu=1}^{k} (-1)^\nu \sum_{\beta=1}^{n_\nu^\ell} \frac{2\sigma}{\alpha_i^\ell} (\Gamma_\beta^\ell)^{k-1} \right\} \quad , \qquad (49)$$

where $\sigma = 1$ if i_β assumes the value i, otherwise $\sigma = 0$.

It is to be noted that the gradients are discontinuous when a vertex satisfies the equation $\underset{\sim}{\phi}^T \underset{\sim}{q}^\ell - c^\ell = 0$. Also, limits exist for the hypervolume formula and its sensitivities when any $\alpha_j^\ell \to \infty$. The formulas (40)-(49) permit the cuts to be functions of the original design constraints. They do not have to be fixed.

The Linear Constraint Case

If the original design constraints are linear we obtain the exact yield and exact yield sensitivities according to the previously mentioned assumptions. In this case

$$g_\ell(\underset{\sim}{\phi}) \equiv \underset{\sim}{\phi}^T \underset{\sim}{q}^\ell - c^\ell \tag{50}$$

and hence the intersections of the hyperplane $g_\ell(\underset{\sim}{\phi}) = 0$ and the orthotope edges are given by

$$\lambda_i = (c^\ell - \sum_{j \neq i} q_j^\ell \phi_j)/q_i^\ell \quad , \qquad i = 1, 2, \ldots, k \, , \tag{51}$$

where

$$\phi_j = \phi_j^0 + \varepsilon_j \mu_j^r \; , \qquad \mu_j^r \in \{-1,1\} \quad , \quad j \neq i. \tag{52}$$

Then

$$\alpha_i^\ell = |\phi_i^r - \lambda_i| = \mu_i^r (\phi_i^r - \lambda_i) \; , \quad i = 1, 2, \ldots, k, \tag{53}$$

where $\underset{\sim}{\phi}^r$ is the reference vertex defined by (43).

Hence,

$$\frac{\partial \alpha_i^\ell}{\partial \phi_j^0} = \mu_i^r \, q_j^\ell / q_i^\ell \; . \tag{54}$$

Equations (53) and (54) are substituted directly into formulas (41), (42), (46), (48) and (49), whichever is relevant.

The Quadratic Constraint Case

There will be many ways of approximately expressing the interaction of quadratic (or nonlinear) constraints with the tolerance orthotope via a linear cut. In this paper, bearing in mind our advocacy of quadratic approximations to the original constraints, we consider an approach which employs the intersections between quadratic functions and the orthotope.

Any of these intersections is obtained by solving a quadratic equation. The quadratic polynomial approximation is expressed along the orthotope edge in the form

$$\phi_i^2 + 2\,\phi_i\,\xi(\phi_1, \phi_2, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_k) + \eta(\phi_1, \phi_2, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_k) = 0, \tag{55}$$

where ξ and η are constant functions, ϕ_i being the only variable along that edge of the orthotope and $\phi_j = \phi_j^0 + \varepsilon_j\,\mu_j^r$, $\mu_j^r \in \{-1,1\}$, $j \neq i$.

Thus,

$$\lambda_i = -\xi \pm \sqrt{\xi^2 - \eta} \quad , \quad \phi_i^0 - \varepsilon_i \leq \lambda_i \leq \phi_i^0 + \varepsilon_i \quad . \tag{56}$$

Consider a hyperplane containing k distinct points of intersection between the boundary of the constraint region and the orthotope edges. The equation of this hyperplane is given by

$$\det \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_k & 1 \\ \phi_1^1 & \phi_2^1 & \cdots & \phi_k^1 & 1 \\ \phi_1^2 & \phi_2^2 & \cdots & \phi_k^2 & 1 \\ \vdots & & & & \\ \phi_1^k & \phi_2^k & \cdots & \phi_k^k & 1 \end{bmatrix} = 0 \quad , \tag{57}$$

where $\underset{\sim}{\phi}^j$, $j = 1, 2, \ldots, k$ are the vectors representing the points of intersection.

The yield sensitivities are calculated according to the gradients of the k intersections:

$$\frac{\partial\lambda_i}{\partial\phi_j} = -\frac{\partial\xi}{\partial\phi_j} \pm \frac{1}{2\sqrt{\xi^2 - \eta}} \left(2\xi\,\frac{\partial\xi}{\partial\phi_j} - \frac{\partial\eta}{\partial\phi_j} \right) \quad , \; j \neq i \;, \tag{58}$$

$$\frac{\partial\lambda_i}{\partial\phi_i} = 0 \quad . \tag{59}$$

Thus, if α_i^ℓ is the distance from the vertex ϕ^r to the point of intersection of the ℓth constraint along the orthotope edge in the ith direction, then

$$\alpha_i^\ell = \mu_i^r(\phi_i^r - \lambda_i), \tag{60}$$

$$\frac{\partial \alpha_i^\ell}{\partial \phi_j^0} = - \mu_i^r \frac{\partial \lambda_i}{\partial \phi_j} , \quad j \neq i , \tag{61}$$

$$\frac{\partial \alpha_i^\ell}{\partial \phi_i^0} = \mu_i^r. \tag{62}$$

Equations (60)-(62) are substituted directly into formulas (41), (42), (46), (48) and (49), whichever is relevant.

V. DESIGN ALGORITHMS

Approximation is done only for complicated functions (objective, responses or constraints) or functions for which gradient information is not available.

Worst-Case Design Algorithm Phase 1

Step 1 Choose initial values for ϕ^0, ε and δ.

Step 2 Until $\delta_i \geq \varepsilon_i$, $i = 1, 2, \ldots, k$, set $\delta_i \leftarrow 4\delta_i$.

Step 3 Set $\bar{\phi}$, the center of the interpolation region, to ϕ^0.

Step 4 Choose base points to satisfy (13) and (14).

Step 5 Find, for each function, an interpolating polynomial of the form of (12) by solving (11).

Step 6 Set ϕ^0 and ε to values obtained by worst-case design.

Step 7 If $|\phi_i^0 - \bar{\phi}_i| > 1.5\ \delta_i$ for any i go to Step 2.

Step 8 Stop if δ is sufficiently small.

Step 9 Set $\underset{\sim}{\delta} \leftarrow \underset{\sim}{\delta}/4$. Go to Step 3 if $\delta_i \geq \varepsilon_i$ for all i.

Step 10 If $\delta_i < \varepsilon_i$ for any i go to Phase 2.

Worst-Case Design Algorithm Phase 2

Step 1 Find interpolating polynomials $P^{\ell}(\underset{\sim}{\phi})$ of the form of (12) by solving (11) around the center points $\overline{\underset{\sim}{\phi}}^{\ell}$ given by

$$\overline{\underset{\sim}{\phi}}^{\ell} \in \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{P}\,\underset{\sim}{E}\,\underset{\sim}{\mu}^s,\ \mu_i^s \in \{-1,1\},\ i = 1, 2, \ldots, k\}, \tag{63}$$

where $\underset{\sim}{P}$ is a k x k diagonal matrix with elements p_i defined by (32), ℓ identifies the center of the interpolation region for which the vertex $\underset{\sim}{\phi}^s$ belongs and where ℓ is given by (31). For each ℓ, choose base points to satisfy (13) and (14), where $\overline{\underset{\sim}{\phi}}$ has been replaced by $\overline{\underset{\sim}{\phi}}^{\ell}$.

Comment Efficiency can be improved by concentrating, in each region, on candidates for active constraints.

Step 2 Set $\underset{\sim}{\phi}^0$ and $\underset{\sim}{\varepsilon}$ to values obtained by worst-case design.

Step 3 Define the set of candidates for active vertices

$$R_{ac} \triangleq \{\underset{\sim}{\phi}^s \mid P_q^{\ell}(\underset{\sim}{\phi}^s) \leq \delta_{ac}\} \quad , \tag{64}$$

where P_q^{ℓ} is the quadratic approximation of the qth constraint in the ℓth interpolation region, ℓ and s are related by (31) and δ_{ac} is a small positive number for selecting the active candidates.

Step 4 If, for any vertex $\underset{\sim}{\phi}^s \in R_{ac}$,

$$|\phi_i^s - \overline{\phi}_i^{\ell}| > 2\delta_i \text{ for any } i,$$

where ℓ is given by (31), go to Step 1.

Step 5 Stop if $\underset{\sim}{\delta}$ is sufficiently small.

Step 6 Set $\underset{\sim}{\delta} \leftarrow \underset{\sim}{\delta}/4$. Go to Step 1.

Algorithm for Yield Less Than 100%

Step 1 According to the expected yield, choose a consistant stopping $\underset{\sim}{\delta}$ for executing Phase 1 and Phase 2 (if necessary) of the worst-case algorithm.

Step 2 Choose a factor κ_i by which each tolerance is expected to increase, i.e., set $\varepsilon_i \leftarrow \kappa_i \varepsilon_i$ for all i.

Step 3 Set $\bar{J}^s \leftarrow \emptyset$ for each vertex s.

Step 4 Let

$$J^s \leftarrow \bar{J}^s \cup \{i \mid P_i^\ell(\underset{\sim}{\phi}^s) < 0,\ \underset{\sim}{\phi}^s = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\, \underset{\sim}{\mu}^s\}\ , \tag{65}$$

where ℓ is given by (31) and $P_i^\ell(\underset{\sim}{\phi})$ is the quadratic approximation of the ith constraint in the ℓth interpolation region.

Step 5 If this is the first optimization go to Step 7.

Step 6 Stop if $J^s = \bar{J}^s$ for all s.

Step 7 Set $\underset{\sim}{\phi}^0$ and $\underset{\sim}{\varepsilon}$ to the values obtained by optimization where yield and yield sensitivities are calculated according to the constraint $g^s(\underset{\sim}{\phi})$ calculated when $J^s \neq \emptyset$ as

$$g^s(\underset{\sim}{\phi}) = m^s \left(\sum_{i \in J} \left(\frac{P_i^\ell(\underset{\sim}{\phi})}{m^s} \right)^q \right)^{1/q} \quad \text{for } m^s \neq 0, \tag{66}$$

where

$$m^s = \min_{i \in J} P_i^\ell(\underset{\sim}{\phi}), \tag{67}$$

$$J = \begin{cases} \{i \mid P_i^\ell(\underset{\sim}{\phi}^s) < 0,\ \ i \in J^s\} & \text{if } m^s < 0, \\ \\ J^s & \text{if } m^s > 0, \end{cases} \tag{68}$$

$$q = -p\ \mathrm{sign}(m^s) \tag{69}$$

and p is given to be > 1. We take $g^s = 0$ when $m^s = 0$.

<u>Comment</u> The single function $g^s = 0$ describes the constraint boundary w.r.t. interpolation region ℓ. Determining its intersections or the intersections of a quadratic approximation of it with the tolerance orthotope reduces to the solution of quadratic equations discussed in Section IV.

<u>Step 8</u> Set $\overline{J^s} \leftarrow J^s$ for all s. Go to Step 4.

VI. EXAMPLES

<u>Example 1</u>

Consider a 2-section 10:1 quarter-wave lossless transmission-line transformer [1]. The worst-case tolerance optimization problem denoted by PO of impedences Z_1 and Z_2 over 100% bandwidth is shown in Table II, for two different objective cost functions. The constraint region and the resulting optimum solutions in the two cases are shown in Fig. 5 and Fig. 6. An equal value of δ_1 and δ_2 was used.

Subsequently, the approximation obtained at the two active vertices shown in Fig. 5 was used for yield optimization. A rough estimate of δ used for stopping Phase 2 was obtained in the following way. For a yield constraint

$$Y \geq 90\%$$

the nonfeasible hypervolume (it is area in this example) is given approximately by

$$A \simeq (1 - 0.9)\ (2\varepsilon_1)\ (2\varepsilon_2).$$

The area cut off by each constraint is

$$A' \simeq \frac{1}{2} A \quad .$$

But, assuming equal intersections $\alpha = \alpha_1 = \alpha_2$,

$$A' = \frac{1}{2}\alpha^2.$$

Hence,

$$\alpha \simeq \sqrt{0.1(2\varepsilon_1)\ (2\varepsilon_2)} = 0.27 \quad ,$$

where ε_1 and ε_2 are the worst-case absolute tolerances. The approximation with $\delta = 0.1$ was used for solving problems P1 and P2 as shown in Table III and Fig. 7. The program used for solving the nonlinear optimization problem is FLNLP2 [13]. Because of the convex feasible region the values of yield obtained are lower bounds for the true yield.

Example 2

A normalized 3-components LC lowpass ladder network, terminated with equal load and source resistances of 1Ω, is considered [1]. Although this filter is symmetric, a 3-dimensional approximation was required in order to perform the yield optimization technique described before.

Using equal step size δ for all components, a worst-case solution was first obtained with final $\delta = 0.01$. The base points used are given by (14) with

$$\underset{\sim}{B} = \begin{bmatrix} 0.5 & -0.5 & 1.0 \\ 0.8 & 0.8 & 1.0 \\ -0.5 & 0.5 & 1.0 \end{bmatrix} .$$

The final solution is given in Table IV. The active frequency point constraints at the solution were 0.55, 1.0 and 2.5 rad./sec. A further optimization was carried out with the constraint $Y \geq 96\%$. In a similar way to the previous example an estimate of $\delta = 0.04$ was obtained. The quadratic approximation obtained with $\delta = 0.04$ after and before averaging symmetric coefficients is shown in Table VI. The diagonal elements of the Hessian matrix suggest a one-dimensionally convex feasible region.

Symmetry between L_1 and L_2 was used for reducing computation in finding the value and the gradients of the intersections between the orthotope edges and the quadratic constraints. The results are shown in Table IV and in Fig. 8.

To check our results a uniformly distributed set of 10,000 points was generated inside the tolerance region. The results are shown in Table V. Also shown is the computation time saving when the approximation is used for statistical analysis instead of the exact constraints.

Example 3

Consider a practical example of a nonideal two-section waveguide transformer [14, 15]. The general situation is illustrated by Fig. 9. The two-section transformer was optimized with a design specification of a reflection coefficient of 0.05 over 500 MHz centered at 6.175 GHz. Table VII shows the dimensions of the input and output waveguides and the width of the two sections. The program given in [15] was used to obtain the reflection coefficient. It should be noted that the program calculates only the reflection coefficient. No sensitivities are provided. An equal absolute tolerance ε was assumed for the heights and lengths of the two sections. The assumption is reasonable if they are machined in the same way. The objective is to maximize ε. All vertices of the tolerance region were considered and the efficient method to obtain the values of the relevant constraints and their gradients was applied. The optimum nominal point and tolerances for the worst-case design is given in Table VIII. The active vertices at the worst-case solution indicate that the reflection coefficient is more sensitive to the error in b_1.

To gain an impression of the utility of our approach we show in Table IX the effect of assuming $\varepsilon = 0.01$, keeping other parameters at the appropriate values in Tables VII and VIII. Based on a uniform distribution, 500 Monte Carlo analyses were conducted with both the quadratic model and with the actual response program. The model yields excellent results 11 times faster.

VII. CONCLUSIONS

It is felt that a significant step has been taken in bridging the gap between available analysis programs, which may or may not be efficiently written and probably do not supply derivative information, and the advancing art of optimal centering, tolerancing, and tuning. Efficient gradient methods, which are essential in such general design problems, can be usefully employed through the use of readily differentiable formulas and approximations.

The yield optimization technique described for quadratic approximations to the constraints can be extended to general nonlinear constraints. The efficient technique for calculation of the function and gradients at the different vertices may then be implemented with a suitable large-change sensitivity algorithm.

Yield estimation for other statistical distributions, different from the uniform distribution, can be done by regionalizing the space and associating a uniform distribution with each region [16]. Relevant optimization in such cases is still under investigation.

APPENDIX

Theorem If there exist three distinct base points $\underset{\sim}{\phi}^1$, $\underset{\sim}{\phi}^2$ and $\underset{\sim}{\phi}^3$ in the ith direction, i.e.,

$$\underset{\sim}{\phi}^j = \underset{\sim}{\phi}^1 + c_j \underset{\sim}{e}_i \tag{A1}$$

where c_j, j = 2, 3 are scalars, and $\underset{\sim}{e}_i$ is the unit vector in the ith direction, then the interpolating polynomial is one-dimensionally convex/concave in the ith variable if the interpolated function is so.

Proof

Assume that there exists a λ such that

$$P(\lambda \underset{\sim}{\phi}^a + (1-\lambda) \underset{\sim}{\phi}^b) \gtrless \lambda P(\underset{\sim}{\phi}^a) + (1-\lambda) P(\underset{\sim}{\phi}^b),\ 0<\lambda<1 \tag{A2}$$

where $\underset{\sim}{\phi}^b = \underset{\sim}{\phi}^a + c \underset{\sim}{e}_i$ and c is a scalar, i.e., $P(\underset{\sim}{\phi})$ is not one-dimensionally convex/concave in the ith variable. Then, for the above λ

$$P(\underset{\sim}{\phi}^a + (1-\lambda) c \underset{\sim}{e}_i) \gtrless \lambda P(\underset{\sim}{\phi}^a) + (1-\lambda) P(\underset{\sim}{\phi}^a + c \underset{\sim}{e}_i) .$$

Expanding both sides of this inequality

$$P(\underset{\sim}{\phi}^a) + (1-\lambda) c \underset{\sim}{e}_i^T \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^a) + \tfrac{1}{2}(1-\lambda)^2 c^2 \underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i$$

$$\gtrless P(\underset{\sim}{\phi}^a) + (1-\lambda) c \underset{\sim}{e}_i^T \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^a) + \tfrac{1}{2}(1-\lambda) c^2 \underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i .$$

Thus,

$$(1-\lambda)^2 \underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i \gtrless (1-\lambda) \underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i ,$$

but since $0 < (1-\lambda) < 1$, hence,

$$\underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i \lessgtr 0 . \tag{A3}$$

Without any loss of generality we can assume the base points to be such that

$$\underset{\sim}{\phi}^3 = \gamma \underset{\sim}{\phi}^1 + (1-\gamma)\underset{\sim}{\phi}^2, \quad 0 < \gamma < 1 . \tag{A4}$$

Then,

$$P(\underset{\sim}{\phi}^3) = P(\gamma\,\underset{\sim}{\phi}^1 + (1-\gamma)\underset{\sim}{\phi}^2)$$

$$= P(\underset{\sim}{\phi}^1 + (1-\gamma)\beta\,\underset{\sim}{e}_i) ,$$

where $\underset{\sim}{\phi}^2 = \underset{\sim}{\phi}^1 + \beta\,\underset{\sim}{e}_i$ and β is a scalar.

$$P(\underset{\sim}{\phi}^3) = P(\underset{\sim}{\phi}^1) + (1-\gamma)\beta\,\underset{\sim}{e}_i^T\,\underset{\sim}{\nabla}\,P(\underset{\sim}{\phi}^1) + \tfrac{1}{2}(1-\gamma)^2\,\beta^2\,\underset{\sim}{e}_i^T\,\underset{\sim}{H}\,\underset{\sim}{e}_i$$

$$= \gamma\,P(\underset{\sim}{\phi}^1) + (1-\gamma)\,P(\underset{\sim}{\phi}^2) - \frac{1}{2}\,\gamma(1-\gamma)\beta^2\,\underset{\sim}{e}_i^T\,\underset{\sim}{H}\,\underset{\sim}{e}_i .$$

But, using (A3),

$$P(\underset{\sim}{\phi}^3) \gtrless \gamma\,P(\underset{\sim}{\phi}^1) + (1-\gamma)\,P(\underset{\sim}{\phi}^2) , \tag{A5}$$

i.e., noting that P = f at base points,

$$f(\underset{\sim}{\phi}^3) \gtrless \gamma\,f(\underset{\sim}{\phi}^1) + (1-\gamma)\,f(\underset{\sim}{\phi}^2) , \tag{A6}$$

which contradicts that $f(\underset{\sim}{\phi})$ is one-dimensionally convex/concave in the ith variable. Hence, the assumption (A2) is never true.

Corollary

A quadratic polynomial is one-dimensionally convex/concave if and only if all of the diagonal elements of the Hessian matrix are nonnegative/nonpositive. The proof follows from inequality (A3).

It is to be noted that the number of base points required to keep the one-dimensional convexity/concavity is 2k+1 which is less than the required number of base points (k+1)(k+2)/2.

This corollary indicates whether the approximate constraint region is one-dimensionally convex or not.

REFERENCES

[1] J.W. Bandler, P.C. Liu and J.H.K. Chen, "Worst case network tolerance optimization", IEEE Trans. Microwave Theory Tech., vol. MTT-23, Aug. 1975, pp. 630-641.

[2] J.W. Bandler, P.C. Liu and H. Tromp, "A nonlinear programming approach to optimal design centering, tolerancing and tuning", IEEE Trans. Circuits and Systems, vol. CAS-23, Mar. 1976, pp. 155-165.

[3] J.F. Pinel and K.A. Roberts, "Tolerance assignment in linear networks using nonlinear programming", IEEE Trans. Circuit Theory, vol. CT-19, Sept. 1972, pp. 475-479.

[4] B.J. Karafin, "The optimum assignment of component tolerances for electrical networks", Bell Syst. Tech. J., vol. 50, April 1971, pp. 1225-1242.

[5] N. Elias, "New statistical methods for assigning device tolerances", Proc. 1975 IEEE Symp. on Circuits and Systems (Newton, Mass., April 1975), pp. 329-332.

[6] S.W. Director and G.D. Hachtel, "The simplicial approximation approach to design centering and tolerance assignment", Proc. 1976 IEEE Symp. on Circuits and Systems (Munich, April 1976), pp. 706-709.

[7] J.W. Bandler, "Optimization of design tolerances using nonlinear programming", J. Optimization Theory and Applic., vol. 14, July 1974, pp. 99-114.

[8] J.W. Bandler and P.C. Liu, "Some implications of biquadratic functions in the tolerance problem", IEEE Trans. Circuits and Systems, vol. CAS-22, May 1975, pp. 385-390.

[9] J.W. Bandler and C. Charalambous, "Practical least pth optimization of networks", IEEE Trans. Microwave Theory Tech., vol. MTT-20, Dec. 1972, pp. 834-840.

[10] S.L. Sobolev, "On the interpolation of functions of n variables", (transl.), Sov. Math. Dokl., vol. 2, 1961, pp. 343-346.

[11] H.C. Thacher, Jr., and W.E. Milne, "Interpolation in several variables", SIAM J., vol. 8, 1960, pp. 33-42.

[12] H.C. Thacher, Jr., "Generalization of concepts related to linear dependence", SIAM J., vol. 6, 1959, pp. 288-299.

[13] W.Y. Chu, "Extrapolation in least pth approximation and nonlinear programming", McMaster Univ., Hamilton, Canada, Report SOC-71, Dec. 1974.

[14] J.W. Bandler, "Computer optimization of inhomogeneous waveguide transformers", IEEE Trans. Microwave Theory Tech., vol. MTT-17, Aug. 1969, pp. 563-571.

[15] J.W. Bandler and P.A. Macdonald, "Response program for an inhomogeneous cascade of rectangular waveguides", IEEE Trans. Microwave Theory Tech., vol. MTT-17, Aug. 1969, pp. 646-649.

[16] H.L. Abdel-Malek and J.W. Bandler, "Yield estimation for efficient design centering assuming arbitrary statistical distributions", Conf. Computer-aided Design of Electronic and Microwave Circuits and Systems (Hull, England, July 1977).

TABLE I

COMPUTATIONAL EFFORT FOR EVALUATION OF THE QUADRATIC POLYNOMIAL AND ITS DERIVATIVES

Description	Number of additions	Number of multiplications
At one vertex only	$\frac{1}{2}k(3k+5)$	$\frac{3}{2}k(k+1)$
At all the vertices using original formula	$2^{k-1}k(3k+5)$	$3 \times 2^{k-1} k(k+1)$
At all the vertices using the efficient scheme	$2^{k-n_i}[\frac{1}{2}k(3k+5)+(k+2)(2^{n_i}-1)]+n_i$	$2^{k-n_i}[\frac{3}{2}k(k+1)+n_i(k+1)+2^{n_i}-1]$
At all the vertices using the efficient scheme when $n_i = k$	$\frac{1}{2}k(3k+7)+(k+2)(2^k-1)$	$\frac{5}{2}k(k+1)+2^k-1$

TABLE II

WORST-CASE DESIGN OF THE TWO-SECTION 10:1 QUARTER-WAVE TRANSFORMER

Cost function	Z_1^0	Z_2^0	ε_1/Z_1^0 (%)	ε_2/Z_2^0 (%)	δ	N.O.F.E.*	CDC Time (sec)
C_1	2.5637	5.5048	14.678	9.007	0.4	18	7.213
	2.5234	5.4379	14.988	9.081	0.1	24	9.533
C_2	2.1515	4.7350	12.715	12.697	0.4	12	2.468
	2.1494	4.7305	12.687	12.700	0.1	18	2.959

Starting values $Z_1^0 = 2.2361$, $Z_2^0 = 4.4721$, $\varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.4$

Frequency points used 0.5, 0.6, ..., 1.5 GHz

Objective cost functions $C_1 = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}$, $C_2 = \frac{Z_1^0}{\varepsilon_1} + \frac{Z_2^0}{\varepsilon_2}$

Reflection coefficient specification $|\rho| \leq 0.55$

*N.O.F.E. denotes the number of function evaluations

TABLE III

YIELD DETERMINATION AND OPTIMIZATION OF THE TWO-SECTION 10:1 QUARTER-WAVE TRANSFORMER

Problem	Z_1^0	Z_2^0	ε_1/Z_1^0 (%)	ε_2/Z_2^0 (%)	Objective	Yield (%)
P1*	2.5273	5.3998	21.09	13.51	3.2465	90.0
P2**	2.5290	5.1513	31.44	22.13	3.2597	65.5

* Minimize $1/\varepsilon_1 + 1/\varepsilon_2$ subject to yield $\geq$ 90%

** Minimize $(1/\varepsilon_1+1/\varepsilon_2)/Y$

TABLE IV

WORST-CASE AND YIELD CONSTRAINED RESULTS OF THE LC LOWPASS FILTER

Yield (%)	L_1^0	L_2^0	C^0	ε_1/L_1^0 (%)	ε_2/L_2^0 (%)	ε_C/C^0 (%)
100	1.999	1.998	0.9058	9.88	9.89	7.60
96	1.997	1.997	0.9033	11.23	11.23	12.46

Frequency points used 0.45, 0.5, 0.55, 1.0 in the passband and 2.5 in the stopband

Objective cost function is $\dfrac{L_1^0}{\varepsilon_1} + \dfrac{L_2^0}{\varepsilon_2} + \dfrac{C^0}{\varepsilon_C}$

Insertion loss specification $\leq$ 1.5 dB in the passband and $\geq$ 25 dB in the stopband

TABLE V

COMPARISON OF METHODS OF YIELD ESTIMATION FOR THE LC LOWPASS FILTER

Description	Yield (%)	CDC Time (sec)
Exact constraints	96.59	20.98
Approximate constraints	96.58	10.43

Yield estimation using a set of 10,000 uniformly distributed points inside the tolerance region for the case of 96% yield according to the hyperplane approximation. All of the five frequency points were used.

TABLE VI

COEFFICIENTS OF THE QUADRATIC APPROXIMATION AROUND ACTIVE VERTICES

Freq. point	State	L_1^2	L_2^2	C^2	L_1L_2	L_1C	L_2C	L_1	L_2	C	-
0.55	before	-0.06847	-0.06847	-0.57056	.33010	0.92247	0.93855	-1.67845	-1.69182	-0.46249	3.83750
	after	-0.06847	-0.06847	-0.57056	.33010	0.93051	0.93051	-1.68513	-1.68513	-0.46249	3.83750
1.00	before	-1.12188	-1.16702	-9.98122	.21439	-8.16357	-8.30295	10.21440	10.51832	44.18607	-33.86206
	after	-1.14445	-1.14445	-9.98122	.21439	-8.23326	-8.23326	10.36637	10.36637	44.18607	-33.86206
2.50	before	-1.38601	-1.42228	-9.90167	.39487	-0.92910	-0.94732	10.19142	10.32736	32.94001	-46.93184
	after	-1.40414	-1.40414	-9.90167	.39487	-0.93821	-0.93821	10.25939	10.25939	32.94001	-46.93184

Coefficients of the quadratic approximations obtained at active vertices with a step $\delta = 0.04$. The table shows the coefficients obtained by the algorithm and the coefficients used for yield determination after averaging symmetric coefficients.

TABLE VII

FIXED PARAMETERS AND SPECIFICATIONS FOR THE TWO-SECTION WAVEGUIDE TRANSFORMER

Description	Width (cm)	Height (cm)	Length (cm)
Input guide	3.48488	0.508	∞
First section	3.6	variable	variable
Second section	3.8	variable	variable
Output guide	4.0386	2.0193	∞

Frequency points used 5.925, 6.175, 6.425 GHz
Reflection coefficient specification $|\rho| \leq 0.05$
Minimax solution (no tolerances) $|\rho| = 0.00443$

TABLE VIII

RESULTS CONTRASTING THE TOLERANCED SOLUTION AND THE MINIMAX SOLUTION WITH NO TOLERANCES FOR THE TWO-SECTION WAVEGUIDE TRANSFORMER

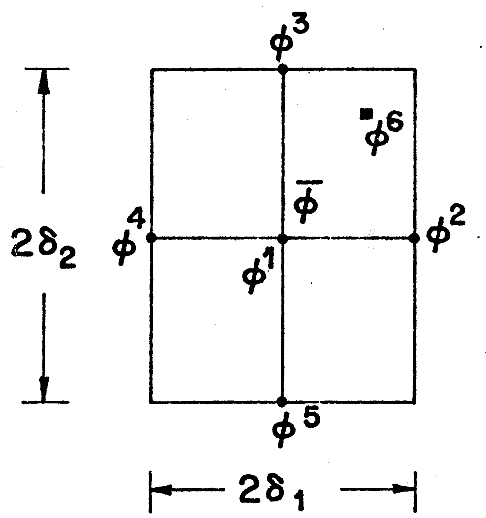
Description	b_1 (cm)	b_2 (cm)	ℓ_1 (cm)	ℓ_2 (cm)	ε (cm)	Number of complete response evaluations	CDC Time (sec)
Toleranced optimum	0.72917	1.41782	1.51317	1.39463	0.00687	45	10
Minimax optimum	0.71315	1.39661	1.56044	1.51621	0	-	-

TABLE IX

COMPARISON OF METHODS OF YIELD ESTIMATION FOR THE TWO-SECTION WAVEGUIDE TRANSFORMER

Number of points	Tolerance ε	Yield(%) Approx.	Yield(%) Actual	CDC Time (sec) Approx.	CDC Time (sec) Actual
500	0.01	99.4	100	< 0.5	5.7

(a)



(b)

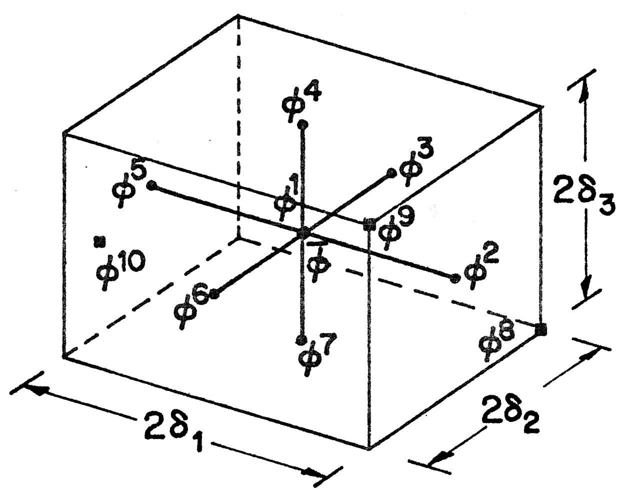


Fig. 1 Arrangement of the base points w.r.t. the centers of the interpolation regions in (a) two dimensions and (b) three dimensions.

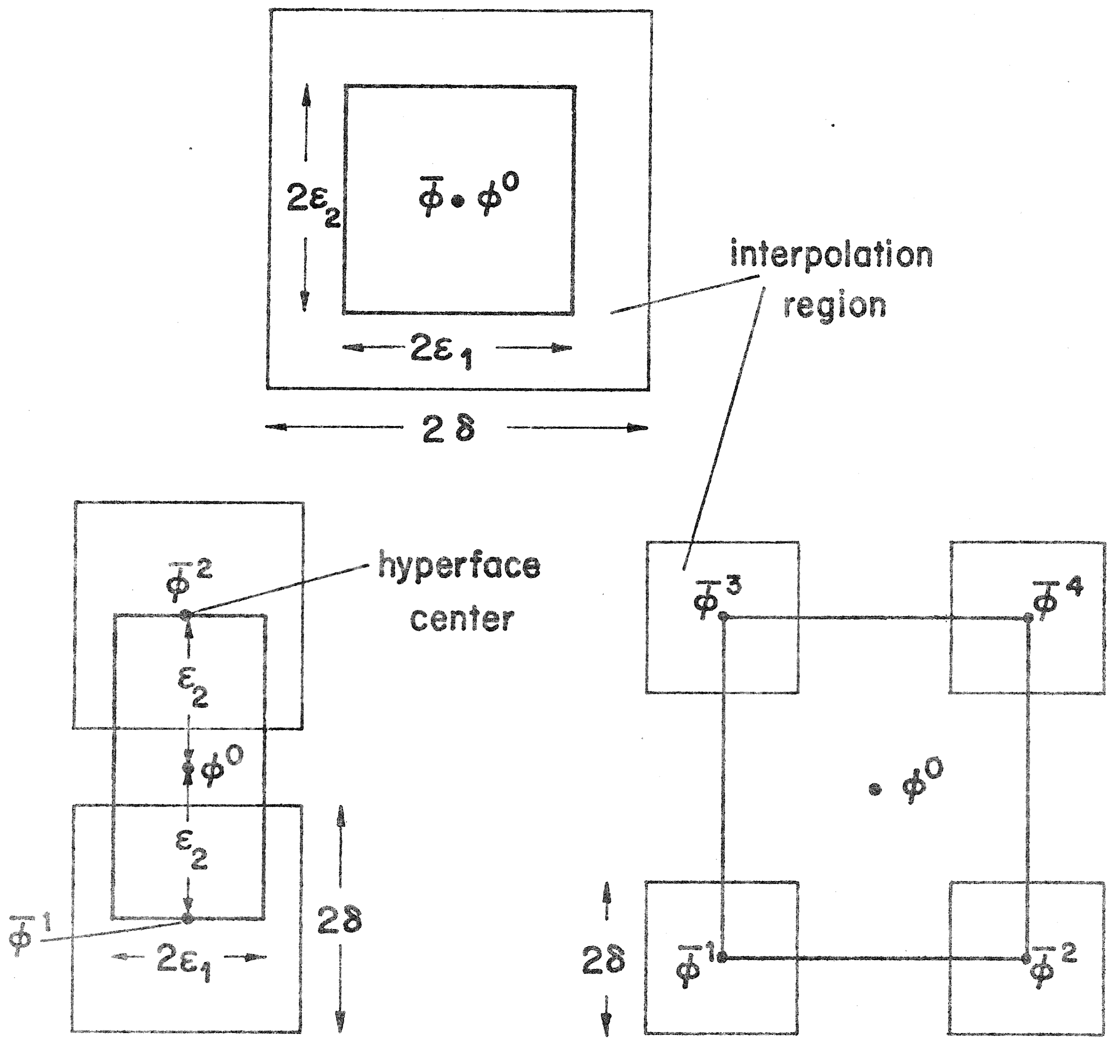


Fig. 2 Three situations created by certain step sizes $\delta = \delta_1 = \delta_2$ and tolerances. The different interpolation regions and their centers are indicated.

(a)

(b)

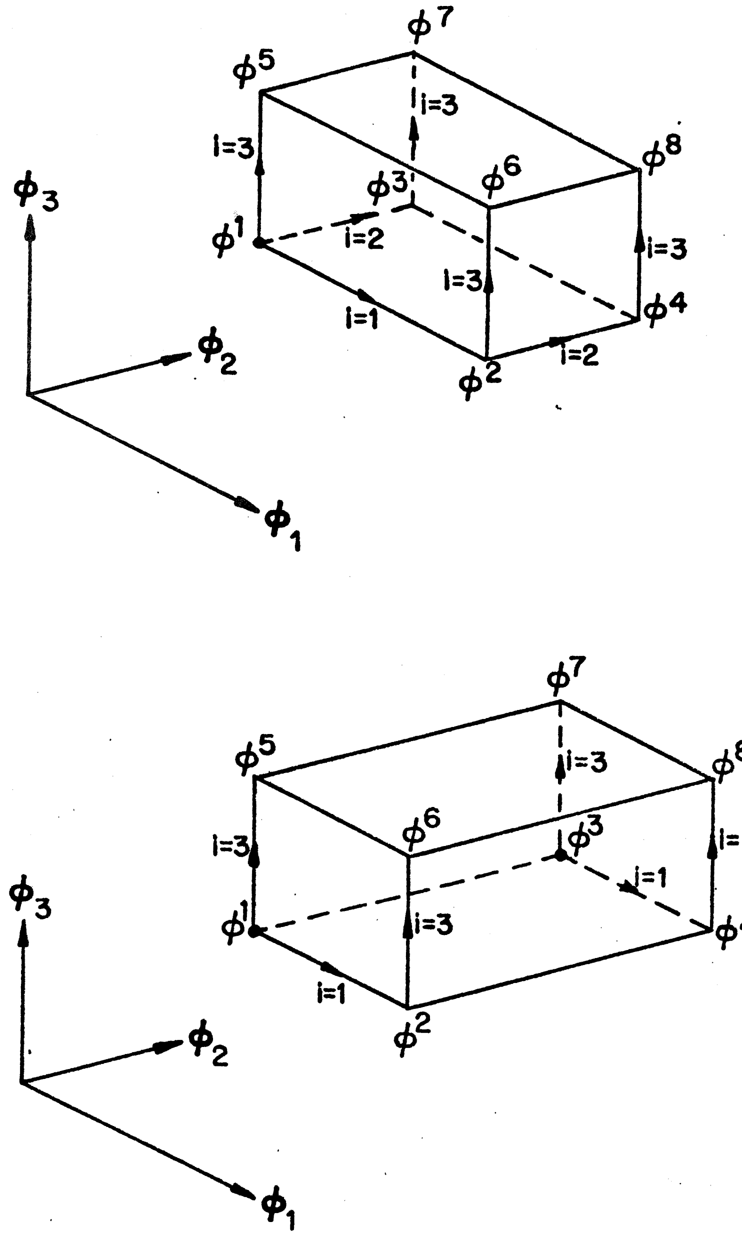


Fig. 3 Illustration of the efficient technique for evaluation of the approximations and their derivatives.

(a) $n_i = 3$, $N_{in} = 1$ and initially $S = \{\underset{\sim}{\phi}^1\}$.

(b) $n_i = 2$, $N_{in} = 2$ and initially $S = \{\underset{\sim}{\phi}^1, \underset{\sim}{\phi}^3\}$.

(a)

(b)

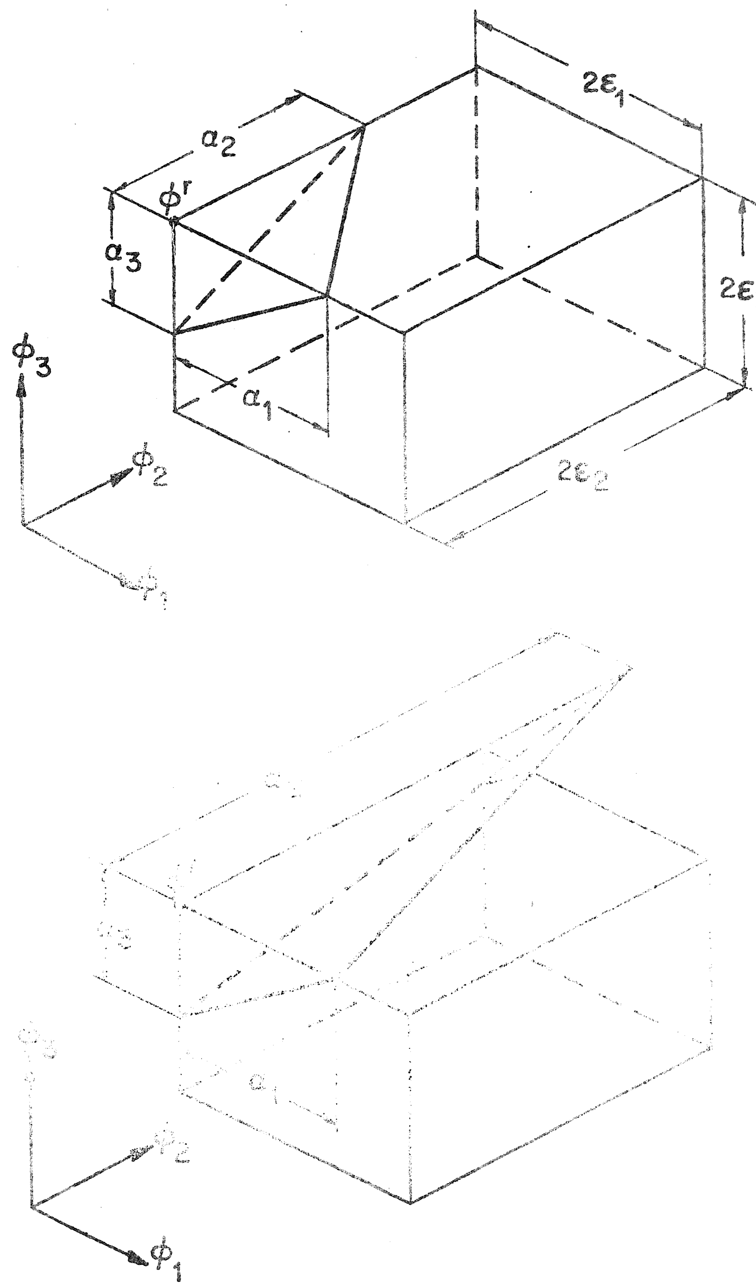


Fig. 4 The nonfeasible volume obtained by a linear cut

(a) $V = \frac{1}{3!} \alpha_1 \alpha_2 \alpha_3$.

(b) $V = (\frac{1}{3!} \alpha_1 \alpha_2 \alpha_3) [1 - (1 - \frac{2\varepsilon_2}{\alpha_2})^3]$.

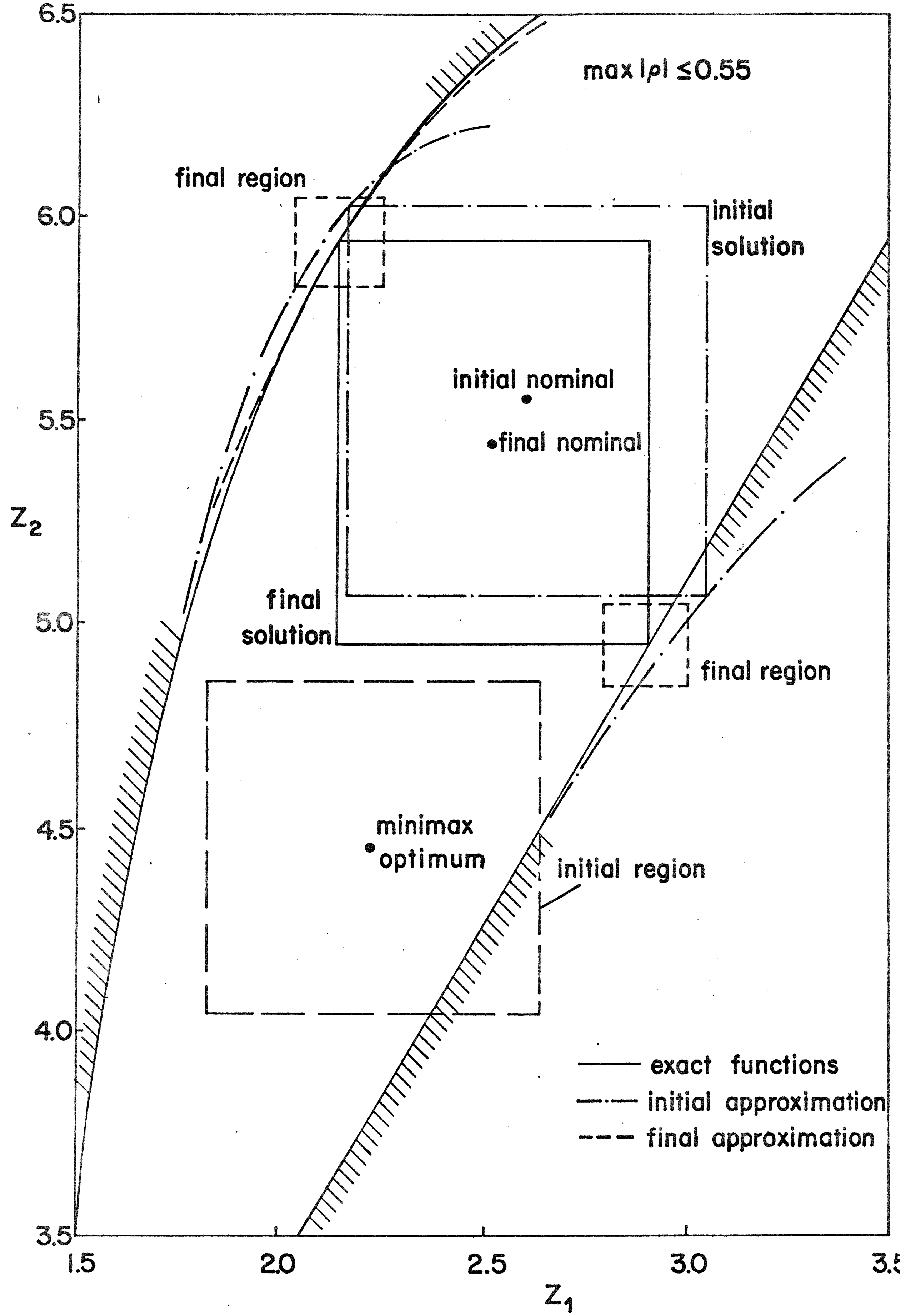


Fig. 5 Minimization of $1/\varepsilon_1 + 1/\varepsilon_2$ for the two-section transformer.

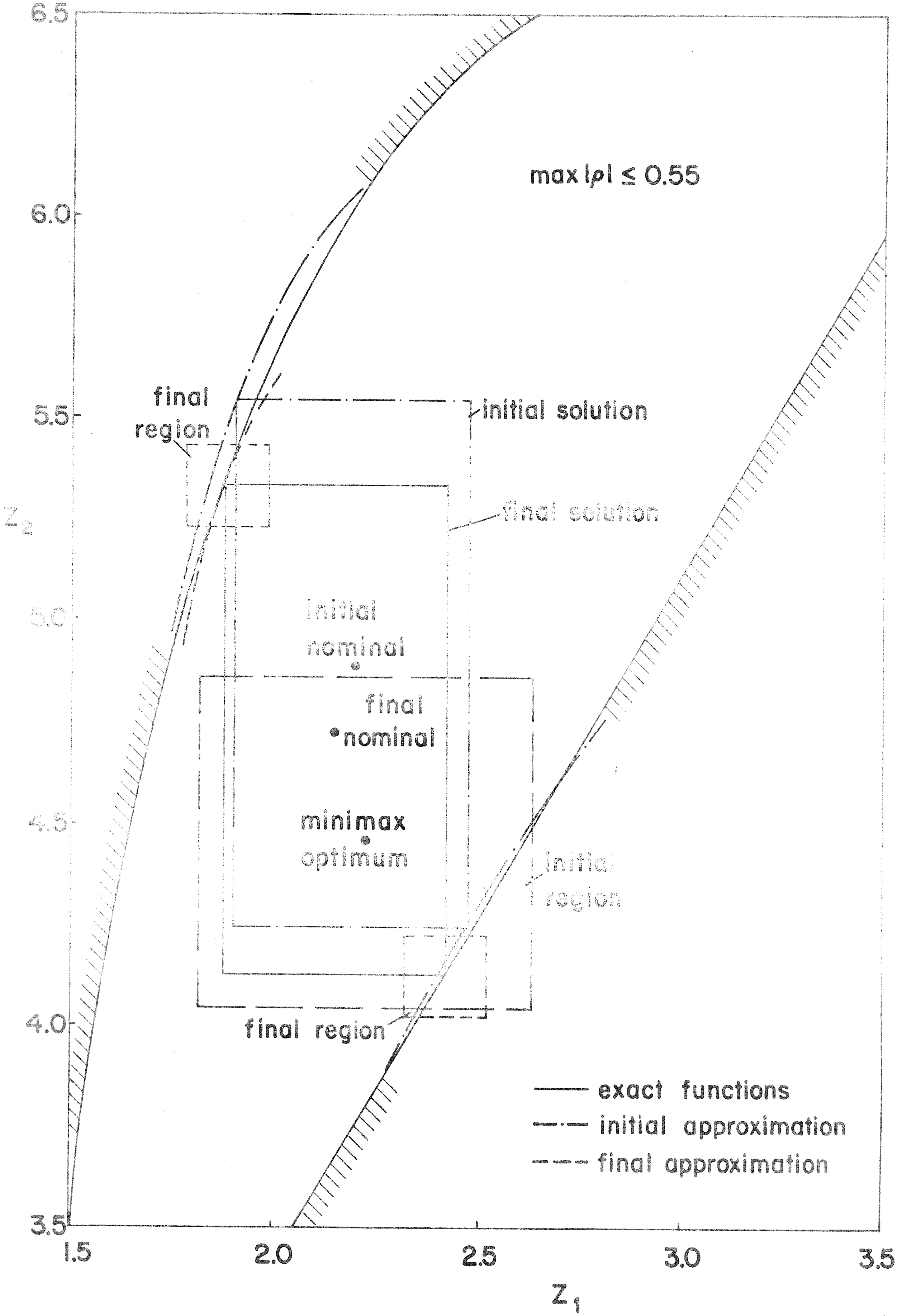


Fig. 6 Minimization of $Z_1^0/\varepsilon_1 + Z_2^0/\varepsilon_2$ for the two-section transformer.

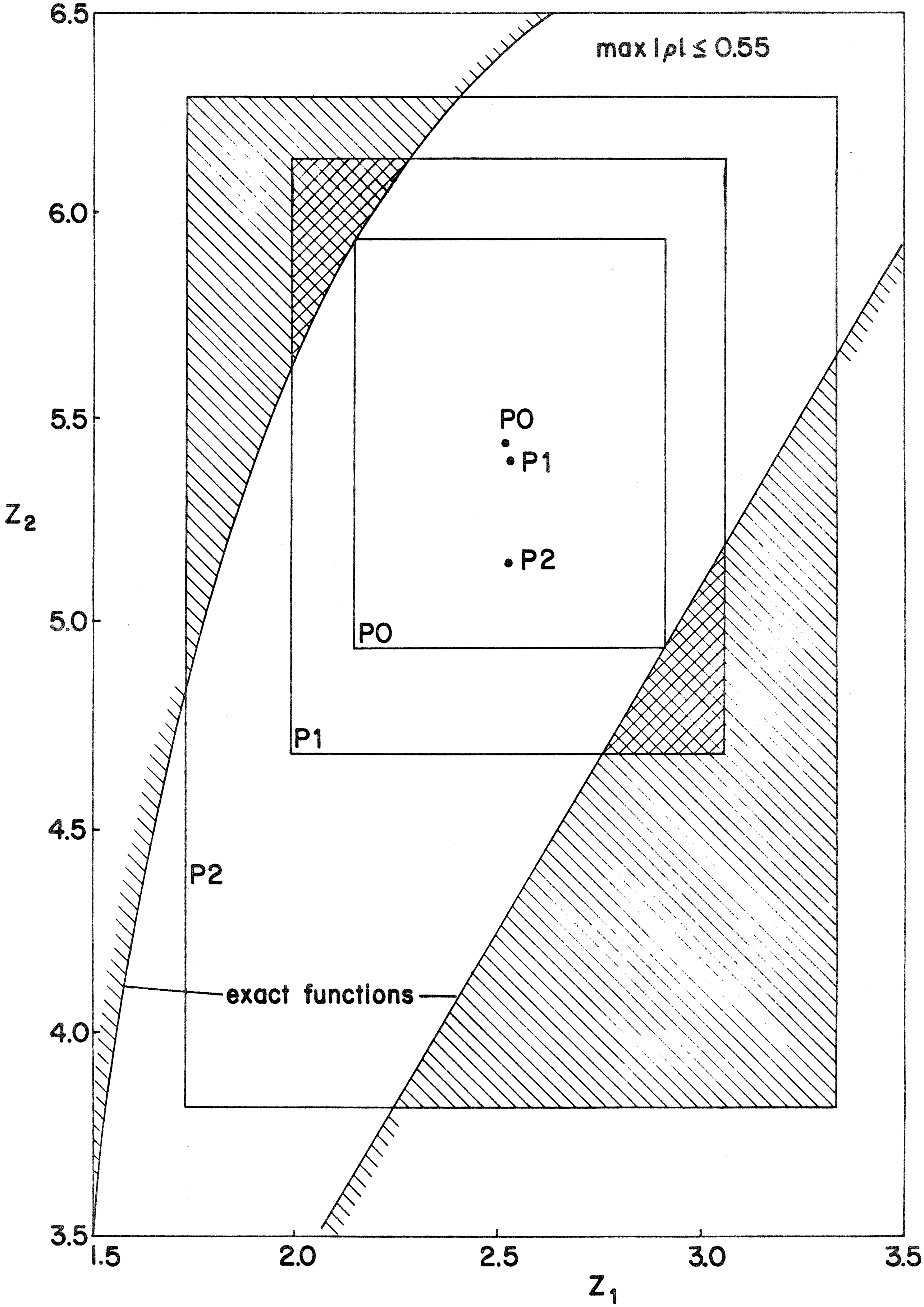


Fig. 7 The optimum tolerance regions and nominal values for the worst-case, 90% yield and optimum yield designs.

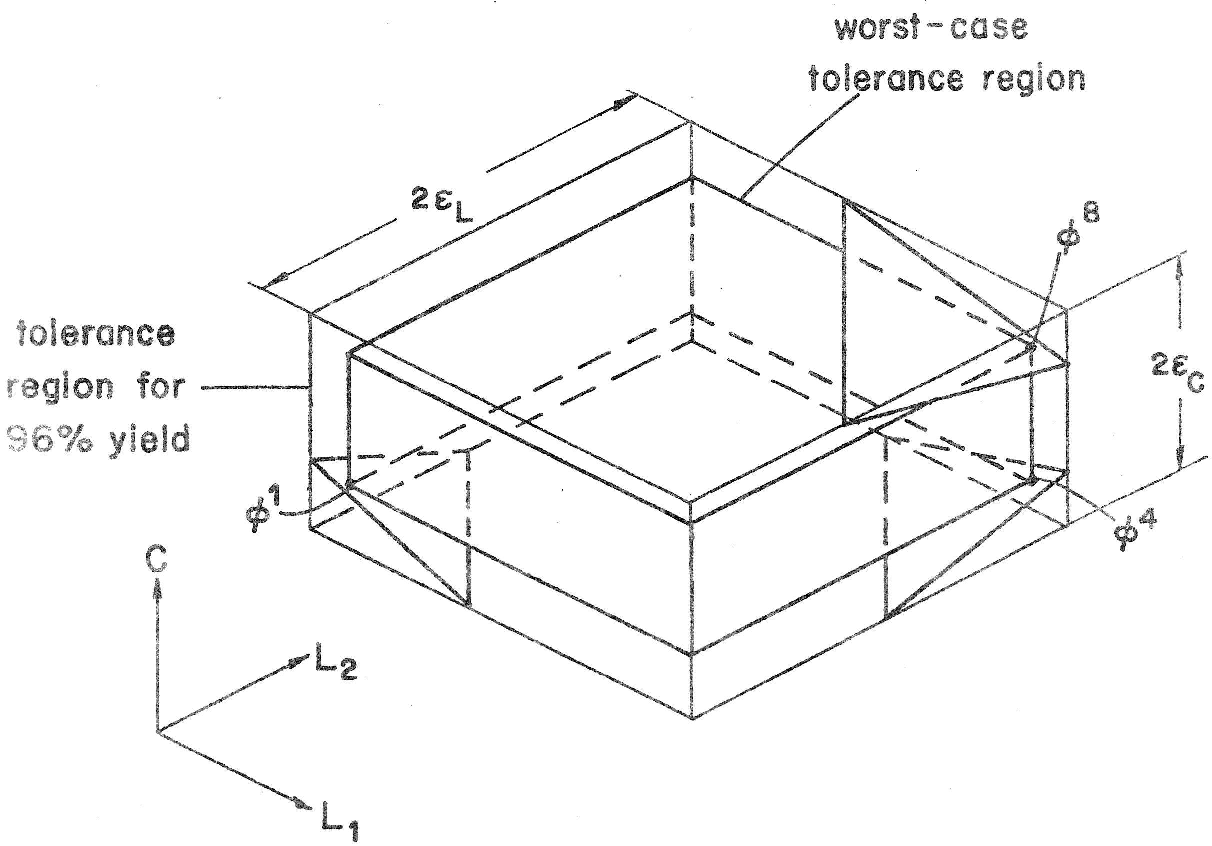


Fig. 8 The tolerance regions for the worst-case design and the 96% yield for the LC filter. The linear cuts shown are based on the intersections of the active quadratic constraint approximations with edges of the tolerance region for 96% yield case.

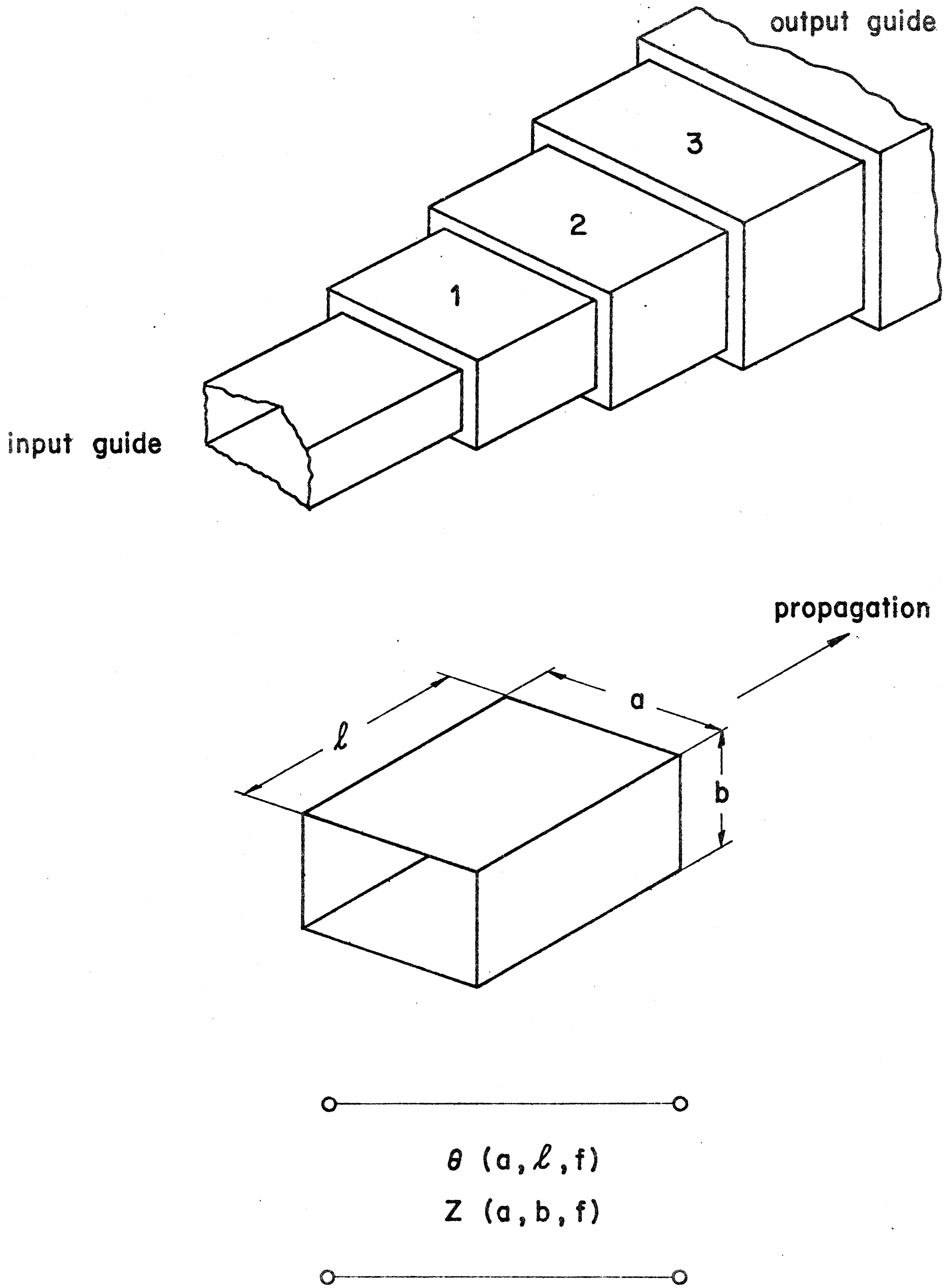


Fig. 9 Illustrations of an inhomogeneous waveguide transformer.

YIELD ESTIMATION FOR EFFICIENT DESIGN CENTERING ASSUMING ARBITRARY STATISTICAL DISTRIBUTIONS

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Abstract

Based upon a uniform distribution inside an orthocell in the toleranced parameter space, it is shown how production yield and yield sensitivities can be evaluated for arbitrary statistical distributions. Formulas for yield and yield sensitivities in the case of a uniform distribution of outcomes between the tolerance extremes are given. A general formula for the yield, which is applicable to any arbitrary statistical distribution, is presented. An illustrative example for verifying the formulas is given. Karafin's bandpass filter has been used for applying the yield formula for a number of different statistical distributions. Uniformly distributed parameters between tolerance extremes, uniformly distributed parameters with accurate components removed and normally distributed parameters were considered. Comparisons with Monte Carlo analysis were made to constrast efficiency.

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I. INTRODUCTION

Design centering and enlarging parameter tolerances, particularly for mass-produced designs such as integrated circuits, is a requirement for cost reduction. It is this aim which emphasizes the problem of yield estimation and makes it an integral part of the design process.

The yield problem has usually been treated through the Monte Carlo method of analysis. Elias [1] presented an approach which applies the Monte Carlo analysis directly to the nonlinear constraints. In an effort to reduce computational time Director and Hachtel [2] suggested applying the Monte Carlo method in conjunction with a polytope describing the constraint region. This polytope (a simplex being a special case [3]) might be defined by quite a large number of hyperplanes. For example, for a space of k dimensions, as described by the algorithm, this number may initially be 2^k. Scott and Walker [4] suggested an efficient technique using Monte Carlo analysis with space regionalization. However, the number of required analyses increases exponentially with the number of variables in order to get the response at the center of each region. Regionalization was later used by Leung and Spence [5] exploiting the technique of systematic exploration. This technique is only applicable to linear circuits.

Karafin [6] used a different approach. The yield was estimated according to truncated Taylor series approximations for the constraints. In the approach presented here we assume a reasonable nominal point and reasonable linear approximations to the constraints. These will usually be available if a centering or a worst-case tolerance assignment problem is solved first. The assumption of a reasonable nominal point was also required by Karafin [6].

The approach is based upon partitioning the region under consideration into a collection of orthotopic cells (orthocells). A weight is assigned to each orthocell and a uniform distribution is assumed inside it. The weights are obtained from tabulated values for known distributions or obtained according to sampling the components used. The freedom in choosing the sizes of the orthocells allows the use of previous information about the problem. A formula for the yield is derived according to these assumptions and it is applicable to any statistical distribution, whether we have independent parameters or correlated parameters with discrete or continuous tolerances.

An illustrative example was used to verify the yield and the yield sensitivity formulas for the uniform case. A comparison with the Monte Carlo analysis method as applied to Karafin's bandpass filter [6] is given for the following statistical distributions:

(a) A uniform distribution of outcomes between tolerance extremes using different values for the tolerances.

(b) A uniform distribution of outcomes between tolerance extremes, but with more accurate components selected out.

(c) Parameters with normal distributions for different values of the standared deviation.

Since the uniform distribution is basic to the presentation, we solve the problem of a uniform distribution first and generalize it for any distribution later.

II. YIELD WITH A UNIFORM DISTRIBUTION

The yield is simply defined by

$$Y \triangleq N/M \quad , \tag{1}$$

where N is the number of outcomes which satisfy the specifications and M is the total number of outcomes.

Define the tolerance region R_ε by

$$R_\varepsilon \triangleq \left\{ \underset{\sim}{\phi} \mid \phi_i^0 - \varepsilon_i \leq \phi_i \leq \phi_i^0 + \varepsilon_i \ , \ i = 1, 2, \ldots, k \right\} \quad , \tag{2}$$

where k is the number of designable parameters, $\underset{\sim}{\phi}^0$ is the nominal parameter vector and $\underset{\sim}{\varepsilon}$ is the vector of absolute tolerances of the corresponding parameters.

Now, define the function V(R) as the hypervolume of the set R. Thus, for the case of independent parameters and assuming a uniform distribution of outcomes between the tolerance extremes, (1) reduces to

$$Y = \frac{V(R_\varepsilon \cap R_c)}{V(R_\varepsilon)} \quad , \tag{3}$$

where

$$R_c \triangleq \left\{ \underset{\sim}{\phi} \mid g_\ell(\underset{\sim}{\phi}) \geq 0 \ , \ \ell = 1, 2, \ldots, m \right\} \tag{4}$$

is the constraint region defined by m linearized constraints

$$g_\ell(\underset{\sim}{\phi}) = \underset{\sim}{\phi}^T \underset{\sim}{q}^\ell - c^\ell \ , \ \ell = 1, 2, \ldots, m \quad . \tag{5}$$

Assuming no overlapping of nonfeasible regions defined by different constraints inside the orthotope R_ε, i.e.,

$$R_i \underset{i \neq j}{\cap} R_j = \emptyset \quad , \tag{6}$$

where

$$R_\ell \triangleq \left\{ \underset{\sim}{\phi} \mid g_\ell(\underset{\sim}{\phi}) < 0 \right\} \cap R_\varepsilon \quad , \tag{7}$$

the yield can be expressed as

$$Y = 1 - \frac{\sum_{\ell=1}^{m} V(R_\ell)}{V(R_\varepsilon)} \quad . \tag{8}$$

Define the set of all vertices of the orthotope R_ε by [7]

$$R_V \triangleq \left\{ \underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu} \ , \ \mu_i \in \{-1, 1\}, \ i = 1, 2, \ldots, k \right\} , \tag{9}$$

where $\underset{\sim}{E}$ is a k x k diagonal matrix with ε_i, i = 1, 2, ..., k along the diagonal and using the following vertex enumeration scheme:

$$r = 1 + \sum_{i=1}^{k} \frac{\mu_i^r + 1}{2} 2^{i-1} \quad . \tag{10}$$

Corresponding to each constraint $g_\ell(\underset{\sim}{\phi}) \geq 0$, let us define a reference vertex

$$\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}^r \quad , \tag{11}$$

where

$$\mu_i^r = - \text{sign } (q_i^\ell) \; , \; i = 1, 2, \ldots, k \quad . \tag{12}$$

If $g_\ell(\underset{\sim}{\phi}^r) \geq 0$, then $V(R_\ell) = 0$. Otherwise we find the distance between the intersection of the hyperplane $g_\ell(\underset{\sim}{\phi}) = 0$ and the reference vertex $\underset{\sim}{\phi}^r$ along an edge of R_ε in the ith direction given by

$$\begin{aligned} \alpha_i^\ell &= \mu_i^r \, g_\ell(\underset{\sim}{\phi}^r) \, / \, q_i^\ell \\ &= \mu_i^r \left\{ \phi_i^0 + \mu_i^r \varepsilon_i - \frac{1}{q_i^\ell} \left[c^\ell - \sum_{\substack{j=1 \\ j \neq i}}^{k} q_j^\ell (\phi_j^0 + \mu_j^r \varepsilon_j) \right] \right\} , \; i=1,2,\ldots,k. \end{aligned} \tag{13}$$

In order to derive an expression for $V^\ell = V(R_\ell)$, consider the two-dimensional examples shown in Fig. 1. The nonfeasible area in Fig. 1(a) is given by

$$\begin{aligned} V &= \Delta\, \phi^r ab - \Delta\, \phi^4 ac - \Delta\, \phi^1 bd \\ &= \frac{1}{2} \alpha_1 \alpha_2 - \frac{1}{2} \left[\alpha_1 \left(1 - \frac{2\varepsilon_1}{\alpha_1} \right) \right] \left[\alpha_2 \left(1 - \frac{2\varepsilon_1}{\alpha_1} \right) \right] \\ &\quad - \frac{1}{2} \left[\alpha_1 \left(1 - \frac{2\varepsilon_2}{\alpha_2} \right) \right] \left[\alpha_2 \left(1 - \frac{2\varepsilon_2}{\alpha_2} \right) \right] \\ &= \frac{1}{2} \alpha_1 \alpha_2 \left[1 - \left(1 - \frac{2\varepsilon_1}{\alpha_1} \right)^2 - \left(1 - \frac{2\varepsilon_2}{\alpha_2} \right)^2 \right] . \end{aligned}$$

Also, in Fig. 1(b), the nonfeasible area is given by

$$V = \Delta\, \phi^r ab - \Delta\, \phi^4 ac - \Delta\, \phi^1 bd + \Delta\, \phi^2 cd$$

$$= \frac{1}{2}\alpha_1 \alpha_2 \left[1 - \left(1 - \frac{2\varepsilon_1}{\alpha_1}\right)^2 - \left(1 - \frac{2\varepsilon_2}{\alpha_2}\right)^2 + \left(1 - \frac{2\varepsilon_1}{\alpha_1} - \frac{2\varepsilon_2}{\alpha_2}\right)^2\right] .$$

A three-dimensional example is shown in Fig. 2. In that example the linear constraint cuts the orthotope at the polygon a b c d e and the volume is given by

$$V = \frac{1}{6}\alpha_1 \alpha_2 \alpha_3 - \frac{1}{6}\alpha_1 \alpha_2 \alpha_3 \left(1 - \frac{2\varepsilon_1}{\alpha_1}\right)^3 - \frac{1}{6}\alpha_1 \alpha_2 \alpha_3 \left(1 - \frac{2\varepsilon_2}{\alpha_2}\right)^3$$

$$- \frac{1}{6}\alpha_1 \alpha_2 \alpha_3 \left(1 - \frac{2\varepsilon_3}{\alpha_3}\right)^3 + \frac{1}{6}\alpha_1 \alpha_2 \alpha_3 \left(1 - \frac{2\varepsilon_1}{\alpha_1} - \frac{2\varepsilon_2}{\alpha_2}\right)^3 .$$

Hence, the general formula can be written as

$$V(R_\ell) = \left\{\frac{1}{k!}\prod_{j=1}^{k} \alpha_j^\ell\right\} \left\{\sum_{s \in S_\ell} (-1)^{\nu^s} (\delta_\ell^s)^k\right\} , \tag{14}$$

where

$$\delta_\ell^s = 1 - \sum_{j=1}^{k} \frac{\varepsilon_j}{\alpha_j^\ell} \left|\mu_j^s - \mu_j^r\right| , \tag{15}$$

$$S_\ell \triangleq \left\{s \mid g_\ell(\underset{\sim}{\phi}^s) < 0, \underset{\sim}{\phi}^s = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}^s\right\} , \tag{16}$$

$$\nu^s = \sum_{i=1}^{k} \left|\mu_i^s - \mu_i^r\right| / 2 . \tag{17}$$

An illustration of (14) for the case of k = 3 is shown in Fig. 2. Since

$$V(R_\varepsilon) = 2^k \prod_{j=1}^{k} \varepsilon_j , \tag{18}$$

the yield sensitivities can be expressed as

$$\frac{\partial Y}{\partial \phi_i^0} = - \sum_{\ell=1}^{m} \frac{\partial V^\ell}{\partial \phi_i^0} \Big/ V(R_\varepsilon) , \tag{19}$$

$$\frac{\partial Y}{\partial \varepsilon_i} = \left(\frac{1}{\varepsilon_i} \sum_{\ell=1}^{m} V^\ell - \sum_{\ell=1}^{m} \frac{\partial V^\ell}{\partial \varepsilon_i} \right) \Big/ V(R_\varepsilon) \ . \tag{20}$$

We take

$$\frac{\partial V^\ell}{\partial \phi_i^0} = \frac{\partial V^\ell}{\partial \varepsilon_i} = 0 \quad \text{if } g_\ell(\underset{\sim}{\phi}^r) \geq 0 \quad ,$$

otherwise

$$\frac{\partial V^\ell}{\partial \phi_i^0} = \left\{ \frac{q_i^\ell}{k!} \sum_{p=1}^{k} \left[\frac{\mu_p^r}{q_p^\ell} \prod_{\substack{j=1 \\ j\neq p}}^{k} \alpha_j^\ell \right] \right\} A$$

$$+ B\left\{ k\ q_i^\ell \sum_{s\in S_\ell} (-1)^{\nu^s} (\delta_\ell^s)^{k-1} \left(\sum_{j=1}^{k} \frac{\mu_j^r}{q_j^\ell} \frac{\varepsilon_j}{(\alpha_j^\ell)^2} \left| \mu_j^s - \mu_j^r \right| \right) \right\} , \tag{21}$$

$$\frac{\partial V^\ell}{\partial \varepsilon_i} = \mu_i^r \frac{\partial V^\ell}{\partial \phi_i^0} - B\left\{ \frac{k}{\alpha_i^\ell} \sum_{s\in S_\ell} \left| \mu_i^s - \mu_i^r \right| (-1)^{\nu^s} (\delta_\ell^s)^{k-1} \right\} , \tag{22}$$

where

$$A = \sum_{s\in S_\ell} (-1)^{\nu^s} (\delta_\ell^s)^k \quad , \tag{23}$$

$$B = \frac{1}{k!} \prod_{j=1}^{k} \alpha_j^\ell \quad . \tag{24}$$

It is to be noted that the yield sensitivities are discontinuous whenever a vertex $\underset{\sim}{\phi}^s$ satisfies the equation $g_\ell(\underset{\sim}{\phi}^s) = 0$ for any $\ell = 1, 2, \ldots, m$. Also for the case of having $\alpha_j \to \infty$ there exists a limit for the hypervolume formula and its sensitivities.

For an alternative way of calculating $V(R_\ell)$ we define a complementary vertex

$$\underset{\sim}{\phi}^{\bar{r}} = \underset{\sim}{\phi}^{0} + \underset{\sim}{E}\,\underset{\sim}{\mu}^{\bar{r}} \quad , \tag{25}$$

where

$$\mu_i^{\bar{r}} = -\,\mu_i^{r} = \text{sign}\,(q_i^{\ell}),\ i = 1, 2, \ldots, k. \tag{26}$$

If $g_\ell(\underset{\sim}{\phi}^{\bar{r}}) \leq 0$, then $V(R_\ell) = V(R_\varepsilon)$. Otherwise we find the distance between the intersection of the hyperplane $g_\ell(\underset{\sim}{\phi}) = 0$ and the complementary vertex $\underset{\sim}{\phi}^{\bar{r}}$ along an edge of R_ε in the ith direction given by

$$\bar{\alpha}_i^{\ell} = \mu_i^{\bar{r}}\, g_\ell(\underset{\sim}{\phi}^{\bar{r}}) \;/\; q_i^{\ell}\ ,\ i = 1, 2, \ldots, k. \tag{27}$$

Hence we find the following equations:

$$V^\ell = V(R_\ell) = 2^k \prod_{j=1}^{k} \varepsilon_j - \left\{\frac{1}{k!} \prod_{j=1}^{k} \bar{\alpha}_j^{\ell}\right\} \left\{ \sum_{s \varepsilon \bar{S}_\ell} (-1)^{\bar{\nu}^s} \, (\bar{\delta}_\ell^s)^k \right\} , \tag{28}$$

where

$$\bar{\delta}_\ell^s = 1 - \sum_{j=1}^{k} \frac{\varepsilon_j}{\bar{\alpha}_j^{\ell}} \left| \mu_j^s - \mu_j^{\bar{r}} \right| \quad , \tag{29}$$

$$\bar{S}_\ell \triangleq \left\{ s \mid g_\ell(\underset{\sim}{\phi}^s) > 0,\ \underset{\sim}{\phi}^s = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}^s \right\} \quad , \tag{30}$$

$$\bar{\nu}^s = \sum_{i=1}^{k} \left| \mu_i^s - \mu_i^{\bar{r}} \right| /2 \quad . \tag{31}$$

Equations (19) and (20) remain as before.

We take

$$\frac{\partial V^\ell}{\partial \phi_i^0} = 0 \quad \text{and} \quad \frac{\partial V^\ell}{\partial \varepsilon_i} = 2^k \prod_{\substack{j=1 \\ j \neq i}}^{k} \varepsilon_j \quad \text{if} \quad g_\ell(\underset{\sim}{\phi}^{\bar{r}}) \leq 0 \quad ,$$

otherwise

$$\frac{\partial V^{\ell}}{\partial \phi_i^0} = - \left\{ \frac{q_i^{\ell}}{k!} \sum_{p=1}^{k} \left[\frac{\mu_p^{\bar{r}}}{q_p^{\ell}} \prod_{\substack{j=1 \\ j \neq p}}^{k} \bar{\alpha}_j^{\ell} \right] \right\} \bar{A}$$

$$- \bar{B} \left\{ k \, q_i^{\ell} \sum_{s \in \bar{S}_{\ell}} (-1)^{\bar{\nu}^s} (\bar{\delta}_{\ell}^s)^{k-1} \left(\sum_{j=1}^{k} \frac{\mu_j^{\bar{r}}}{q_j^{\ell}} \frac{\varepsilon_j}{(\bar{\alpha}_j^{\ell})^2} \left| \mu_j^s - \mu_j^{\bar{r}} \right| \right) \right\} , \tag{32}$$

$$\frac{\partial V^{\ell}}{\partial \varepsilon_i} = 2^k \prod_{\substack{j=1 \\ j \neq i}}^{k} \varepsilon_j + \mu_i^{\bar{r}} \frac{\partial V^{\ell}}{\partial \phi_i^0} + \bar{B} \left\{ \frac{k}{\alpha_i^{\ell}} \sum_{s \in \bar{S}_{\ell}} \left| \mu_i^s - \mu_i^{\bar{r}} \right| (-1)^{\bar{\nu}^s} (\bar{\delta}_{\ell}^s)^{k-1} \right\} , \tag{33}$$

where

$$\bar{A} = \sum_{s \in \bar{S}_{\ell}} (-1)^{\bar{\nu}^s} (\bar{\delta}_{\ell}^s)^k , \tag{34}$$

$$\bar{B} = \frac{1}{k!} \prod_{j=1}^{k} \bar{\alpha}_j^{\ell} . \tag{35}$$

In order to obtain the hypervolume and its sensitivities efficiently we use the following criteria:

i) If $g_{\ell}(\underset{\sim}{\phi}^r) \geq 0$, use reference vertex approach.

ii) If $g_{\ell}(\underset{\sim}{\phi}^{\bar{r}}) \leq 0$, use complementary vertex approach.

iii) If $g_{\ell}(\underset{\sim}{\phi}^r) < 0$ and $g_{\ell}(\underset{\sim}{\phi}^{\bar{r}}) > 0$, then

if $|g_{\ell}(\underset{\sim}{\phi}^r)| \leq |g_{\ell}(\underset{\sim}{\phi}^{\bar{r}})|$, use reference vertex approach,

if $|g_{\ell}(\underset{\sim}{\phi}^r)| > |g_{\ell}(\underset{\sim}{\phi}^{\bar{r}})|$, use complementary vertex approach.

The cases i) and ii) are clear since the hypervolume will be either completely feasible or completely nonfeasible, respectively. Case iii) follows from the theorem in the Appendix.

Example 1

Consider the following four-dimensional example, with a linear constraint

$$g(\tilde{\phi}) = \frac{\phi_1}{24} + \frac{\phi_2}{15} + \frac{\phi_3}{60} + \frac{\phi_4}{240} - 1 \geq 0 \quad ,$$

and where

$$\tilde{\phi}^0 = \begin{bmatrix} 9 \\ 7 \\ 9 \\ 26 \end{bmatrix} \quad , \quad \tilde{\varepsilon} = \begin{bmatrix} 5 \\ 2 \\ 4 \\ 6 \end{bmatrix} .$$

Hence,

$$\tilde{\phi}^r = \begin{bmatrix} 9 \\ 7 \\ 9 \\ 26 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 5 \\ 20 \end{bmatrix}$$

and

$$V = \left[\frac{1}{4!} 8 \times 5 \times 20 \times 80\right] \left[1 - \left(1 - \frac{4}{5}\right)^4 - \left(1 - \frac{8}{20}\right)^4 - \left(1 - \frac{12}{80}\right)^4 + \left(1 - \frac{4}{5} - \frac{12}{80}\right)^4 + \left(1 - \frac{8}{20} - \frac{12}{80}\right)^4\right]$$

$$= 1034.15 \quad .$$

Table I shows the nonfeasible vertices. A check for the analytical formulas for the gradients and the numerical gradients obtained by central differences is shown in Table II.

The alternative approach will lead to

$$\tilde{\phi}^{\overline{r}} = \begin{bmatrix} 9 \\ 7 \\ 9 \\ 26 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 13 \\ 32 \end{bmatrix}$$

and

$$V = 2^4 \times 5 \times 2 \times 4 \times 6 - \left[\frac{1}{4!}(8\text{x}1.6)(5\text{x}1.6)(20\text{x}1.6)(80\text{x}1.6)\right]$$

$$\cdot \left[1 - \left(1 - \frac{10}{8\text{x}1.6}\right)^4 - \left(1 - \frac{4}{5\text{x}1.6}\right)^4 - \left(1 - \frac{8}{20\text{x}1.6}\right)^4 + \left(1 - \frac{4}{5\text{x}1.6} - \frac{8}{20\text{x}1.6}\right)^4\right.$$

$$- \left(1 - \frac{12}{80\text{x}1.6}\right)^4 + \left(1 - \frac{10}{8\text{x}1.6} - \frac{12}{80\text{x}1.6}\right)^4 + \left(1 - \frac{4}{5\text{x}1.6} - \frac{12}{80\text{x}1.6}\right)^4$$

$$\left. + \left(1 - \frac{8}{20\text{x}1.6} - \frac{12}{80\text{x}1.6}\right)^4 - \left(1 - \frac{4}{5\text{x}1.6} - \frac{8}{20\text{x}1.6} - \frac{12}{80\text{x}1.6}\right)^4\right]$$

$$= 3840 - 2805.85 = 1034.15 \quad .$$

III. YIELD WITH STATISTICAL DISTRIBUTIONS

The probability distribution function (PDF) might extend as far as $(-\infty, \infty)$, however, for all practical cases we consider a tolerance region R_ε such that

$$\int_{R_\varepsilon} F(\underset{\sim}{\phi})\, d\phi_1\, d\phi_2 \ldots d\phi_k \simeq 1 \quad , \tag{36}$$

where $F(\underset{\sim}{\phi})$ is the PDF.

The orthotope R_ε is now partitioned into a set of orthocells $R(i_1, i_2, \ldots, i_k)$ as in Fig. 3, where $i_j = 1, 2, \ldots, n_j$, n_j is the number of intervals in the jth direction and $j = 1, 2, \ldots, k$. A weighting factor $W(i_1, i_2, \ldots, i_k)$ is assigned to each orthocell and is given by

$$W(i_1, i_2, \ldots, i_k) = w(i_1, i_2, \ldots, i_k) \;/\; V(R(i_1, i_2, \ldots, i_k)) \quad , \tag{37}$$

where

$$w(i_1, i_2, \ldots, i_k) = \int_{R(i_1,i_2,\ldots,i_k)} F(\underset{\sim}{\phi})\, dv \quad , \tag{38}$$

$$V(R(i_1, i_2, \ldots, i_k)) = \int_{R(i_1,i_2,\ldots,i_k)} dv = \prod_{j=1}^{k} \varepsilon_{j,i_j} \quad , \tag{39}$$

$$dv = d\phi_1 \, d\phi_2 \, \ldots \, d\phi_k \tag{40}$$

and $\varepsilon_{1,i_1}, \varepsilon_{2,i_2}, \ldots, \varepsilon_{k,i_k}$ are the dimensions of the orthocell.

In principle, the problem of finding the yield is now reduced to finding the contribution to the yield given by any of these orthocells. However, it will be a tedious job to consider $\prod_{j=1}^{k} n_j$ orthocells. By exploiting the way (14) is constructed, a formula for the weighted nonfeasible hypervolume with respect to the ℓth constraint is constructed and is given by

$$V^{\ell} = \left[\frac{1}{k!} \prod_{j=1}^{k} \alpha_j^{\ell}\right] \left[\sum_{i_1=1}^{n_1+1} \sum_{i_2=1}^{n_2+1} \ldots \sum_{i_k=1}^{n_k+1} \Delta W(i_1,i_2,\ldots,i_k)\,(\delta_{\ell}(i_1,i_2,\ldots,i_k))^k\right] , \tag{41}$$

where, for <u>indexing with respect to $\underset{\sim}{\phi}^r$</u> (see Fig. 3), α_j^{ℓ} = the distance from the reference vertex to the point of intersection in the jth direction,

$$\delta_{\ell}(i_1, i_2, \ldots, i_k) = \max \left[0, \left(1 - \sum_{j=1}^{k} \frac{1}{\alpha_j^{\ell}} \sum_{p=1}^{i_j} \varepsilon_{j,p-1}\right)\right] , \tag{42}$$

$$\varepsilon_{j,0} = 0 \quad , \; j = 1, 2, \ldots, k \tag{43}$$

$$\begin{aligned} \Delta W(i_1,i_2,\ldots,i_k) = {} & W(i_1,i_2,\ldots,i_k) - \sum_{j=1}^{k} W(i_1,i_2,\ldots,i_{j-1},i_j-1,i_{j+1},\ldots,i_k) \\ & + \sum_{j=1}^{k-1} \sum_{p=j+1}^{k} W(i_1, i_2,\ldots,i_j-1,\ldots,i_p-1,\ldots,i_k) - \ldots \\ & + (-1)^k \, W(i_1-1,i_2-1,\ldots,i_k-1) \end{aligned} \tag{44}$$

$$W(i_1,i_2,\ldots,i_k) = 0 \;\text{ if }\; i_j = 0 \;\text{ or }\; i_j = n_j+1 \text{ for any } j. \tag{45}$$

For the case of independent parameters (41) can be written as

$$V^{\ell} = \left[\frac{1}{k!} \prod_{j=1}^{k} \alpha_j^{\ell}\right]\left[\sum_{i_1=1}^{n_1+1} \Delta W_1(i_1) \sum_{i_2=1}^{n_2+1} \Delta W_2(i_2) \ldots \right.$$

$$\left. \sum_{i_k=1}^{n_k+1} \Delta W_k(i_k) \; (\delta_{\ell}(i_1,i_2,\ldots,i_k))^k \right] \quad (46)$$

where

$$\Delta W_j(i_j) = W_j(i_j) - W_j(i_j-1) \quad , \quad (47)$$

$$W_j(0) = W_j(n_j+1) = 0 \quad , \quad (48)$$

$$W_j(i_j) = \int_{R_j(i_j)} f_j(\phi_j)\, d\phi_j \Big/ \varepsilon_{j,i_j} \quad , \; i_j=1,2,\ldots,n_j \; , \quad (49)$$

$f_j(\phi_j)$ is the PDF of the jth parameter and $R_j(i_j)$ is the ith interval for that parameter. Table III illustrates the calculation of weighted hypervolume.

Again, assuming nonoverlapping, nonfeasible regions defined by different constraints inside the orthotope R_ε, the yield can be expressed as

$$Y = 1 - \sum_{\ell=1}^{m} V^{\ell} \quad . \quad (50)$$

In short, the method approximates the integration of the PDF over the feasible region. It allows freedom in discretizing the PDF which is an advantage particularly if a worst-case solution is already known.

Example 2

The bandpass filter [6, 8], shown in Fig. 4, was used for verification of the yield formula. The specifications are shown in Table IV. All inductors have the same Q at the nominal value given in [8] as the corresponding inductors in [6]. The results given in [8] as indicated by the authors violates the specifications at unconsidered frequency points. The adjoint network technique was used for evaluating the sensitivities and, hence, linearizing the constraints at these frequency points. The linearization was done at the worst violating vertex, i.e., the vertex which gives the most negative value for that particular constraint. The yields obtained by the present approach and applying the Monte Carlo method with the nonlinear constraints for a uniform distribution are shown in Table V. Further, as the tolerances were increased more frequency points were considered. In order to avoid overlapping constraints, for each nonfeasible vertex the frequency point corresponding to the worst violated constraint is considered.

In addition, a uniform distribution of outcomes was considered but with the more accurate components removed. This gives $w_i(1) = w_i(3) = 0.5$ and $w_i(2) = 0$. The problem is equivalent to having 2^8 different orthotopes. The results are shown in Table VI.

Consider now the case of a normal distribution which has a probability distribution function [9]

$$F(\underset{\sim}{\phi}) = \frac{1}{(2\pi)^{k/2}} \frac{1}{\sqrt{|COV|}} \exp\left[-\frac{1}{2}(\underset{\sim}{\phi} - \underset{\sim}{\phi}^0)^T (COV)^{-1} (\underset{\sim}{\phi} - \underset{\sim}{\phi}^0)\right] ,$$

where

k is the number of parameters,

$\underset{\sim}{\phi}^0$ is the mean value of the parameter vector $\underset{\sim}{\phi}$,

COV is the covariance matrix.

In the case of no correlation, COV is a diagonal matrix with variances σ_i^2, $i = 1, 2, \ldots, k$, along the diagonal. Hence,

$$F(\underset{\sim}{\phi}) = \frac{1}{(2\pi)^{k/2}} \frac{1}{\prod_{i=1}^{k} \sigma_i} \exp\left[-\sum_{i=1}^{k} \left(\frac{\phi_i - \phi_i^0}{\sigma_i}\right)^2\right] .$$

Using the described approach and dividing the interval $\left[\phi_i^0 - 2\sigma_i, \phi_i^0 + 2\sigma_i\right]$ for each parameter into three different subintervals the weights are obtained in the following manner. Let [10]

$$I_1 = \frac{1}{\sqrt{2\pi}\,\sigma_i} \int_{-2\sigma_i}^{-2\sigma_i/3} \exp\left[-\left(\frac{\phi_i - \phi_i^0}{\sigma_i}\right)^2\right] d\phi_i = 0.2298 \quad ,$$

$$I_2 = \frac{1}{\sqrt{2\pi}\,\sigma_i} \int_{-2\sigma_i/3}^{2\sigma_i/3} \exp\left[-\left(\frac{\phi_i - \phi_i^0}{\sigma_i}\right)^2\right] d\phi_i = 0.4950 \ ,$$

$$I_3 = \frac{1}{\sqrt{2\pi}\,\sigma_i} \int_{2\sigma_i/3}^{2\sigma_i} \exp\left[-\left(\frac{\phi_i - \phi_i^0}{\sigma_i}\right)^2\right] d\phi_i = 0.2298 \quad .$$

Considering a probability of unity for finding ϕ_i in the interval $[\phi_i-2\sigma_i, \phi_i+2\sigma_i]$, the weights for each interval are given by (see Fig. 5)

$$w_1 = w_3 = 0.2298/(I_1+I_2+I_3) \quad ,$$

$$w_2 = 0.4950/(I_1+I_2+I_3) \ .$$

The results are shown in Table VII for equal standard deviations for all of the eight parameters and for two values, namely, 5% and 6%. Table VIII shows the execution time if Monte Carlo analysis is applied to the linear constraints for the case of normally distributed parameters.

IV. CONCLUSIONS

It has been shown how yield may be estimated for arbitrary statistical distributions in an efficient way without recourse to the Monte Carlo method. Examples involving a number of distributions have been presented and the results contrasted with those given by the Monte Carlo method.

For the case of a uniform distribution between tolerance extremes yield sensitivity formulas have been derived with respect to nominal parameter values and tolerances assuming independent variables. These can be useful in optimization [11,12]. Since the uniform distribution is basic to the subsequent consideration of arbitrary distributions, it is felt that the ideas on sensitivity could be carried through to effect design centering with respect to given distributions.

As usual in iterative schemes the choice of starting point may be important. In the present work it is recommended that a rough solution to a worst-case centering and tolerance assignment problem be used to provide and identify suitable active constraints. This allows only essential constraints to be considered and provides some justification for a worst-case solution even if less than 100% yield is subsequently contemplated [11,12].

APPENDIX

Theorem

If $g_\ell(\underset{\sim}{\phi}^r) < 0$, $g_\ell(\underset{\sim}{\phi}^{\bar{r}}) > 0$ and $|g_\ell(\underset{\sim}{\phi}^r)| \leq |g_\ell(\underset{\sim}{\phi}^{\bar{r}})|$, then

Order $(S_\ell) \leq$ Order $(\bar{S}_\ell)$.

Proof

In the case under consideration the order of a set is simply the number of its elements. Assume that $s \in S_\ell$, then

$$g_\ell(\underset{\sim}{\phi}^s) = g_\ell(\underset{\sim}{\phi}^r) + (\underset{\sim}{\phi}^s - \underset{\sim}{\phi}^r)^T \underset{\sim}{\nabla} g_\ell(\underset{\sim}{\phi}^r) < 0 ,$$

$$= g_\ell(\underset{\sim}{\phi}^r) + \sum_{i=1}^{k} \varepsilon_i \, (\mu_i^s - \mu_i^r) \, q_i^\ell < 0 ,$$

or

$$- g_\ell(\underset{\sim}{\phi}^r) + \sum_{i=1}^{k} \varepsilon_i \, (-\mu_i^s + \mu_i^r) \, q_i^\ell > 0 .$$

But, since

$$- g_\ell(\underset{\sim}{\phi}^r) \leq g_\ell(\underset{\sim}{\phi}^{\bar{r}}) \quad \text{and} \quad \mu_i^{\bar{r}} = -\mu_i^r ,$$

then

$$g_\ell(\tilde{\phi}^{\bar{r}}) + \sum_{i=1}^{k} \varepsilon_i \; (-\mu_i^s - \mu_i^{\bar{r}}) \; q_i^\ell > 0,$$

i.e.,

$$g_\ell(\tilde{\phi}^{\bar{s}}) > 0 \quad ,$$

where

$$\tilde{\phi}^{\bar{s}} = \tilde{\phi}^0 - \tilde{E} \, \tilde{\mu}^s \quad .$$

Hence,

$$\bar{s} \in \bar{S}_\ell \quad .$$

This means that for each vertex $s \in S_\ell$ there exists a vertex $\bar{s} \in \bar{S}_\ell$, thus

$$\text{Order } (S_\ell) \leq \text{Order } (\bar{S}_\ell) \quad .$$

REFERENCES

[1] N.J. Elias, "New statistical methods for assigning device tolerances", Proc. 1975 IEEE Int. Symp. on Circuits and Systems (Newton, Mass., April 1975), pp. 329-332.

[2] S.W. Director and G.D. Hachtel, "The simplicial approximation approach to design centering and tolerance assignment", Proc. 1976 IEEE Symp. on Circuits and Systems (Munich, April 1976), pp. 706-709.

[3] H.S.M. Coxeter, Regular Polytopes (2nd. Ed.). New York: MacMillan, 1963, Chap. 7.

[4] T.R. Scott and T.P. Walker, "Regionalization: a method for generating joint density estimates", IEEE Trans. Circuits and Systems, vol. CAS-23, April 1976, pp. 229-234.

[5] K.H. Leung and R. Spence, "Idealized statistical models for low-cost linear circuit yield analysis", IEEE Trans. Circuits and Systems, vol. CAS-24, Feb. 1977, pp. 62-66.

[6] B.J. Karafin, "The general component tolerance assignment problem in electrical networks", Ph.D. Thesis, Univ. of Pennsylvania, Philadelphia, Penn., 1974.

[7] J.W. Bandler, "Optimization of design tolerances using nonlinear programming", J. Optimization Theory and Appl., vol. 14, July 1974, pp. 99-114.

[8] J.W. Bandler and P.C. Liu, "Automated network design with optimal tolerances", IEEE Trans. Circuits and Systems, vol. CAS-21, March 1974, pp. 219-222.

[9] M.F. Neuts, Probability. Boston, Mass.: Allyn and Bacon, 1973, pp. 189-192.

[10] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions. New York: Dover Publications, 1965, pp. 976-977.

[11] J.W. Bandler and H.L. Abdel-Malek, "Optimal centering, tolerancing and yield determination using multidimensional approximations", Proc. 1977 IEEE Symp. on Circuits and Systems (Phoenix, April 1977), pp. 219-222.

[12] J.W. Bandler and H.L. Abdel-Malek, "Optimal centering, tolerancing and yield determination using updated approximations and cuts", Faculty of Engineering, McMaster University, Hamilton, Canada, Report SOC-173, June 1977.

TABLE I

NONFEASIBLE VERTICES FOR EXAMPLE 1

Vertex	ϕ_1	ϕ_2	ϕ_3	ϕ_4	μ_1	μ_2	μ_3	μ_4	Nonfeasible vertices
1	4	5	5	20	-1	-1	-1	-1	X
2	14	5	5	20	1	-1	-1	-1	
3	4	9	5	20	-1	1	-1	-1	X
4	14	9	5	20	1	1	-1	-1	
5	4	5	13	20	-1	-1	1	-1	X
6	14	5	13	20	1	-1	1	-1	
7	4	9	13	20	-1	1	1	-1	
8	14	9	13	20	1	1	1	-1	
9	4	5	5	32	-1	-1	-1	1	X
10	14	5	5	32	1	-1	-1	1	
11	4	9	5	32	-1	1	-1	1	X
12	14	9	5	32	1	1	-1	1	
13	4	5	13	32	-1	-1	1	1	X
14	14	5	13	32	1	-1	1	1	
15	4	9	13	32	-1	1	1	1	
16	14	9	13	32	1	1	1	1	

TABLE II

HYPERVOLUME GRADIENT CHECK FOR EXAMPLE 1

Parameters	Analytical gradients	Numerical gradients
ϕ_1^0	-337.50	-337.50
ϕ_2^0	-540.00	-540.00
ϕ_3^0	-135.00	-135.00
ϕ_4^0	- 33.75	- 33.75
ε_1	337.50	337.50
ε_2	573.60	573.60
ε_3	268.20	268.20
ε_4	173.18	173.18

TABLE III

EXAMPLE OF CALCULATION OF WEIGHTED HYPERVOLUME BY THE GENERAL FORMULA

Orthocell dimensions		i_1	0	1	2	3	4
		ε_{1,i_1}	0	3.0	3.0	2.0	-
i_2	ε_{2,i_2}						
0	0	w,W	0	0	0	0	0
1	2.0	w	0	18/100	12/100	3/10	0
		W	0	3/100	1/50	3/40	0
		ΔW	-	3/100	-1/100	11/200	-3/40
		δ	-	1	3/4	1/2	1/3
2	3.0	w	0	12/100	8/100	2/10	0
		W	0	1/75	2/225	1/30	0
		ΔW	-	-1/60	1/180	-11/360	1/24
		δ	-	1/3	1/12	0	0
3	-	w,W	0	0	0	0	0
		ΔW	-	-1/75	1/225	-11/450	1/30
		δ	-	0	0	0	0

Reference vertex $\underset{\sim}{\phi}^r$ given by $\mu_1^r = -1$, $\mu_2^r = 1$

Intersections of the linear constraint are $\alpha_1 = 12$, $\alpha_2 = 3$

Weighted volume V = 1813/3600

TABLE IV

SPECIFICATIONS FOR THE BANDPASS FILTER

Frequency range (Hz)	Relative insertion loss (dB)	Type
0 - 240	35	lower (stopband)
360 - 490	3	upper (passband)
700 - 1000	35	lower (stopband)

Reference frequency 420 Hz (fixed, therefore, ripples higher than 3 dB are to be expected in the passband)

Nominal values L_1^0=3.0142, C_2^0=4.975 x 10^{-8}, L_3^0=2.902, C_4^0=5.0729 x 10^{-8}, L_5^0=0.82836, C_6^0=5.5531 x 10^{-7}, L_7^0=0.30319 and C_8^0=1.6377 x 10^{-7}

TABLE V

COMPARISON WITH THE MONTE CARLO ANALYSIS FOR UNIFORM DISTRIBUTION BETWEEN TOLERANCE EXTREMES

Tolerances (%)								Sample points (Hz)	Yield (%)		CDC Time (sec)	
ε_1/L_1^0	ε_2/C_2^0	ε_3/L_3^0	ε_4/C_4^0	ε_5/L_5^0	ε_6/C_6^0	ε_7/L_7^0	ε_8/C_8^0		Approx.	M.C.	Approx.*	M.C.**
6.99	6.52	6.97	6.55	4.36	5.69	6.80	5.25	188, 700, 876	100.00	99.75	0.67	24.0
7.00	7.00	7.00	7.00	5.00	6.00	7.00	6.00	188, 700, 876	100.00	99.65	0.66	24.2
8.00	8.00	8.00	8.00	6.00	7.00	8.00	7.00	188, 700, 876	99.99	99.60	0.67	24.4
								190, 240, 360, 480, 490, 700, 860	99.94	99.35	1.56	52.4
10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	190, 240, 360, 480, 490, 700, 860	92.62	93.00	1.67	51.4

CDC time for selecting frequency points = 7.65 sec

* This time includes the linearization time

** 2000 points were used in Monte Carlo (M.C.) analyses with the nonlinear constraints

TABLE VI

COMPARISON WITH THE MONTE CARLO ANALYSIS FOR ACCURATE COMPONENTS REMOVED

$\frac{\phi_i - \phi_i^0}{\phi_i^0}$ (%)	Yield (%)		CDC Time (sec)	
	Approx.	M.C.	Approx.	M.C.
[-10,-5], [5,10]	68.9	71.0	4.9	45.6

Frequency points used are 190, 240, 360, 480, 490, 700 and 860 Hz

TABLE VII

COMPARISON WITH MONTE CARLO ANALYSIS FOR NORMALLY DISTRIBUTED COMPONENTS

$\frac{\sigma_i}{\phi_i^0}$ (%)	Yield (%)		CDC Time (sec)	
	Approx.	M.C.	Approx.	M.C.
5.0	96.5	95.1	4.9	69.2
6.0	88.4	87.0	7.4	68.0

TABLE VIII

EFFECT OF NUMBER OF MONTE CARLO ANALYSES ON THE YIELD BASED UPON THE LINEARIZED CONSTRAINTS

$\frac{\sigma_i}{\phi_i^0}$ (%)	N.O.M.P.*	Yield (%)	CDC Time (sec)
5.0	2000	94.4	24.6
	500	94.2	7.0
	200	91.5	2.8
6.0	2000	86.6	24.3
	500	85.2	6.9
	200	84.0	2.8

* N.O.M.P. denotes the number of Monte Carlo points used

(a)

(b)

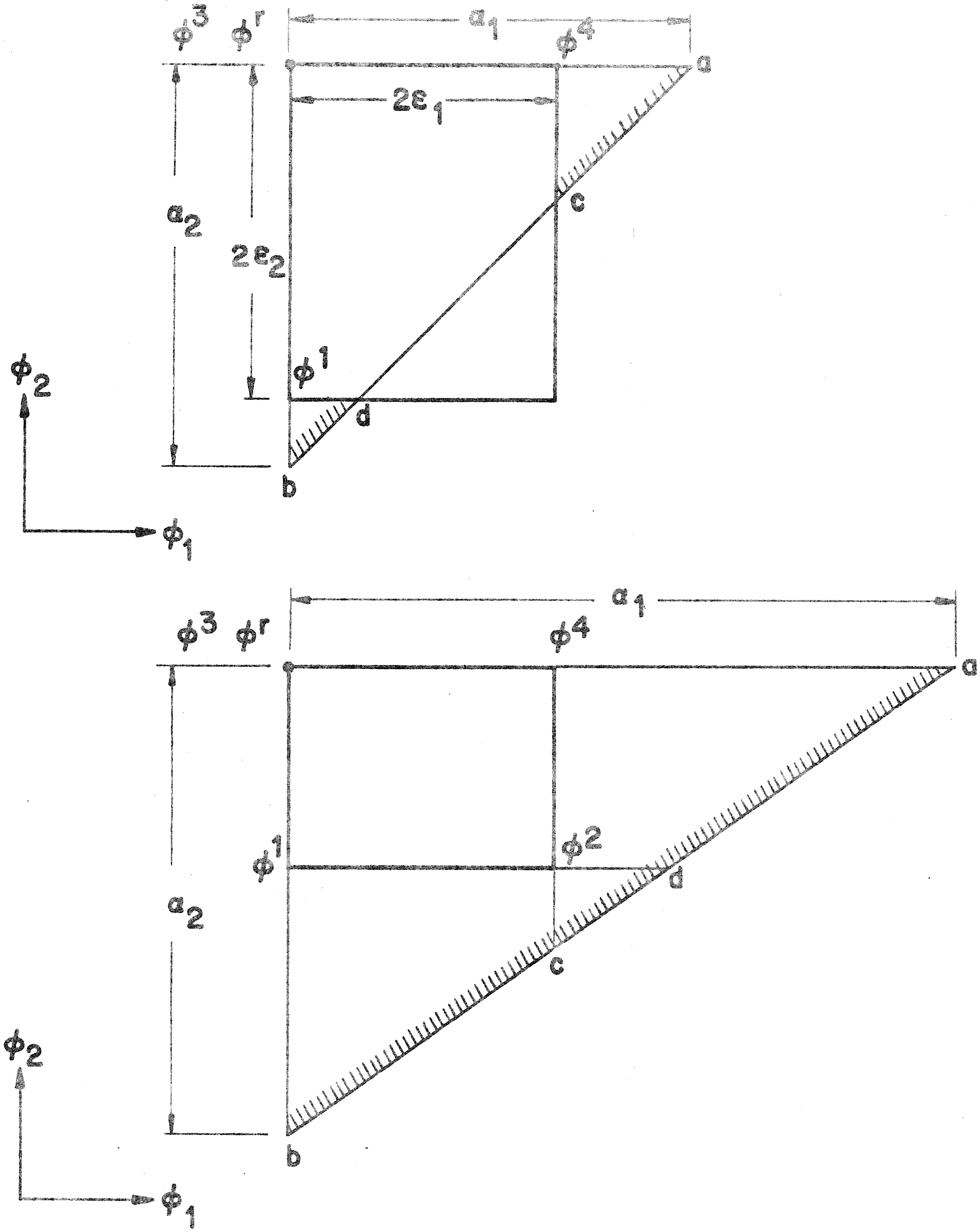

Fig. 1 Two-dimensional examples illustrating the calculation of the nonfeasible hypervolumes, (a) tolerance region partially feasible, (b) tolerance region nonfeasible.

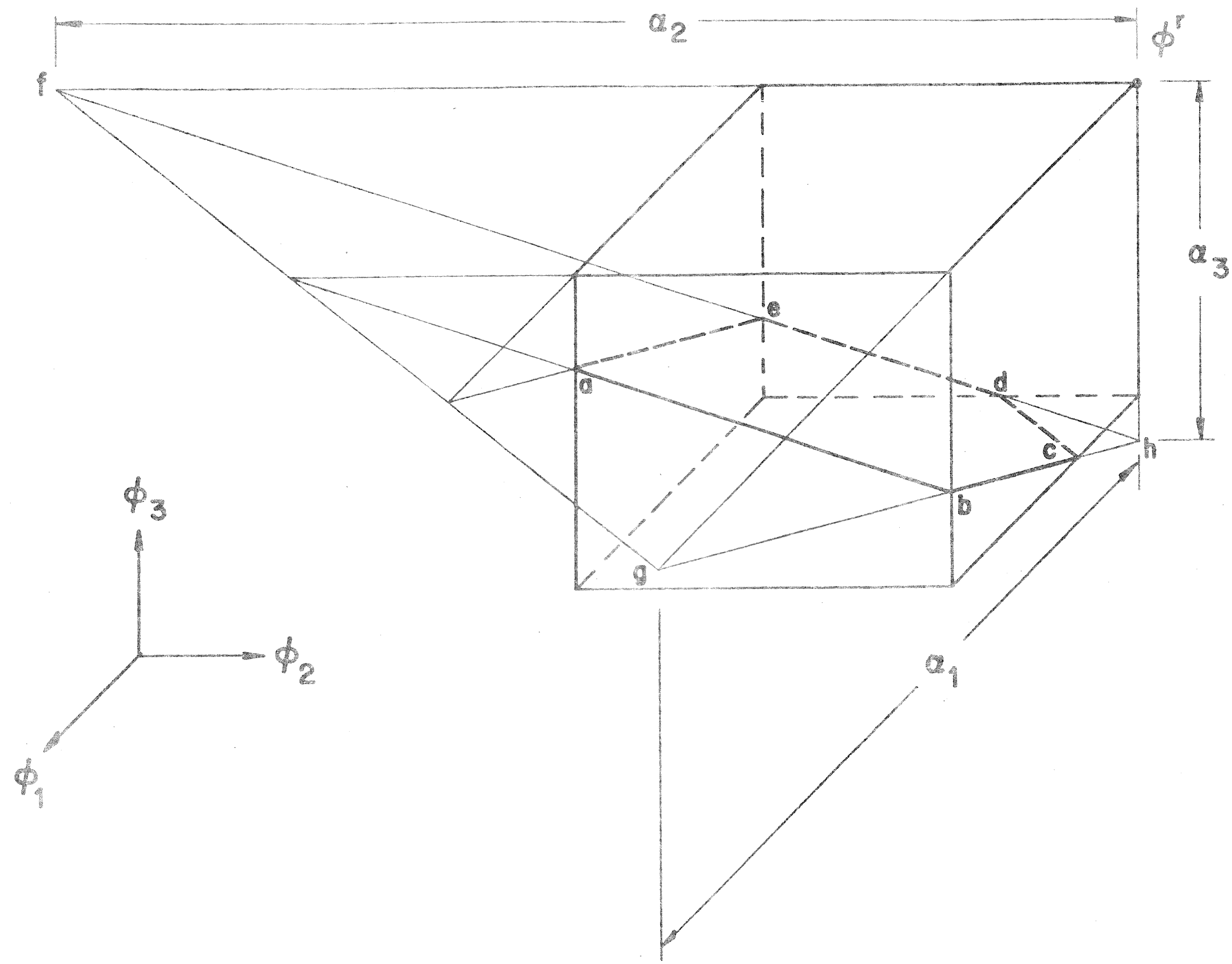

Fig. 2 Three-dimensional example illustrating the calculation of nonfeasible hypervolumes in the case of a partially feasible tolerance region.

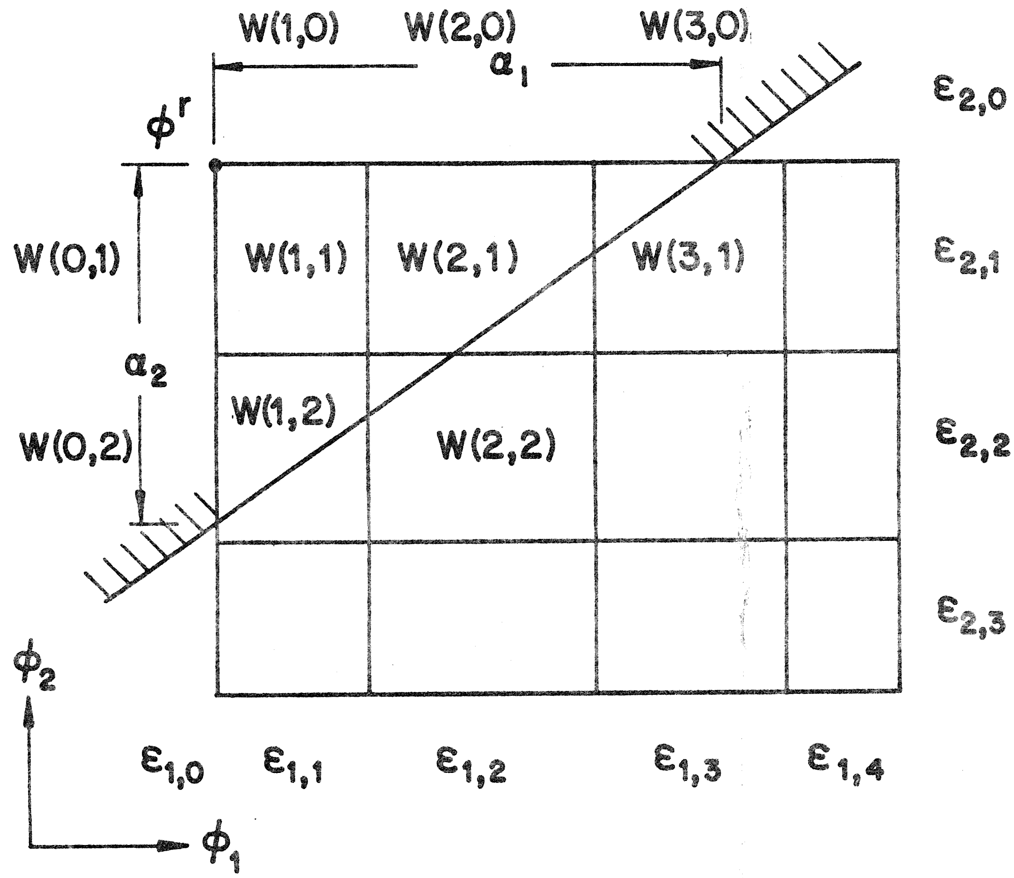

Fig. 3 Two-dimensional illustration of the partitioning of the tolerance region into cells indicating the dimensions and weighting of those cells relevant to the calculation of the weighted nonfeasible hypervolume.

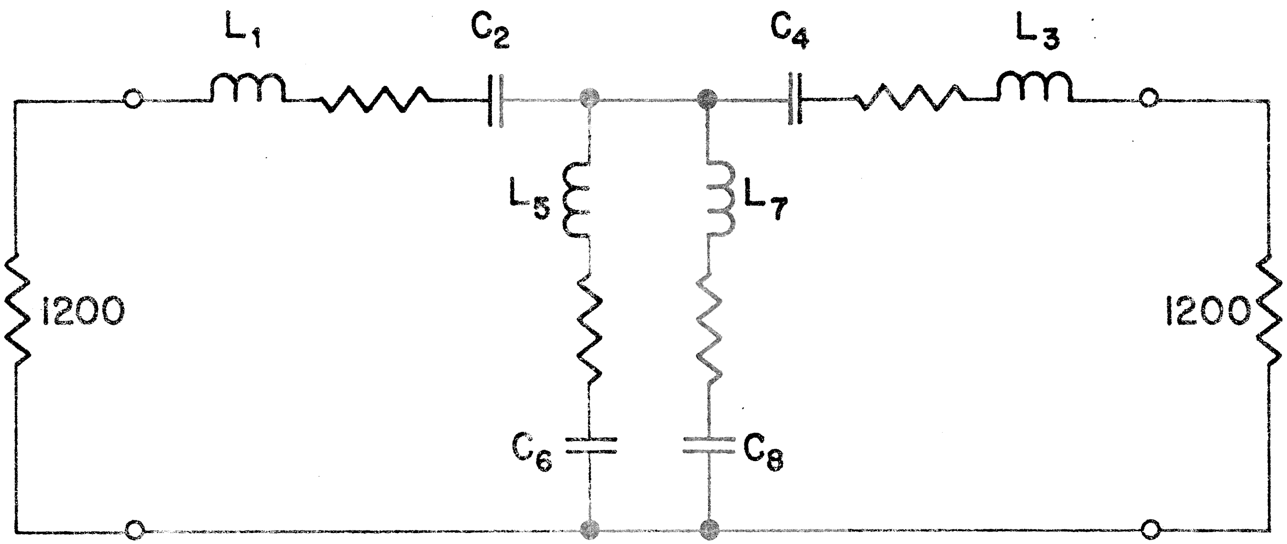

Fig. 4 Karafin's bandpass filter. The values of the resistances are related to nominal values of the corresponding inductances by the same ratio used by Karafin [6, p. 112].

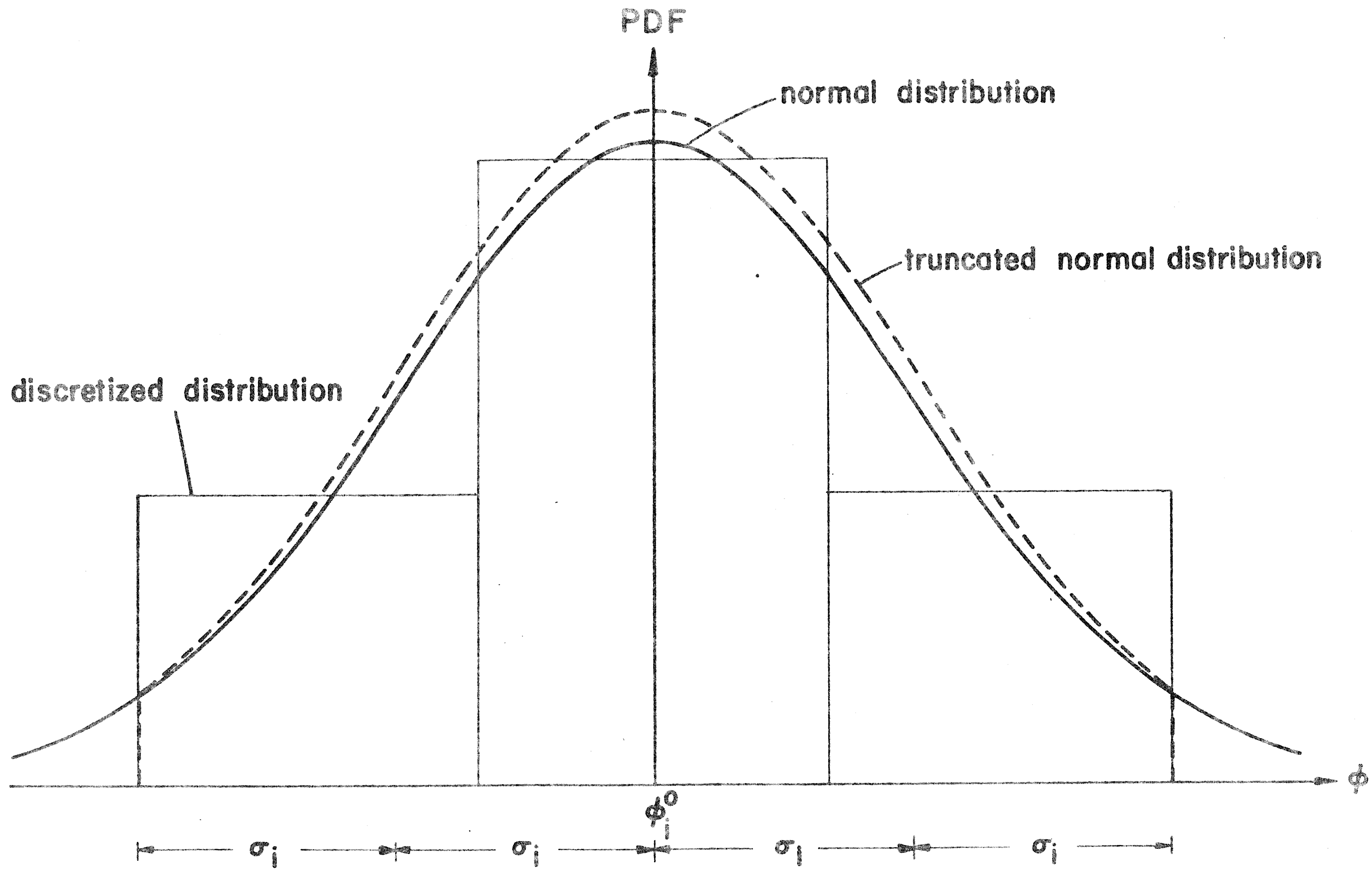

Fig. 5 Normal distribution, truncated normal distribution and discretized normal distribution.

MODELING AND APPROXIMATION FOR STATISTICAL EVALUATION AND OPTIMIZATION OF MICROWAVE DESIGNS

J.W. BANDLER AND H.L. ABDEL-MALEK

This paper shows how, by suitably updated approximations, one may surmount the obstacle of expensive experimental tuning or repeated computer simulations of trial designs when optimal designs or statistical analyses are required. The authors address themselves to the efficient use of available software by designers wishing to exploit the current state of the art in techniques of statistical design, tolerance assignment and optimal tuning. The ideas and results are new to microwave design.

INTRODUCTION

We present recent ideas and new results developed by the authors involving concepts of modeling and approximation as applied to problems in microwave design and circuit design in general. A principal aim is to utilize as far as possible available software for simulating microwave structures. There need be no restriction on the nature of the simulations. The simulations may, indeed, be performed experimentally, by finite element analysis of fields or by programs employing lumped and distributed circuit models, etc. The goal of this work is to facilitate rapid and accurate determination of design solutions or to perform efficiently the mass of calculations involved in statistical and worst-case evaluations. As a result, estimation of production yield, tolerance assignment, design centering and the evaluation of effects of parasitics and other uncertainties can be effectively handled at low computational cost.

The present work is the logical culmination of developments presented by Bandler at three previous European Microwave Conferences [1-3]. Complicated design problems such as nonideal, inhomogeneous rectangular waveguide transformers [1] involving sensitivity evaluation [2] and, furthermore, design centering and tolerance assignment [3] are treated here. Justification of the importance of design centering and/or tolerance assignment has already received ample exposure [3-6], in particular in the microwave area [7,8]. Furthermore, the use of multidimensional approximations of the performance or responses with respect to designable parameters has already begun to be described in the literature [9-12].

It is shown how low-order multidimensional approximations are readily obtained by suitably choosing sets of parameter values and calculating or measuring the corresponding responses. These approximations are then used in statistical simulations or optimal design. They are updated as necessary if candidates for optimal parameter solutions drift too far from the region in which the responses were evaluated. We have developed algorithms which minimize computational or experimental effort in the sense that a minimal number of straightforward parameter settings are initially chosen by the designer and subsequently updated by the computer.

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DESIGN APPROACHES

In this section a brief description of the different problems in computer-aided circuit design and suitable approaches used for solving them are given. The important aspects of modern computer-aided design are shown in Fig. 1. They are arranged approximately in the order of increasing complexity and sophistication as one proceeds down the tree. Circuit response analysis capability given all design parameters and all other relevant factors is assumed available. The ultimate solutions obtained should be verified by checking, e.g., convexity assumptions and/or updating of the multidimensional approximations and/or by the Monte Carlo method of analysis.

No work has yet been described which includes all the topics shown at the bottom of Fig. 1 in an integrated fashion. Some specialized combinations of these topics have been considered, e.g., see [6,8,11].

Nominal Approximation

This is the most well-known and widely used design technique. By least squares or other measure a best nominal (single design) approximation is obtained. If the specifications cannot be met by this single solution one cannot proceed to better or more realistic designs. An improved approximation, e.g., a minimax or a Chebyshev solution can be found if tolerances are not involved. If explicit assignment of tolerances is not being sought one could carry out a sensitivity minimization for the nominal design, whose sensitivities are included in the objective function.

Our modeling and approximation approach is to choose an interpolation region centered at the initial guess of the nominal design. The simulation program is used to provide the values of the response functions at certain base points, defined by the values of the design parameters, inside the interpolation region. Based upon the responses, low-order multidimensional approximations (quadratic polynomials) are constructed. In order to make efficient use of the simulation program, the number of simulations required is the minimum to fully describe the responses by multidimensional quadratic polynomials. An optimization technique, e.g., FLOPT4 [13], is used in conjunction with these approximations to provide the nominal design. The multidimensional approximation is to be updated in different regions in the space or in smaller interpolation regions as indicated by the optimization or to obtain higher accuracy, respectively.

Tolerance Problem

If parameter tolerances or uncertainties are to be explicitly considered, then the following questions typically arise. Is a worst-case solution desired? Are the tolerances to be fixed? Are some parameters to be chosen from discrete values? Can a yield less than 100% be allowed? Is the multidimensional statistical distribution describing possible parameter fluctuations fixed? Is the design fixed or tunable?

Worst-Case Design

All outcomes of the fabrication procedure must satisfy the specifications, after tuning if necessary. Candidates for worst-case must be used during the optimization process. Assuming a one-dimensionally convex feasible region [14], the vertices of the tolerance orthotope supply the candidates for the worst case. Hence, our attention for formulating reliable approximations (models) is directed to the vertices. Subsequently, more than one interpolation region might be used in order to achieve accurate solutions. The procedure attempts to minimize the number of simulations by collecting many vertices within each interpolation region.

Design With Yield Less Than 100%

Accurate yield estimation is based upon the determination of the boundary of the feasible region of designs. Thus, approximations must be directed to critical regions at the boundary where violation of the specifications might occur. Explicit yield and yield sensitivity formulas [12] are used with the multidimensional approximations and discretized multidimensional statistical distribution. The detection of the critical regions, where approximation of the boundary is required, is a difficult problem. For high yield, however, a worst-case design should provide a good indication of these critical regions and is, therefore, worthwhile investigating as a preliminary exercise to statistical design (see Fig. 1).

Examples

Several examples have been solved by our approach. A two-section transmission-line transformer has been used to illustrate the procedure for the case of cost minimization and design with a yield constraint [9,11]. An LC lowpass filter was considered [11]. Some tolerances were approximately doubled by allowing the yield to drop to 96% from 100% for the worst-case design. A bandpass filter [12] was used for applying our yield formulas. The yield was estimated according to different statistical distributions. Excellent agreement with the Monte Carlo method validated the results. We considered the optimal assignment of tolerances on the physical dimensions of a two-section nonideal inhomogeneous waveguide transformer [11]. Applying Monte Carlo analysis to the final updated approximations instead of the actual functions, which are more expensive to compute [1], reduced the execution time by a factor of 11. SPICE2 [15] was used for simulation of a time-domain current switch emitter follower and was used to provide the responses at the base points. Design centering and tolerance assignment for a worst-case design were successfully performed.

CONCLUSIONS

In the near future, at least, it is felt that there will be no significant advance in the art of microwave design on a large scale unless such modeling and approximation techniques are adopted. They are capable of bridging the widening gap between available simulation techniques and the advancing art of statistical analysis, design centering, tolerance assignment and optimal tuning where a whole production line rather than an individual realization is to be considered, in particular, since they do not require sensitivity calculations.

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REFERENCES

[1] J.W. Bandler, "Computer optimization of inhomogeneous waveguide transformers," European Microwave Conf. (London, Sept. 1969). Also IEEE Trans. Microwave Theory Tech., vol. MTT-17, Aug. 1969, pp. 563-571.

[2] J.W. Bandler, "Computer optimization of microwave circuits," (invited), Proc. European Microwave Conf. (Stockholm, Aug. 1971), pp.

B8/S:1 - S:8.

[3] J.W. Bandler, "The tolerance problem in optimal design," (invited), Proc. European Microwave Conf. (Brussels, Sept. 1973), Paper A.13.1 (I).

[4] B.J. Karafin, "The optimum assignment of component tolerances for electrical networks", Bell Syst. Tech. J., vol. 50, April 1971, pp. 1225-1242.

[5] J.F. Pinel and K.A. Roberts, "Tolerance assignment in linear networks using nonlinear programming", IEEE Trans. Circuit Theory, vol. CT-19, Sept. 1972, pp. 475-479.

[6] J.W. Bandler, P.C. Liu and H. Tromp, "A nonlinear programming approach to optimal design centering, tolerancing and tuning", IEEE Trans. Circuits and Systems, vol. CAS-23, Mar. 1976, pp. 155-165.

[7] J.W. Bandler, P.C. Liu and J.H.K. Chen, "Worst case network tolerance optimization", IEEE Trans. Microwave Theory Tech., vol. MTT-23, Aug. 1975, pp. 630-641.

[8] J.W. Bandler, P.C. Liu and H. Tromp, "Integrated approach to microwave design", IEEE Trans. Microwave Theory Tech., vol. MTT-24, Sept. 1976, pp. 584-591.

[9] J.W. Bandler, H.L. Abdel-Malek, P.B. Johns and M.R.M. Rizk, "Optimal design via modeling and approximation", Proc. 1976 IEEE Symp. on Circuits and Systems (Munich, April 1976), pp. 767-770.

[10] S.W. Director and G.D. Hachtel, "The simplicial approximation approach to design centering and tolerance assignment", Proc. 1976 IEEE Symp. on Circuits and Systems (Munich, April 1976), pp. 706-709. Also IEEE Trans. Circuits and Systems, to appear.

[11] J.W. Bandler and H.L. Abdel-Malek, "Optimal centering, tolerancing and yield determination using multidimensional approximations", Proc. 1977 IEEE Symp. on Circuits and Systems (Phoenix, April 1977), pp. 219-222. Also Faculty of Engineering, McMaster University, Hamilton, Canada, Report SOC-132, Sept. 1976.

[12] H.L. Abdel-Malek and J.W. Bandler, "Yield estimation for efficient design centering assuming arbitrary statistical distributions", Conf. Computer-aided Design of Electronic and Microwave Circuits and Systems (Hull, England, July 1977). Also Int. J. Circuit Theory and Applications, to appear.

[13] J.W. Bandler and D. Sinha, "FLOPT4 - a program for least pth optimization with extrapolation to minimax solutions", Faculty of Engineering, McMaster University, Hamilton, Canada, Report SOC-151, Jan. 1977.

[14] J.W. Bandler, "Optimization of design tolerances using nonlinear programming", J. Optimization Theory and Applications, vol. 14, July 1974, pp. 99-114.

[15] L.W. Nagel, "SPICE2: a computer program to simulate semiconductor circuits", Electronics Research Laboratory, University of California, Berkeley, Memorandum No. ERL-M520, May 1975.

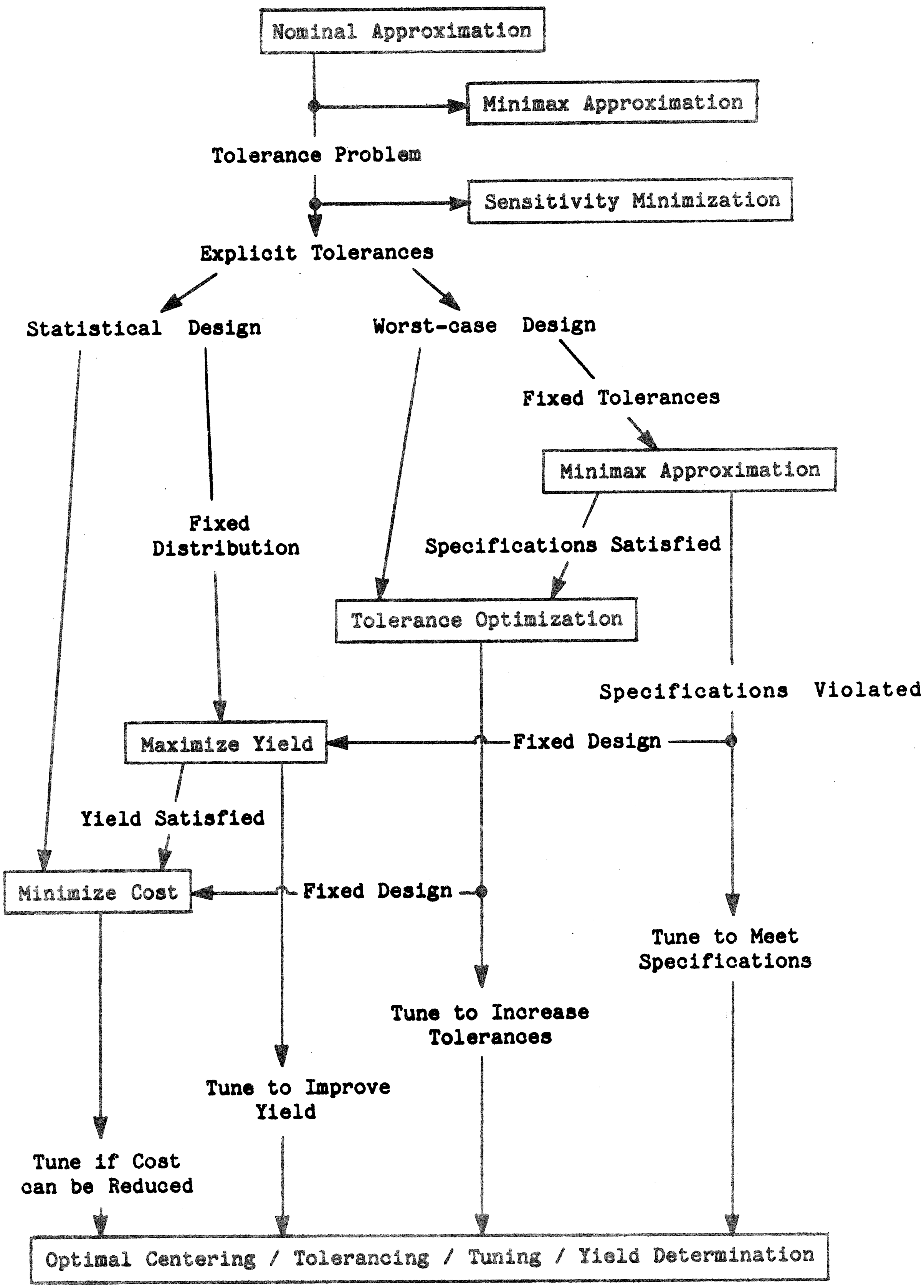

Figure 1 Typical sequence of problems in modern computer-aided design shown in approximate order of increasing complexity.

SOC-177

OPTIMAL CENTERING, TOLERANCING AND TUNING IN ENGINEERING DESIGN

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Key Words: Centering, tolerancing, tuning, engineering design, optimization

Abstract: This collection sixteen papers and reports represents the development of work in computer-aided design oriented around optimal centering, tolerancing and tuning carried out by the editor and his colleagues over several years. This report essentially contains the material of the following reports: SOC-1, SOC-18, SOC-24, SOC-37, SOC-49, SOC-62, SOC-65, SOC-110, SOC-111, SOC-118, SOC-119, SOC-120, SOC-124, SOC-131, SOC-132, SOC-142, SOC-167 and SOC-173.

Description:

Related Work: SOC-87, SOC-105, SOC-113.

Price: $ 30.00.