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OPTIMIZATION OF ELECTRICAL CIRCUITS

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Abstract This paper reviews applications of optimization methods in the area of electrical circuit design. It is addressed to engineers in general as well as mathematical programmers. As a consequence, a brief introduction to electrical circuits is presented, including analog, digital and power concepts. Network analysis techniques along with response evaluation and the determination of partial derivatives (useful in gradient methods of optimization) provide the nonelectrical reader with some necessary background. Objective functions aimed at improving network performance are presented, including least pth and minimax criteria. The approaches by many contributors to optimal circuit design are outlined, concentrating on general methods within the domain of nonlinear programming, nonlinear approximation and nonlinear discrete optimization techniques. A complete section is devoted to recent work in design centering, optimal assignment of manufacturing tolerances and postproduction tuning. The inclusion of model and environmental uncertainties is discussed. Practical examples illustrate the current

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state of the art. Difficulties facing the design optimizer as well as directions of possible future research are elaborated on. A long but by no means exhaustive list of references is appended.

1. Introduction

This paper reviews applications of optimization methods in the area of electrical circuit design. The discussion is focussed on general methods within the domain of nonlinear programming, nonlinear approximation and nonlinear discrete optimization techniques. In some ten years of serious attention given by circuit designers to such general methods, implementations have ranged from the relatively unsophisticated application of least squares approximation to specified frequency or time responses of a single circuit to optimal assignment of component tolerances, optimal design centering and the consideration of post-production tuning with respect to a large number of circuits simultaneously.

The paper highlights work in optimization formulations and algorithms developed by electrical engineers which may not be widely known to mathematical programmers in general. It is shown how the circuit design problem may be reformulated as a nonlinear programming problem (minimizing an objective function subject to inequality constraints) where the objective and constraints embody the design criteria. The objective (error) function itself is usually of the least squares, least pth or minimax form.

Section 2 discusses the basic formulation of suitable objective functions to force the response (performance) of the circuit to meet some desired specifications. The circuit to be optimized is of a known topology (configuration) and contains known component types. The variables are usually some or all of the independently adjustable parameters of this circuit. For some circuits performance specifications are easily defined as, for example, in filter design, while for others such as switching circuits defining the specifications explicitly and in advance may not be easy. In Section 2 we are interested in finding an optimum solution represented by one point in the feasible region, if such a point exists.

Section 3 sketches briefly, bearing in mind that the reader of this paper might not be an electrical engineer, the methods of analysis, and hence obtaining the response, of different circuits in different domains. It also discusses the sensitivity evaluation of these responses with respect to design variables.

Lumped and distributed, active and passive, linear and nonlinear, analog and digital, frequency domain and time domain, transient and steady state concepts are presented to encompass the analysis of filters, amplifiers, switching circuits, microwave circuits, power systems, etc.

Section 4 reviews the optimization approaches which have been used in the design of electronic analog, digital and power networks. The approaches discussed are those which the authors feel have contributed directly to the development of optimization formulations and techniques applied to electrical circuits. They include penalty and barrier function methods, reduced gradient methods, least pth and minimax approaches, methods based on linear programming and extensions of least squares, as well as discrete optimization.

Section 5 deals with optimal design when certain additional practical engineering problems are considered. The centering problem, which involves finding the center of a constraint region in the parameter space in order to maximize the parameter tolerances or production yield, formulated in a nonlinear programming form is presented. Further practical considerations such as tuning, tolerance assignment under model and environmental uncertainties and discrete optimization in tolerance assignment are considered. The discussion includes worst-case design and also cases when the constraints are relaxed to obtain a yield less than 100%.

The difficulties facing the designer wishing to avail himself of efficient nonlinear programming aids are elaborated on. Section 6, in particular, reviews some practical examples which have been solved using techniques and algorithms discussed in the paper. Future directions of research, and further development of available algorithms and problem formulations which can, in our opinion, improve the state of the art are suggested in the last section. Finally, a list of references is appended to lead the reader to further details. This list is by no means exhaustive but should provide a balanced and accurate reflection of the current state of the art.

2. The classical design problem

The classical circuit design problem can be stated as follows: given a circuit with a fixed topology, find a single set of designable parameter values which let the circuit response or performance optimally meet some given specifications.

The circuit response $F(\underset{\sim}{\phi}, \underset{\sim}{\psi})$ is a function of the network parameters $\underset{\sim}{\phi}$ (resistors, capacitors, node voltages of power circuits, etc.) and of other independent variables $\underset{\sim}{\psi}$ (frequency, time, temperature, tunable network elements, etc.). The function $F(\underset{\sim}{\phi}, \underset{\sim}{\psi})$ is usually assumed to be continuous in the ranges of $\underset{\sim}{\phi}$ and $\underset{\sim}{\psi}$ of interest. Performance specifications are usually functions of $\underset{\sim}{\psi}$ only, whereas design constraints are generally functions of $\underset{\sim}{\phi}$. This distinction, however, is sometimes blurred.

The design problem considered here is basically an approximation problem, where the method of approximation depends heavily on what the designed circuit has to achieve, the nature of the specifications, the existence of constraints in general and parameter bounds in particular. It is necessary in practice (on a digital computer) to consider a discrete set of samples of $\underset{\sim}{\psi}$, such that satisfying the specifications at these sample points implies satisfying them almost everywhere.

2.1 Single specification

Consider the problem where the response function has to meet a single specification function $S(\psi)$, in the interval $[\psi_\ell, \psi_u]$ along a ψ axis. Let us define the error function as

$$e(\underset{\sim}{\phi}, \psi) \stackrel{\Delta}{=} w(\psi)\ (F(\underset{\sim}{\phi},\psi)-S(\psi)), \tag{1}$$

where $w(\psi)$ is a positive weighting function. Figure 1(a) depicts the functions involved in evaluating the error function shown in Fig. 1(b). Since we are considering discrete approximation we define

$$e_i(\underset{\sim}{\phi}) \stackrel{\Delta}{=} e(\underset{\sim}{\phi}, \psi_i), \qquad i \in I , \tag{2}$$

where

ψ_i is the value of ψ at the ith sample point,

$e_i(\underset{\sim}{\phi})$ is the error function evaluated at ψ_i,

I is a given index set of sample points.

The number of sample points and their distribution along the ψ axis require the experience of the designer. They might be equally spaced or nonuniform, being dense on some subintervals and well-scattered on others.

The single specification problem can be solved by least pth or minimax approximation. In the least pth approximation, a simple objective for real functions e_i suggested by Temes and Zai [162] takes the form

$$U = \sum_{i \in I} [e_i(\underset{\sim}{\phi})]^p \quad , \tag{3}$$

where $p \geq 2$ is any even number. For large values of p accuracy and convergence problems arise due to very large and very small numbers involved in the calculations.

Bandler and Charalambous [19] alleviated this ill-conditioning by considering the objective

$$U_p = M(\underset{\sim}{\phi}) \left(\sum_{i \in I} \left| \frac{e_i(\underset{\sim}{\phi})}{M(\underset{\sim}{\phi})} \right|^p \right)^{1/p} , \text{ for } 1 < p < \infty \quad , \tag{4}$$

where

$$M(\underset{\sim}{\phi}) \stackrel{\Delta}{=} \max_{i \in I} |e_i(\underset{\sim}{\phi})| \quad . \qquad (5)$$

The error functions, in general, can be real or complex functions. Hebden [108] employed this type of scaling in some related work.

The minimax approximation might be appropriate for solving the single specification problem. The objective function in this case will be of the form of (5) which is commonly known as the Chebyshev type of objective. This objective might lead to difficulties because discontinuous derivatives are generated when the maximum error function is suddenly switched from one sample point to another [36].

The design of an amplifier can serve as an example of the single specification problem. Consider Fig. 2(a), in which $V_1(j\omega)$ is the input voltage (voltage of the source) to the amplifier at frequency ω and $V_2(j\omega)$ is the output voltage at the same frequency. The gain of the amplifier, which is a linear circuit, is usually given by

$$F(\underset{\sim}{\phi}, \psi) = G(\underset{\sim}{\phi}, \omega) \stackrel{\Delta}{=} 20 \log_{10} \left| \frac{V_2(j\omega)}{V_1(j\omega)} \right| . \qquad (6)$$

The problem is to obtain $\underset{\sim}{\phi}$ which results in a gain as close as possible, in some sense, to a desired gain, for example, such as the one shown in Fig. 2(b).

2.2 Upper and lower specifications

Another situation which is frequently encountered in practice is the problem defined by upper and lower specifications. In filter design, for example, we are generally interested in two band types (consisting of intervals of frequency ω), namely the stopband and the passband. In the stopband the signal is to be prevented from passing through the filter by making the losses as high as possible. This can be expressed by a lower specification (or bound) of large value. In the passband the situation is reversed and it is expressed by an upper specification (or bound) of a small value. Figure 3(a) shows the upper and lower specifications of a bandpass filter and a response function violating these specifications on the interval $[\psi_\ell, \psi_u]$. Figure 3(b) depicts a response function satisfying arbitrary upper and lower specifications. In such a case the error functions are defined as [8]

$$e_u(\underset{\sim}{\phi}, \psi) \triangleq w_u(\psi)\,(F(\underset{\sim}{\phi}, \psi) - S_u(\psi)), \tag{7}$$

$$e_\ell(\underset{\sim}{\phi}, \psi) \triangleq w_\ell(\psi)\,(F(\underset{\sim}{\phi}, \psi) - S_\ell(\psi)), \tag{8}$$

where

$w_u(\psi)$ is a weighting function for $S_u(\psi)$,

$w_\ell(\psi)$ is a weighting function for $S_\ell(\psi)$,

and, for the discrete set of ψ, the error functions are

$$e_{ui}(\underset{\sim}{\phi}) \triangleq e_u(\underset{\sim}{\phi}, \psi_i) = w_{ui}(F_i(\underset{\sim}{\phi}) - S_{ui}),\ i \in I_u \ , \tag{9}$$

$$e_{\ell i}(\underset{\sim}{\phi}) \triangleq e_\ell(\underset{\sim}{\phi}, \psi_i) = w_{\ell i}(F_i(\underset{\sim}{\phi}) - S_{\ell i}),\ i \in I_\ell \ , \tag{10}$$

where the subscript i denotes the ith sample point, ψ_i the value of ψ at this point, w_{ui}, $w_{\ell i}$, $F_i(\underset{\sim}{\phi})$, S_{ui} and $S_{\ell i}$ are the appropriate functions evaluated at ψ_i. I_u and I_ℓ are index sets, not necessarily disjoint, which contain the values of i. The subscripts u and ℓ are for upper and lower specifications, respectively.

Bandler and Charalambous [21,22] introduced a general and practical least pth objective for the approximation problem of this type. This objective is given by

$$U_p(\underset{\sim}{\phi}) = \begin{cases} M(\underset{\sim}{\phi}) \left[\sum_{i \in K_u} \left(\frac{e_{ui}(\underset{\sim}{\phi})}{M(\underset{\sim}{\phi})} \right)^q + \sum_{i \in K_\ell} \left(\frac{-e_{\ell i}(\underset{\sim}{\phi})}{M(\underset{\sim}{\phi})} \right)^q \right]^{1/q} & \text{for } M \neq 0 , \\ 0 & \text{for } M = 0, \end{cases} \tag{11}$$

where

$$M(\underset{\sim}{\phi}) \triangleq \max_{i,j} \left[e_{ui}(\underset{\sim}{\phi}), -e_{\ell j}(\underset{\sim}{\phi}) \right] , \; i \in I_u, \; j \in I_\ell , \tag{12}$$

$$J_u \triangleq \left\{ i \;\middle|\; e_{ui}(\underset{\sim}{\phi}) \geq 0, \; i \in I_u \right\}, \tag{13}$$

$$J_\ell \triangleq \left\{ i \;\middle|\; -e_{\ell i}(\underset{\sim}{\phi}) \geq 0, \; i \in I_\ell \right\}, \tag{14}$$

$$K_u \triangleq \begin{cases} J_u \text{ if } M(\underset{\sim}{\phi}) > 0 , \\ I_u \text{ if } M(\underset{\sim}{\phi}) < 0 , \end{cases} \tag{15}$$

$$K_\ell \triangleq \begin{cases} J_\ell \text{ if } M(\underset{\sim}{\phi}) > 0 , \\ I_\ell \text{ if } M(\underset{\sim}{\phi}) < 0 , \end{cases} \tag{16}$$

and

$$q \triangleq \frac{M(\underset{\sim}{\phi})}{|M(\underset{\sim}{\phi})|} \, p \quad \begin{cases} 1 < p < \infty & \text{for } M > 0 , \\ 1 \leq p < \infty & \text{for } M < 0 . \end{cases} \tag{17}$$

A minimax objective for a problem with upper and lower specifications can be of the form of (12). This objective will tend to minimize the maximum amount by which the actual response fails to meet the specifications, or to maximize the minimum amount by which the circuit response exceeds the specifications.

2.3 Inequality constraints

The problem with upper and lower specifications can be expressed in terms of inequality constraints given by

$$F_i(\underset{\sim}{\phi}) \leq S_{ui}, \; i \in I_u \; , \tag{18}$$

$$F_i(\underset{\sim}{\phi}) \geq S_{\ell i}, \; i \in I_\ell \; . \tag{19}$$

By defining an additional independent variable ϕ_{k+1}, where k is the number of variables, Waren et al. [167] formulated the problem as the nonlinear program

$$\text{minimize} \quad \phi_{k+1}$$

subject to

$$\phi_{k+1} \geq e_{ui} \; , \qquad i \in I_u \; , \tag{20}$$

$$\phi_{k+1} \geq -e_{\ell i} , \qquad i \in I_\ell \; , \tag{21}$$

plus all other constraints. At least one of the constraints has to be active at the optimum, otherwise ϕ_{k+1} could be further minimized without violating any of the constraints. If the optimum ϕ_{k+1} is negative then the specifications are satisfied, while if it is positive the specifications are violated.

Ishizaki and Watanabe [116] used essentially the same formulation but for the case of the single specification.

2.4 Multiple objectives

In electrical circuit design more than one response function might have to meet given specifications. As an example, a circuit can be designed to meet desired specifications in both frequency and time domains. In this case we have more than one independent variable ψ, namely ψ^1, ψ^2, ..., ψ^n, where n is the number of these independent variables. Accordingly, we have n response functions $F^1(\underset{\sim}{\phi}, \psi^1)$, $F^2(\underset{\sim}{\phi}, \psi^2)$, ..., $F^n(\underset{\sim}{\phi}, \psi^n)$ and n specifications $S^1(\psi^1)$, $S^2(\psi^2)$, ..., $S^n(\psi^n)$. The corresponding error functions are given by

$$e^j(\underset{\sim}{\phi}, \psi^j) = w^j(\psi^j)\ (F^j(\underset{\sim}{\phi}, \psi^j) - S^j(\psi^j)),\ j = 1, 2, \ldots, n\ , \tag{22}$$

and, for the discrete case, taking I^j as the index set for the jth functions,

$$e_i^j(\underset{\sim}{\phi}) \triangleq e^j(\underset{\sim}{\phi}, \psi_i^j) = w_i^j(F_i^j(\underset{\sim}{\phi}) - S_i^j),\ i \in I^j\ , \tag{23}$$

is the jth error function evaluated at the ith sample point along the ψ^j axis.

In general, we can have upper and lower specifications for each ψ^j. In the design of a lowpass filter, for example, we can have upper and lower specifications in the frequency domain, and a single specification in the time domain. The error functions will be of the form

$$e_u^1(\underset{\sim}{\phi}, \psi^1) = w_u^1(\psi^1)\ (F^1(\underset{\sim}{\phi}, \psi^1) - S_u^1(\psi^1))\ , \tag{24}$$

$$e_\ell^1(\underset{\sim}{\phi}, \psi^1) = w_\ell^1(\psi^1)\ (F^1(\underset{\sim}{\phi}, \psi^1) - S_\ell^1(\psi^1))\ , \tag{25}$$

$$e^2(\underset{\sim}{\phi}, \psi^2) = w^2(\psi^2)\ (F^2(\underset{\sim}{\phi}, \psi^2) - S^2(\psi^2))\ , \tag{26}$$

where ψ^1 is the frequency ω and ψ^2 is the time t. Figures 4(a) and 4(b) show the specifications in the frequency and time domains, respectively.

An objective function can be evaluated for each error function, and a combined objective consisting of n objectives is obtained, namely,

$$U = U^1 + U^2 + \ldots + U^n \; . \tag{27}$$

A more general objective is of the form

$$U = \sum_{k=1}^{n} \alpha_k \, U^k \; , \tag{28}$$

where the α parameters are factors serving to emphasize the important objectives. These objectives do not necessarily have to be of the same type; one might be a minimax objective and the others can be least pth objectives [8].

2.5 Multidimensional specifications

In the discussion of multiple objectives we considered that each response function and each specification is a function of one independent variable ψ^j. In some cases we are confronted with response functions and specifications which are functions of the n independent variables. These variables can for instance, be time and temperature; frequency and a tunable circuit parameter; or frequencies in a two-dimensional frequency response of a two-dimensional digital filter [131,144]. The response function and the specifications will be $F(\underset{\sim}{\phi}, \underset{\sim}{\psi})$ and $S(\underset{\sim}{\psi})$, respectively, where

$$\underset{\sim}{\psi} \triangleq \begin{bmatrix} \psi^1 \\ \psi^2 \\ \cdot \\ \cdot \\ \cdot \\ \psi^n \end{bmatrix} \; . \tag{29}$$

The frequency response function of a two-dimensional lowpass digital filter, for example, of a symmetrically constrained finite impulse response (zero phase) is given by [144]

$$H(e^{j\omega_1}, e^{j\omega_2}) = e^{-j(n_1\omega_1+n_2\omega_2)} \sum_{k=0}^{n_1} \sum_{\ell=0}^{n_2} a(k, \ell) \cos k\omega_1 \cos \ell\omega_2 , \quad (30)$$

where $a(k, \ell)$ are the filter coefficients, and the specifications are

$$S(\omega_1, \omega_2) = \begin{cases} 1 & \omega_1^2 + \omega_2^2 \leq \omega_p^2 , \\ 0 & \omega_1^2 + \omega_2^2 \geq \omega_s^2 , \end{cases} \quad (31)$$

where ω_p and ω_s are the edges of passband and stopband, respectively. In the discrete case the response function evaluated at the ith sample point is denoted by

$$F_i(\underset{\sim}{\phi}) \triangleq F(\underset{\sim}{\phi}, \underset{\sim}{\psi}_i) , \quad (32)$$

for

$$\underset{\sim}{\psi}_i = \begin{bmatrix} \psi_i^1 \\ \psi_i^2 \\ \cdot \\ \cdot \\ \cdot \\ \psi_i^n \end{bmatrix} , \quad i \in I , \quad (33)$$

where $\psi_i^1, \psi_i^2, \ldots, \psi_i^n$ are the values of the independent variables at the ith sample point in the index set I.

In general, where we have upper and lower specifications, the error functions are generalized to

$$e_{ui}(\underset{\sim}{\phi}) \triangleq e_u(\underset{\sim}{\phi}, \underset{\sim}{\psi}_i) = w_{ui} (F_i(\underset{\sim}{\phi}) - S_{ui}), \; i \in I_u , \quad (34)$$

$$e_{\ell i}(\underset{\sim}{\phi}) \triangleq e_\ell(\underset{\sim}{\phi}, \underset{\sim}{\psi}_i) = w_{\ell i}\ (F_i(\underset{\sim}{\phi}) - S_{\ell i}),\ i \in I_\ell\ , \tag{35}$$

which can be used in a suitable objective for the approximation problem. Figures 5(a) and 5(b) show two possible cases in two dimensions.

3. Evaluation of the response function and its derivatives

The evaluation of any of the objective functions mentioned in Section 2 involves the evaluation of the response function at a certain point $\underset{\sim}{\phi}$. The response function is obtained by numerically analyzing the circuit, where the method of analysis depends on several criteria, namely,

a) the size of the circuit,

b) the equations describing the circuit (linear, nonlinear, algebraic, differential, etc.).

This section of the paper is concerned very briefly with the evaluation of the response function of different types of electrical circuits in the D.C. (direct current), frequency and time domains.

3.1 Introductory concepts and definitions

Consider a linear network (consisting of linear elements, i.e., the relation between the voltage across an element and the current passing through it is linear) which has a single input u(t) and a single output y(t), where u(t) and y(t) are continuous functions of time t. The two functions are related by the convolution integral

$$y(t) = \int_0^t h(t - \tau)\, u(\tau)\, d\tau, \quad t \geq 0 , \tag{36}$$

where h(t) is designated as the network function.

This relation can be converted for lumped, linear, time-invariant circuits to the frequency domain by the Laplace transform, giving

$$Y(s) = H(s)\, U(s) , \tag{37}$$

where H(s) is a rational function of the complex frequency variable s.

The poles of $H(s)$ will be the natural frequencies of the circuit, which have to lie in the left half of the complex s-plane for the circuit to be stable (to have a bounded output for a bounded input). As an example the linear, time-invariant RLC circuit shown in Fig. 6 has the transfer function (voltage across C)

$$H(s) = \frac{1}{LC} \frac{1}{s^2 + (R/L)s + 1/LC} . \tag{38}$$

The dependence of the network function on R, L and C elements can be in the form of a ratio of linear polynomials, i.e., a bilinear relation [103]

$$F(\phi, s) = \frac{A(s) + \phi B(s)}{C(s) + \phi D(s)} . \tag{39}$$

This bilinear property of linear network functions w.r.t. each variable ϕ is very important in relating differential and large change network sensitivities [96], which are potentially useful in design optimization.

Bandler and Liu [31] investigated the validity of certain assumptions considered in tolerance optimization problems for networks which possess bilinear dependence on each parameter. These assumptions are that the worst cases (certain extreme points) occur at the boundaries of the constraint region, if the region is one-dimensionally convex [10]. In their investigation they studied the behavior of the modulus squared of the bilinear network function, which is a biquadratic function given by

$$\frac{c\phi^2 + 2d\phi + e}{\phi^2 + 2a\phi + b} , \tag{40}$$

and they proved that the worst case assumptions they considered are often valid in the frequency domain case.

Brayton et al. [57] proved, for linear D.C. networks, that if each parameter is at its extreme value the currents and voltages of the network will be at their local or optimal extrema. The investigation of this kind of problem in nonlinear networks or in the time domain has not yet been reported.

The network function, or transfer function, is only obtained in an analytical form for very small circuits. For a medium or large network the output is necessarily obtained by numerically analyzing the network. Methods for circuit analysis are quite numerous and in this section some of the important methods are briefly discussed. In all methods the network equations are formulated, which are basically Kirchoff's current and/or voltage laws [84], and then solved appropriately [54].

For certain circuits special methods may be more efficient than general methods of analysis. As an example, cascaded networks, such as the one shown in Fig. 7, are analyzed by the transmission or chain matrix, where each element is considered as a two-port subnetwork described by a 2 x 2 matrix of the form

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} , \tag{41}$$

which relates the input to the output of each two-port subnetwork [38, 105, 136].

Table 1 summarizes different types of electrical networks and indicates the nature of the equations describing these networks along with common methods of solution.

3.2 Linear networks in the frequency domain (the A.C. case)

A linear network is described by a set of linear equations of the form

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b} \quad , \tag{42}$$

where $\underset{\sim}{A}$ is the matrix describing the circuit and can be the nodal admittance matrix $\underset{\sim}{Y}$, the mesh impedance matrix $\underset{\sim}{Z}$ or the tableau matrix [106]. $\underset{\sim}{x}$ is the unknown vector which can be voltages, currents or both, $\underset{\sim}{b}$ is a known vector consisting essentially of sources exciting the circuit. A three node linear circuit and its network equations at a frequency ω are shown in Fig. 8. Note that the equations

$$\underset{\sim}{Y}\,\underset{\sim}{V} = \underset{\sim}{I} \quad , \tag{43}$$

where

$\underset{\sim}{V}$ is the vector of node voltages (with ground node as reference),

$\underset{\sim}{I}$ is the current excitation vector,

have complex coefficients.

An important feature of the matrix $\underset{\sim}{A}$ is that it is sparse for large networks. The sparsity of the matrix increases with the size of the network. Sparse matrix techniques [46,53,91,145,148,163,164], for storing the matrix $\underset{\sim}{A}$ and for the near-optimum ordering of the equations, are usually used. The reordering of the equations is performed so as to preserve the sparsity and to reduce the number of fill-ins (created nonzero elements which were formerly zeros) during the LU decomposition, which is often used to solve these equations. At each frequency point of interest the matrix $\underset{\sim}{A}$ is rebuilt and the set of equations resolved. Only the numerical values of the entries of the $\underset{\sim}{L}$ and $\underset{\sim}{U}$ matrices, where

$\underset{\sim}{A} = \underset{\sim}{L}\underset{\sim}{U}$, are changing but their structures remain fixed.

As a special case of the linear A.C. analysis is the D.C. analysis of resistive networks (independent of frequency). The equations, which are real in this case, are set up in the same way as in the A.C. case and then solved once.

3.3 Linear networks in the time domain

In some problems we are interested in the transients of the circuit and the analysis has to be carried out in the time domain. The network equations describing the linear network (using the state variable approach [79] which is commonly used) are

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}\underset{\sim}{u} , \tag{44}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\underset{\sim}{x} + \underset{\sim}{D}\underset{\sim}{u} , \tag{45}$$

where $\underset{\sim}{A}$ is a coefficient matrix relating the state vector $\underset{\sim}{x}$ (capacitor voltages and inductor currents, for example) to its time derivative $\dot{\underset{\sim}{x}}$, and $\underset{\sim}{B}$ is a coefficient matrix coupling the effects of the independent source vector $\underset{\sim}{u}$. Equation (45) gives the output vector $\underset{\sim}{y}$, where $\underset{\sim}{C}$ and $\underset{\sim}{D}$ are coefficient matrices. Equation (44) is a set of first-order differential equations whose solution is given by

$$\underset{\sim}{x}(t) = e^{\underset{\sim}{A}t} \int_{t_o}^{t} e^{-\underset{\sim}{A}\tau} \underset{\sim}{B}\, \underset{\sim}{u}(\tau)\, d\tau + e^{\underset{\sim}{A}(t-t_o)} \underset{\sim}{x}(t_o) , \tag{46}$$

and the output vector is

$$\underset{\sim}{y}(t) = \underset{\sim}{C}\, e^{\underset{\sim}{A}(t-t_o)} \underset{\sim}{x}(t_o) + \left\{ \underset{\sim}{C}\, e^{\underset{\sim}{A}t} \int_{t_o}^{t} e^{-\underset{\sim}{A}\tau} \underset{\sim}{B}\, \underset{\sim}{u}(\tau)\, d\tau + \underset{\sim}{D}\, \underset{\sim}{u}(t) \right\} . \tag{47}$$

Different approaches to evaluating $e^{\underset{\sim}{A}t}$ and the integrals in (46) and (47) exist [61,79,83].

3.4 Nonlinear networks: the D.C. case

In the nonlinear D.C. case the network equations are expressed in the form

$$\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{0} \quad . \tag{48}$$

Figure 9 [79] is an example of a small nonlinear circuit composed of diodes and linear resistances where the relation between the voltage v across a diode and the current i passing through it is given by the nonlinear relation

$$i = I_s(e^{\lambda v} - 1) \quad , \tag{49}$$

where I_s and λ are constants. The nonlinear network equations are shown in Fig. 9 where v_1, v_2 and v_3 are the unknowns. These equations are usually solved by the Newton-Raphson algorithm (see Table 1).

Another method is to linearize the equations describing the nonlinear elements of the circuit, for example, (49) is linearized as

$$i^{r+1} = i^r + \frac{\partial i}{\partial v}(v^r)\ (v^{r+1} - v^r) \ , \tag{50}$$

where r+1 is the present iteration step. The linearized formulas are then represented by linear elements, called the discrete or the companion elements [61,79] and the resulting linear circuit is analyzed successively until convergence is reached.

Piecewise-linear analysis is also used in solving nonlinear networks [77]. Other approaches dealing with circuits with multiple solutions are described in [55,75,78].

3.5 Nonlinear networks in the time domain

Nonlinear transient networks may be analyzed by different methods. One method is to formulate the state equations of the network, which are ordinary differential equations in the normal form

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, t), \tag{51}$$

where $\tilde{x}$ is the vector of state variables. Equation (51) is then solved by a numerical integration scheme. Stability of the integration and its ability to deal with stiff equations [100,101] are some criteria for choosing the integration scheme for the analysis. The tableau approach [106] is another method for solving nonlinear networks. The method discretizes, at the circuit component (branch) level, the derivative operator d/dt, obtaining nonlinear algebraic difference equations solved by the Newton-Raphson algorithm. The process proceeds in two loops, one for solving the nonlinear algebraic difference equations and the next for the time iteration. In the Newton-Raphson iteration a set of linear equations are repeatedly solved and the sparsity of the coefficient matrix of these equations should be taken into consideration.

Another method for solving nonlinear networks in the time domain, is to reduce the problem to a sequence of D.C. analyses. This is achieved by discretizing the time derivative operator, then replacing the nonlinear elements by their corresponding companion (linearized) elements and solve a D.C. network. The difference between the tableau metnod and this method is that the latter solves a linear system iteratively (until convergence is reached) while the former solves a nonlinear system.

3.6 Response function derivatives

It is well known that optimization techniques which use derivatives are superior to nongradient techniques if first-order sensitivities are readily available. In order to get the derivatives of the response function $F(\underset{\sim}{\phi}, \underset{\sim}{\psi})$, which is a function of certain voltages and/or currents of the circuit, sensitivities of these voltages and/or currents with respect to the variable parameters have to be evaluated. One of the most commonly used approaches to evaluate these sensitivities is the adjoint-network approach [88]. In this approach an adjoint network is constructed, having the same topology as the original network, and analyzed. The results of both analyses are used to evaluate the required sensitivities.

As an example, in the frequency domain, if the network is represented by its admittance matrix $\underset{\sim}{Y}$ at a frequency point and the equations are (43), then the equations representing the adjoint network are

$$\underset{\sim}{Y}^T \underset{\sim}{\hat{V}} = \underset{\sim}{\hat{I}} \quad , \tag{52}$$

where

T denotes transpose,

$\underset{\sim}{\hat{V}}$ is the vector of node voltages of the adjoint network,

$\underset{\sim}{\hat{I}}$ is the current excitation vector of the adjoint network.

$\underset{\sim}{V}$ and $\underset{\sim}{\hat{V}}$, for example, are substituted into some derived formulas to evaluate the sensitivities [39,89].

Branin [56] demonstrated that the sensitivities, in general, can be obtained by matrix manipulation without the need of defining what is termed the adjoint network. Note also that at each frequency two sets of equations are solved. Using the LU decomposition we can achieve some saving by avoiding the decomposition of the matrix transpose [85]. In the linear D.C. case the adjoint network is linear and both original and adjoint networks are analyzed once to calculate the sensitivities. A nonlinear D.C. network will have an associated linear adjoint network which has to be analyzed.

In the time-domain case sensitivities are much more difficult to evaluate because the equations are in the form of ordinary differential equations. Hachtel and Rohrer [107] used variational techniques to get an adjoint set of equations which, when solved along with the original set, allow sensitivities to be evaluated. In the adjoint-network approach, if the original network is analyzed in the interval $t = [0, t_f]$, the adjoint network is analyzed in the interval $\tau = [0, t_f]$, where $\tau = t_f - t$. The integration involving the adjoint network is backward on the time axis. The formulas for the sensitivities are integral formulas, i.e., in evaluating the sensitivities with respect to k variables, k integrations have to be performed after analyzing the original and adjoint networks. Other methods can be used to evaluate the sensitivities [135] but they do not appear easier or more efficient than the adjoint-network approach.

An approach developed by Bandler and Abdel-Malek [12,13] avoids the evaluation of the exact response function derivatives. Multidimensional polynomial approximations of the response functions are performed using a minimal number of evaluations of the actual functions within an interpolation region. The approximations are used in the optimization process instead of the actual functions. The derivatives of the approximations are efficiently and rapidly obtained. During optimization the approximation is updated in different regions in the space or in smaller interpolation regions as indicated by the optimization or to obtain higher accuracy, respectively.

3.7 Digital filters

Digital filters [104,134,137] differ from analog circuits in that the inputs and outputs are sequences of numbers having a finite number of digits. A special-purpose computer or general-purpose computer along with a stored program can serve as a digital filter. There are two types of digital filters, namely the infinite impulse response (IIR) filter which needs a recursive computational algorithm for its realization and the finite impulse response (FIR) filter which needs a nonrecursive computational algorithm.

In the design procedure of digital filters by optimization, the transfer function H(z) is given, where z is the complex variable of the z transform [118]. The transfer function H(z) of the IIR filter is a rational function in z of the form

$$H(z) = g + \sum_{i=1}^{s} \frac{a_i + b_i z^{-1}}{1+c_i z^{-1}+d_i z^{-2}} , \qquad (53)$$

when the s elementary sections, Fig. 10, are connected in parallel (Fig. 11), and

$$H(z) = g \prod_{i=1}^{s} \frac{1 + a_i z^{-1} + b_i z^{-2}}{1 + c_i z^{-1} + d_i z^{-2}} \qquad (54)$$

when the s sections are connected in cascade. The transfer function of a FIR filter is a polynomial in z of the form

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} \quad . \tag{55}$$

In optimization problems, the coefficients of the transfer function are to be found so that the function best meets the given specifications. Usually, for stability reasons the values of these coefficientts are constrained. Practical considerations usually constrain the coefficients of the digital filter to a certain number of bits since they are stored in a binary format with finite-length registers. Quantization of the coefficients might violate the stability criteria or deteriorate the response unacceptably. Discrete optimization is then more reliable [5,18,73,158,160]. In this case, the coefficients multiplied by 2^{-Q}, where Q+1 is the number of bits used, have to be integers.

Since the transfer function is known analytically in terms of the coefficients, the derivatives with respect to these coefficients are easily obtained [17,157].

3.8 Network equations of power systems

Power systems are basically composed of generating plants, transmission lines and loads. Figure 12 [52] depicts a small power network. The network equations or the load flow equations are solved for power system planning, operation, security and control. The solution of the load flow equations [156,159] determines the voltages and powers in each branch of the network under steady-state operating conditions.

The power system network, like any electrical network obeys Kirchoff's laws and hence its nodal admittance equations are of the form of (43). The ith element of the vector $\underset{\sim}{I}$, corresponding to the ith node is

$$I_i = S_i^*/V_i^* \quad , \tag{56}$$

where S_i is the total power at the busbar node, and * denotes the complex conjugate. This S_i is given by

$$S_i = (P_{Gi} - P_{Ci}) + j(Q_{Gi} - Q_{Ci}) \quad , \tag{57}$$

where

P_{Gi} and Q_{Gi} are the active and reactive powers, respectively, generated at node i,

P_{Ci} and Q_{Ci} are the active and reactive powers, respectively, consumed at node i.

Substituting (56) in the nodal equations we get for node i

$$S_i = V_i \sum_{j=1}^{n} Y_{ij}^* \, V_j^* \quad , \tag{58}$$

where n is the number of nodes. This equation can be rewritten in a more detailed form

$$P_{Gi} - P_{Ci} - \sum_{k \in K_i} |V_i||V_k||Y_{ik}|\sin(\theta_i - \theta_k - \delta_{ik})$$

$$- \sum_{k \in K_i} |V_i|^2 |Y_{ik}|\sin \delta_{ik} = 0 \quad , \tag{59}$$

$$Q_{Gi} - Q_{Ci} + \sum_{k \in K_i} |V_i||V_k||Y_{ik}|\cos(\theta_i - \theta_k - \delta_{ik})$$

$$- \sum_{k \in K_i} |V_i|^2 |Y_{ik}|\cos \delta_{ik} + |Y_{ii}||V_i|^2 = 0 \; , \tag{60}$$

for i = 1, 2, ..., n, and $i \notin K_i$, where

K_i is the set of nodes directly connected to node i,

θ_i, $|V_i|$ are the phase angle and magnitude, respectively, of the voltage at node i,

$|Y_{ik}|$ is the admittance magnitude of line ik,

δ_{ik} is the loss angle of line ik,

$|Y_{ii}|$ is the admittance magnitude with respect to ground at node i.

For each node two variables out of the four, namely, P, Q, $|V|$ and θ are given. P and Q are given for load nodes, P and $|V|$ for generation nodes. In one of the generation nodes $|V|$ and θ are given, where this θ is set to zero as a reference. This node is called the slack node. The unknown variables in the equations include the voltage variables, also called the state variables of the system.

Equations (59) and (60) are in the form of $\underset{\sim}{f}(\underset{\sim}{\phi}) = \underset{\sim}{0}$ and they are usually solved by the Newton-Raphson algorithm [159]. Once the voltages are obtained all other unknowns can be obtained. The power networks are very large, with thousands of nodes and the $\underset{\sim}{Y}$ matrix is very sparse. The Jacobian matrix set up in the Newton iteration is, accordingly, very sparse. Sparse matrix techniques are consequently essential in solving the large set of load-flow equations.

4. Optimization approaches in circuit design

The work discussed in this section has been performed by a variety of principal contributors to the field of optimization of electrical circuits. A brief review of different optimization methods and approaches applied to circuit design problems is given.

4.1 General review

Lasdon and Waren [122] formulated the filter design problem as a nonlinear program and, by means of an interior penalty function formulation, obtained filters optimal in a minimax sense. Nonlinear programming has been applied by Lasdon, Waren and their colleagues to the computer design of cascade crystal-realizable lattice filters, optical filters and antenna arrays [121,123,166,167].

Temes and Calahan [161] have reviewed the application of optimization to filter design and to the modelling of active devices. Temes and Zai [162] considered least pth approximation in the design of active equalizers. Their objective formulation limited them to p not much greater than 10, as was the work of Deczky [81] on digital filters. Bandler and Charalambous [21,22] introduced the generalized least pth approximation, where any value of p can be used, and several minimax problems were subsequently solved either by using a large value of p or by a sequence of least pth approximations [17-21,23-29,38-41,68-72].

Ishizaki and Watanabe [116] treated the nonlinear programming problem by successively solving linear programming problems, which are derived by locally linearizing the original nonlinear programming problem. Madsen et al. [127,129] developed two minimax network optimization algorithms, one of them not requiring derivatives. They are based on successive linear approximation to the nonlinear functions defining the problem. The method has been used for transmission-line transformers, microwave filter design and a practical transferred-electron reflection-type amplifier. Charalambous and Conn [74] developed a minimax algorithm where the discontinuities in the first derivatives can be characterized by projections.

In the field of digital filter design optimization techniques have been used extensively [5,17,18,65,68,73,81,82,109,144,157,158,160]. Helms [109] reviewed the techniques used to get equiripple or minimax errors in the design of nonrecursive filters. These techniques included the simplex method of linear programming, nonlinear programming and integer programming. Steiglitz [157] described some practical methods used in the design of recursive digital filters with arbitrary prescribed magnitude characteristics. Rabiner et al. [144] discussed various approaches to designing finite impulse response digital filters using the theory of weighted Chebyshev approximation. Charalambous and Best [73] applied the branch and bound technique for nonlinear discrete optimization of recursive digital filters with finite word length. Bandler et al. [18] studied optimum word length problems in a similar vein.

Optimization techniques are essential in the design and operation of power networks [3,4,50-52,62,90,99,112,117,133,138,146,149-152,154]. They have been applied, for example, to the economic dispatch, which is the problem of minimizing the cost of fuel of thermal plants. With modern technology, economic dispatch should be solved on-line every few minutes and the results used to continually adjust the power outputs of generating stations. Another problem where optimization has been applied is the hydrothermal dispatch which is much larger than the economic dispatch [99]. In this case the hydro generation is not defined and the solution of the problem decides on the generators to be connected to the system and their level of operation for a certain period of time. Heuck [112] discussed the nonlinear programming formulation of this problem.

Existing power plants have to be expanded in order to satisfy the increasing demand. The generation expansion planning determines, up to a certain time, if new generating plants have to be built, which of the existing ones have to increase their generating capacity and when this can be performed. The objective is to expand the plant with minimum cost such that the demands are satisfied, while the generating capacity serves as constraints. Several other problems have been solved by optimization. These problems are very large, with large numbers of variables, the system equations being nonlinear and sparse. The techniques involved are linear, nonlinear, quadratic, integer and dynamic programming. Sasson and Merrill [151] reviewed some of these techniques and their applications.

4.2 Approach due to Lasdon and Waren

Optimal design of filters has been treated as a nonlinear programming problem by Lasdon and Waren [122]. The problem is defined by the inequalities (18), (19) and lower and upper bounds on the variable parameters. The problem is reformulated essentially as the problem given by (20) and (21). Lasdon and Waren applied the interior penalty sequential unconstrained minimization technique by Fiacco and McCormick [95] along with the Fletcher-Powell variable metric method

[98] to solve this type of problem. This technique has been applied to the design of cascade crystal-realizable lattice filters, linear arrays [121], planar arrays [167] and acoustic sonar transducer arrays [123].

An excellent example of the results achieved by Lasdon and Waren [122] is shown in Figure 13. The specifications they employed are of the form of Fig. 3(a).

4.3 The GRG method

Waren et al. [166] developed a generalized reduced gradient (GRG) algorithm for solving the nonlinear program

$$\text{minimize } U(\underset{\sim}{\phi})$$

subject to

$$h_i(\underset{\sim}{\phi}) = 0, \quad i = 1, 2, \ldots, n_h , \tag{61}$$

$$g_i(\underset{\sim}{\phi}) \geq 0, \quad i = 1, 2, \ldots, n_g , \tag{62}$$

by converting it to

$$\text{minimize } U(\underset{\sim}{\phi})$$

subject to

$$h_i(\underset{\sim}{\phi}) - \phi_{k+i} = 0, \quad i = 1, 2, \ldots, n_h , \tag{63}$$

$$g_i(\underset{\sim}{\phi}) - \phi_{k+n_h+i} = 0, \quad i = 1, 2, \ldots, n_g , \tag{64}$$

$$\phi_{\ell i} \leq \phi_i \leq \phi_{ui}, \quad i = 1, \ldots, k + n_h , \tag{65}$$

$$\phi_{\ell i} = \phi_{ui} = 0, \quad i = k + 1, \ldots, k + n_h , \tag{66}$$

$$\phi_{k+n_h+i} \geq 0, \quad i = 1, 2, \ldots, n_g , \tag{67}$$

where

n_h is the number of equality constraints,

n_g is the number of inequality constraints,

k is the number of variables,

$\phi_{k+1}, \ldots, \phi_{k+n_h+n_g}$ are nonnegative slack variables.

At each stage of the optimization process the variables are separated into dependent and independent variables. The number of natural dependent variables is the number of active constraints n_a. The slack variables of the nonactive constraints are the additional dependent variables. All the remaining ones are taken as independent variables. The active constraints are then solved for the natural dependent variables n_a in terms of the natural independent ones $k - n_a$. This reduces the objective function to a function of $k - n_a$ variables only. The generalized reduced gradient algorithm solves the original problem as a sequence of reduced problems. The reduced problems are solved using a variable metric gradient method.

Waren et al. used the GRG method in the design of dielectric interference filters. The problem, defined by inequalities, is reformulated as a nonlinear program (as in Section 4.2). The numbers of variables and constraints are considerable. The GRG method apparently handles this large problem efficiently and yields satisfactory results.

4.4 Sasson's approaches

Sasson [149] used the Fiacco-McCormick and Lootsma methods [126] and the Zangwill transformation [169] along with the Fletcher-Powell method to solve nonlinear programs associated with power system optimization. The Zangwill penalty function

$$P(\underset{\sim}{\phi}, r) = U(\underset{\sim}{\phi}) + 1/r \sum_{i=1}^{m} [X_i(C_i(\underset{\sim}{\phi}))]^2 , \qquad (68)$$

where

$$X_i(C_i(\underset{\sim}{\phi})) = [\min(0, C_i(\underset{\sim}{\phi}))], \text{ if } C_i(\underset{\sim}{\phi}) = g_i(\underset{\sim}{\phi}) , \qquad (69)$$

and

$$X_i(C_i(\underset{\sim}{\phi})) = C_i(\underset{\sim}{\phi}), \qquad \text{if } C_i(\underset{\sim}{\phi}) = h_i(\underset{\sim}{\phi}) , \qquad (70)$$

$$m = n_h + n_g , \qquad (71)$$

has the advantage of not requiring an initial feasible point and the ability to handle equality constraints. The three methods are sensitive to the initial choice of r, and ill-conditioning arises when r approaches zero.

Sasson [150] also used the Powell extension [142] to the Zangwill transformation

$$P(\underset{\sim}{\phi}, \underset{\sim}{r}, \underset{\sim}{s}) = U(\underset{\sim}{\phi}) + \sum_{i=1}^{m} \frac{(X_i(C_i(\underset{\sim}{\phi}))+s_i)^2}{r_i} \qquad (72)$$

where s_i and r_i are constants during each sequential optimization and $X_i(C_i(\underset{\sim}{\phi}))$ is as defined by (69) and (70).

The value of s_i is updated by [142]

$$s_i^{j+1} = s_i^j + g_i , \qquad (73)$$

where j is the present iteration number, and the values of r_i form a decreasing set approaching zero.

(The ill-conditioning problem which arises in penalty function methods when r tends to zero has been studied by Charalambous [67], where he extended the work by Powell. The approach is based on the simple idea of perturbing the constraints outwards for the interior penalty function, and inwards for the exterior penalty function by a certain amount so that the r parameter does not have to tend to zero at the optimum. The factor by which the constraints are perturbed and the updating formula are similar to the s_i factor and its updating formula in Powell's transformation.)

In power system problems the Hessian matrix can be obtained from the Jacobian and is sparse enough so that sparse matrix techniques can be used. Sasson et al. [152] calculated the correction vector $\Delta\underset{\sim}{\phi}$, of the Newton step, using Gaussian elimination taking sparsity into account. We have to note that in these problems the vector of unknowns $\underset{\sim}{\phi}$ represents the state variables of the whole system.

4.5 Decomposition and reduction in power systems

Billinton and Sachdeva [51] solved the real and reactive power optimization problem by decomposing the problem to two parts. After a load flow solution, an optimum voltage evaluation is obtained by minimizing the system losses considering the reactive powers as equality constraints and the voltage magnitudes as variables. Then the real power optimization is carried out with the obtained system voltages kept fixed and voltage angles as variables. The process iterates between the two suboptimizations until the final solution is reached. The penalty functions discussed earlier in Section 4.4 were used to solve the two problems.

Another approach which attempts to reduce the problem was used by Dommel and Tinney [90], where they solved the optimal power flow problem by separating the variables into two sets: the vector $\underset{\sim}{x}$ of all unknown variables (state variables) and $\underset{\sim}{y}$ the vector of all specified variables. The vector $\underset{\sim}{y}$ is further partitioned into the vector $\underset{\sim}{u}$ of control variables and vector $\underset{\sim}{p}$ of fixed parameters. The problem in its simplest form is set up as

$$\min_{\underset{\sim}{u}} U(\underset{\sim}{x}, \underset{\sim}{u}) \tag{74}$$

subject to the equality constraints

$$\underset{\sim}{h}(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{p}) = \underset{\sim}{0}. \tag{75}$$

The Lagrangian function

$$L(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{p}, \underset{\sim}{\lambda}) = U(\underset{\sim}{x}, \underset{\sim}{u}) + \sum_{i=1}^{n_h} \lambda_i \, h_i(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{p}) \tag{76}$$

and the necessary conditions for an optimum

$$\nabla_{\underset{\sim}{x}} L = \nabla_{\underset{\sim}{x}} U + J_{\underset{\sim}{x}}^T \underset{\sim}{\lambda} = \underset{\sim}{0} , \tag{77}$$

$$\nabla_{\underset{\sim}{u}} L = \nabla_{\underset{\sim}{u}} U + J_{\underset{\sim}{u}}^T \underset{\sim}{\lambda} = \underset{\sim}{0} , \tag{78}$$

$$\nabla_{\underset{\sim}{\lambda}} L = \underset{\sim}{h}(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{p}) = \underset{\sim}{0} , \tag{79}$$

where $J_{\underset{\sim}{x}}$ and $J_{\underset{\sim}{u}}$ are the Jacobians w.r.t. the $\underset{\sim}{x}$ and $\underset{\sim}{u}$ variables, respectively. The basic steps of the algorithm used to solve this problem are

1) assume a set of control variables $\underset{\sim}{u}$,

2) find a feasible power flow solution, i.e., solve (79)

3) solve (77) to get $\underset{\sim}{\lambda}$

4) substitute $\underset{\sim}{\lambda}$ in (78) and use $\nabla_{\underset{\sim}{u}} L$ to determine $\Delta \underset{\sim}{u}$.

Practically, the control variables are bounded. This introduces inequality constraints into the previously mentioned problem. If a correction step $\Delta \underset{\sim}{u}$ lets one of the variables go outside its bounds, the value of this variable is set to the allowable limit and the process is continued. The problem, in general, will have inequality constraint functions of $\underset{\sim}{x}$ and $\underset{\sim}{u}$ and bounds on the state variables $\underset{\sim}{x}$. These inequalities are handled as penalties. Alsac and Stott [4] used the same technique for solving the problem considering the security constraints to obtain a secure optimal load flow.

The general problem with equality constraints and inequality constraints on $\underset{\sim}{u}$ and $\underset{\sim}{x}$ has been solved [138] by the generalized reduced gradient method discussed in Section 4.3. When a state variable reaches one of its bounds it is changed to an independent variable, and one of the independent variables far from its bounds becomes a dependent variable. Carpentier [62] used the generalized reduced gradient to solve a reduced problem which is equivalent to but much smaller than the original one. After solving the network equations a set of violated and nearly-active constraints are chosen to form the constraints of the reduced problem. These constraints are expressed in terms of the control variables. At the solution of the reduced problem the complete problem is examined and if any constraint violation exists, new constraints are added and the process is repeated. Adielson [3] applied the generalized reduced gradient method to solve a decomposed problem similar to the one in [90].

Snyder and Sasson [154] recently developed the modified decoupled Hessian technique which concentrates on minimizing the constraint violations. The method seemed appropriate for security load flow problems. The objective function is

$$P(\underset{\sim}{u}) = U(\underset{\sim}{x}, \underset{\sim}{u}) + \sum_{i=1}^{n_g} w_i (g_i(\underset{\sim}{x}, \underset{\sim}{u}))^2 \qquad (80)$$

subject to $\underset{\sim}{h}(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}$, which are the network equality constraints. All the inequality constraints are considered in (80) where the w_i are weighting factors which are proportional to the constraint violation. They established an incremental relation between P and $\underset{\sim}{u}$ which becomes the new objective to be minimzed. The correction vector $\Delta \underset{\sim}{u}$ is calculated through a Newton step. The weighting factors are changed throughout the iteration process to emphasize the violating constraints. The problem was set up so as to take advantage of the existing decoupled relationships which exist among the load flow variables [4].

4.6 Quadratic programming in power system optimization

Reid and Hasdorff [146] used quadratic programming for solving the economic dispatch. The problem is reformulated to suit the Wolfe simplex method for quadratic programming [168]. A quadratic objective function is set up, containing some new variables representing the real power component at each generating bus. This increases the number of variables and leads to some new equality constraints. The constraints are linearized using a Taylor series expansion and slack variables are introduced to the inequality constraints to transform them to equality contraints. Transformation of variables is also applied to restrict the variables to be positive, which is one the requirements of the Wolfe algorithm. Nicholson and Sterling [133] decomposed the problem where one of the subproblems has a quadratic objective and linear constraints. This subproblem is solved by the method of Beale [44] while the other is solved by a steepest descent gradient technique.

Biggs and Laughton developed the REQP (Recursive Equality Quadratic Programming) algorithm for solving the economic dispatch [50]. The minimum point $\underset{\sim}{\phi} + \underset{\sim}{d}$ of a penalty function of the form of (68) is estimated by calculating the search direction $\underset{\sim}{d}$. This search direction is obtained by solving a quadratic programming problem. The calculation of $\underset{\sim}{d}$ involves the solution of an $\ell \times \ell$ system of equations (ℓ is usually less than the number of variables k) and an approximation to the inverse of the Lagrangian's Hessian by the Broyden-Fletcher-Shanno formula. More details on the algorithm are given in [49,50].

4.7 Least pth optimization

Temes and Zai [162] generalized the least squares method of Marquardt [130] with appropriate damping in the spirit of Levenberg [124] to a least pth method, where p is any positive even integer. The method was applied to the optimization of a four-variable RC active equalizer, where p was equal to 10. The maximum deviation from the desired specification for p = 2 was found to be 33 percent higher. They also demonstrated the nonuniqueness of the optimum in that particular problem. They obtained different solutions with different starting points.

Deczky [81] used least pth error criteria in the synthesis of recursive digital filters along with the Fletcher-Powell method. In his problems p did not exceed the value of 10.

4.8 Generalized least pth objective

Bandler and Charalambous generalized the least pth objective [21,22], as given in (11), for the design problem with upper and lower specifications described in Section 2.2. In general, we have

$$U_p = \begin{cases} M \left(\sum_{i \in K} \left(\frac{\Phi_i}{M} \right)^q \right)^{1/q} & \text{for } M \neq 0 , \\ 0 & \text{for } M = 0, \end{cases} \tag{81}$$

where the Φ_i are n real, nonlinear functions (assumed differentiable) identified by an index set I,

$$M = \max_{i \in I} \Phi_i , \tag{82}$$

and

$$\text{if } M > 0 \text{ then } K = J \text{ and } q = p ,$$

$$\text{if } M < 0 \text{ then } K = I \text{ and } q = -p ,$$

where

$$J = \{i \mid \Phi_i \geq 0\} . \tag{83}$$

The gradient vector of the function is given by

$$\nabla U_p = \left(\sum_{i \in K} \left(\frac{\Phi_i}{M} \right)^q \right)^{1/q - 1} \sum_{i \in K} \left(\frac{\Phi_i}{M} \right)^{q-1} \nabla \Phi_i , \quad \text{for } M \neq 0 . \tag{84}$$

Minimization algorithms which require derivatives can thus be used. We have to note that for $p > 1$ the first-order partial derivatives of the objective function are continuous if the functions Φ_i are continuous with continuous first-order partial derivatives. In the case when $M = 0$ and two or more maxima are equal the objective's first-order derivatives are discontinuous. (The obvious consequences in gradient algorithms can be alleviated, as shown later.)

This generalized least pth form leads to the development of algorithms for solving minimax and nonlinear programming problems [24-27,29,40].

4.9 Near minimax optimization

Consider for example,

$$\Phi_i = f_i(\underset{\sim}{\phi}) \ , \ i \in I \ , \qquad (85)$$

and let

$$M_f = M \qquad (86)$$

in (82). One simple approach attempts to reach a point close to the solution $\check{\underset{\sim}{\phi}}$ of the minimax problem by minimizing U_p of (81) w.r.t. $\underset{\sim}{\phi}$ using a single large value of p. Typically this value is larger than 100. The solution reached is denoted $\check{\underset{\sim}{\phi}}_p$.

Bandler and Charalambous considered necessary and sufficient conditions for optimality in generalized least pth optimization for $p \to \infty$ [20,23] and related them to the conditions for minimax optimality [9]. The multipliers obtained from the necessary conditions are

$$\mu_i = \lim_{p\to\infty} \left(\frac{[f_i(\check{\underset{\sim}{\phi}}_p)/M_f(\check{\underset{\sim}{\phi}}_p)]^q}{\sum\limits_{i\in L} [f_i(\check{\underset{\sim}{\phi}}_p)/M_f(\check{\underset{\sim}{\phi}}_p)]^q} \right) , \qquad (87)$$

where

$$L \triangleq \{i \mid f_i(\check{\underset{\sim}{\phi}}) = M_f(\check{\underset{\sim}{\phi}})\}. \qquad (88)$$

Discontinuity of the objective function's first partial derivatives, for the reason we mentioned earlier, suggests the introduction of an artificial margin ξ, so that the M to be minimized is

$$M = \max_{i\in I} \ (f_i(\underset{\sim}{\phi}) - \xi) = M_f(\underset{\sim}{\phi}) - \xi \ . \qquad (89)$$

The corresponding functions Φ_i in this case are

$$\Phi_i = f_i(\underset{\sim}{\phi}) - \xi \ , \qquad i \in I \ . \tag{90}$$

U_p is minimized with sufficiently large p and constant ξ. The true minimax optimum $M_f(\check{\underset{\sim}{\phi}})$, of course, is not affected by the margin ξ.

This approach has been used by Bandler et al. in the design of microwave circuits [22,38,64] and digital filters [17].

4.10 Sequential least pth algorithms

Two algorithms have been presented [72] in which a sequence of least pth optimization problems is constructed. While they can be used with large p they are designed for a moderate value.

The objective function to be minimized w.r.t. $\underset{\sim}{\phi}$ is

$$U_p = U_p(\underset{\sim}{\phi}, \xi^r) \ , \tag{91}$$

where

$$\Phi_i = f_i(\underset{\sim}{\phi}) - \xi^r \ , \qquad i \in I, \tag{92}$$

$$M = M(\underset{\sim}{\phi}, \xi^r) = M_f(\underset{\sim}{\phi}) - \xi^r \ , \tag{93}$$

where r is the optimization number and ξ^r is the margin at the rth optimization. The margin is updated at the end of each optimization to be

$$\xi^{r+1} = M_f(\check{\underset{\sim}{\phi}}^r) + \varepsilon \ , \tag{94}$$

where $\check{\underset{\sim}{\phi}}^r$ is the optimum at the rth optimization and ξ is a small number. The second algorithm updates the margin as in the first algorithm if $M_f(\check{\underset{\sim}{\phi}}^r, \xi^r)$ is negative, otherwise

$$\xi^{r+1} = (1 - \lambda^r)\xi^r + \lambda^r M_f(\check{\underset{\sim}{\phi}}^r) \ , \tag{95}$$

where

$$0 < \lambda^r < 1 . \qquad (96)$$

In both algorithms, for the first optimization the margin ξ^1 is min $[0, M_f(\underset{\sim}{\phi}^0) + \varepsilon]$, where $\underset{\sim}{\phi}^0$ is the starting point. For $r > 1$ the first algorithm will let all the Φ_i be negative and considered in the objective function and the maximum is to be moved away from the margin. In the second algorithm, ξ starts with zero and increases approaching $M_f(\check{\underset{\sim}{\phi}})$. The small number ε is introduced to avoid $M = 0$. It is well known that the minimax solution will not change if a constant is added to all the functions f_i. If this constant is greater than $|M_f(\check{\underset{\sim}{\phi}})|$ the second algorithm will be used throughout the whole optimization even if $M_f(\check{\underset{\sim}{\phi}}) \leq 0$.

Useful design information can be extracted from these algorithms. Suppose that we are considering a design problem with upper and lower specifications, so that

$$f_i = \begin{cases} e_{uj}, & j \in I_u, \\ -e_{\ell k}, & k \in I_\ell, \end{cases} \qquad i \in I , \qquad (97)$$

where

$$I_u = \{1, 2, \ldots, n_u\} , \qquad (98)$$

$$I_\ell = \{1, 2, \ldots, n_\ell\} , \qquad (99)$$

$$I = \{1, 2, \ldots, n_u + n_\ell\} , \qquad (100)$$

and according to a numbering scheme where the error functions for upper specifications are considered first:

$$\begin{aligned} j &= i && \text{if } i \leq n_u , \\ k &= i - n_u && \text{if } i > n_u . \end{aligned} \qquad (101)$$

The sign of $M_f(\underset{\sim}{\phi})$ indicates whether the specifications are satisfied or violated. That is, if

$$M_f(\underset{\sim}{\phi}) \begin{cases} > 0 & \text{the specifications are violated ,} \\ = 0 & \text{the specifications are just met,} \\ < 0 & \text{the specifications are satisfied.} \end{cases}$$

The reason for letting ξ^1 in the first step be min[0, $M_f(\underset{\sim}{\phi}^0)$ is that the first optimization (with any value of p) will indicate whether the specifications can be met or not. If the specifications cannot be met an appropriate change in the topology of the design or a relaxation of the given specifications must be made.

A design problem with a single specification such that

$$f_i(\underset{\sim}{\phi}) = |e_i(\underset{\sim}{\phi})| \ , \quad i \in I \ , \tag{102}$$

does not appear to yield the foregoing information too readily.

Parameter constraints can be treated in the same way as the performance specifications. For example, the upper bound ϕ_{uj} on the parameter ϕ_j can be considered in the objective function by considering an additional function $f_{n+1} = \phi_j - \phi_{uj}$. Figure 14(a) [38] shows possible contours of the least pth objective function without parameter constraints and Fig. 14(b) shows possible contours for the same problem when the upper bound on one parameter is considered. Suppose that one of the aforementioned algorithms is used in a design problem where we considered the parameter constraints and performance specifications in the objective function. If after the first optimization $M_f > 0$ a tradeoff between performance specifications and parameter constraints must be made in order to achieve a feasible design.

Charalambous [68] and Bandler et al. [25,26] took this margin ξ^{r+1} to be the lower bound on the maximum predicted under convexity assumptions after each optimization. The margin ξ^r is used as a lower bound for the (r+1)th optimization so that all the functions less than this margin are discarded and considered inactive. The associated algorithm is called the ξ algorithm.

Recently, Bandler et al. [26,40,76] used extrapolation to $p = \infty$, after performing least pth approximation with different values of p, to obtain the minimax solution. The algorithm, called the p algorithm, has been implemented in a user-oriented package called FLOPT4 [40], which is currently used for solving many engineering design problems. More recently, Charalambous suggested acceleration of the least pth approach by including in the objective estimates of the multipliers at the minimax solution [70].

A three-section 100-percent relative-bandwidth 10:1 transmission-line transformer is shown in Fig. 15. It is a special case of an N-section transmission-line transformer. Originally studied by Bandler [7,37] and developed into a family of test problems by Bandler and Macdonald [36], this type of test problem is now widely considered [22,23,26,36,37,40,43,74,92,93]. The variables are the lengths and the characteristic impedances of the transmission lines.

Another microwave example is the design of the bandpass filter shown in Fig. 16 with given specifications on the insertion loss in the passband and the stopband. Bandler et al. [26] took the lengths of the transmission lines to be fixed while the characteristic impedances were variables. Twenty-one uniformly spaced passband points were considered but, due to symmetry, only the first ten were used. For the p algorithm the values 2, 12, 72, 432 were used and the minimax solution was obtained using third-order extrapolation. The optimum response is shown in Fig. 17(a). The response is not exactly equal ripple because of the uniformly spaced sample points.

The ξ algorithm generated the results of Table 2 employing sample points corresponding to the ripple maxima of the optimum response obtained by the p algorithm, for 4 sets of specifications. The value of ξ being zero for the first least pth approximation (with p=2) the first maximum error $M_f(\check{\phi}^1)$ gives an indication as to whether the specifications will be met at the minimax solution or not. For the first set of specifications $M_f(\check{\phi}^1) < 0$, i.e., the specifications can be satisfied at the minimax solution, while for the other sets the specifications will be violated.

The ripple maxima in the passband were found by quadratic approximations based on sets of three adjacent sample points taken from 101 uniformly spaced candidates. Fig. 17(b) shows responses corresponding to the 4 sets of specifications after only two optimizations with p = 2 of the ξ algorithm.

In this, as in many examples solved by the least pth approach, the Fletcher program [97] was used.

4.11 Least pth objective and nonlinear programming

Bandler and Charalambous [24,63] suggested that the nonlinear programming problem could be solved using minimax techniques by transforming the problem to minimizing w.r.t. ϕ the unconstrained function

$$M(\underset{\sim}{\phi}, \underset{\sim}{\alpha}) = \max_{1 \leq i \leq n_g} [U(\underset{\sim}{\phi}), U(\underset{\sim}{\phi}) - \alpha_i g_i(\underset{\sim}{\phi})] , \tag{103}$$

where

$$\underset{\sim}{\alpha} = [\alpha_1 \ \ \alpha_2 \ \dots \ \alpha_{n_g}]^T \tag{104}$$

and

$$\alpha_i > 0 , \qquad i = 1, 2, \dots, n_g . \tag{105}$$

(Equality constraints can be transformed to two inequality constraints). They related the Kuhn-Tucker necessary conditions for optimality of the nonlinear programming problem to the necessary conditions for optimality of $M(\underset{\sim}{\phi}, \underset{\sim}{\alpha})$. These conditions require that the α parameters be positive and satisfy

$$\sum_{i=1}^{n_g} \frac{\mu_i}{\alpha_i} < 1 , \tag{106}$$

where the μ_i are the Kuhn-Tucker multipliers (not known a priori). Sufficiently large values should be assigned to $\underset{\sim}{\alpha}$ to ensure that the inequality (106) is satisfied.

This minimax problem can be solved by least pth optimization [64], with large values of p, by letting

$$\Phi_1 = U(\underset{\sim}{\phi}) , \tag{107}$$

$$\Phi_{j+1} = U(\underset{\sim}{\phi}) - \alpha_j g_j , \qquad j = 1, 2, \dots, n_g, \tag{108}$$

$$M = M(\underset{\sim}{\phi}, \underset{\sim}{\alpha}) . \tag{109}$$

Ill-conditioning can arise when the minimax solution is approached because of the tendency of the first partial derivatives to be discontinuous. Charalambous [69] attacked the problem by defining a sequence of least pth optimizations where the objective function to be minimized w.r.t. $\underset{\sim}{\phi}$ is

$$U_p = U_p(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r, \xi^r) \; , \tag{110}$$

where

$$\Phi_1 = \Phi_1(\underset{\sim}{\phi}, \xi^r) = U(\underset{\sim}{\phi}) - \xi^r \tag{111}$$

$$\Phi_{j+1} = \Phi_{j+1}(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r, \xi^r) = \Phi_1 - \alpha_j^r g_j \; , \quad j = 1, 2, \ldots, n_g \; , \tag{112}$$

$$M = M(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r, \xi^r) = M(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r) - \xi^r \; , \tag{113}$$

where r is the optimization number. He proved that if

$$\underset{\sim}{\alpha} = \underset{\sim}{\alpha}^r = (n_1 + 1)\, \check{\underset{\sim}{\mu}} \; , \tag{114}$$

the point $\check{\underset{\sim}{\phi}}$ is a stationary point of the function $U_p(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r, \xi^r)$ for any p and ξ, where

- $\check{\underset{\sim}{\phi}}$ is the optimum of the nonlinear programming problem,
- $\check{\underset{\sim}{\mu}}$ are the multipliers at the optimum $\check{\underset{\sim}{\phi}}$,
- n_1 is the number of constraints with multipliers greater than or equal to a certain small number ε_1.

An approximation to the multipliers which is an estimate to $\check{\underset{\sim}{\mu}}$ (since $\check{\underset{\sim}{\mu}}$ cannot be known beforehand) is used in updating $\underset{\sim}{\alpha}$. These approximations for the Kuhn-Tucker multipliers are calculated after each optimization, and are given by

$$\mu_i^r = \begin{cases} \dfrac{v_i^r}{v_1^r + \sum_{i \in K} v_i^r} & i \in K , \\ \\ 0 & i \notin K , \end{cases} \tag{115}$$

$$v_1^r = \left(\frac{\phi_1(\check{\underset{\sim}{\phi}}^r, \xi^r)}{M(\check{\underset{\sim}{\phi}}^r, \underset{\sim}{\alpha}^r, \xi^r)} \right)^{q-1} , \tag{116}$$

$$v_i^r = \left(\frac{\phi_i(\check{\underset{\sim}{\phi}}^r, \underset{\sim}{\alpha}^r, \xi^r)}{M(\check{\underset{\sim}{\phi}}^r, \underset{\sim}{\alpha}^r, \xi^r)} \right)^{q-1} , \quad i \in K, \tag{117}$$

where $\check{\underset{\sim}{\phi}}^r$ is the optimum point for the rth optimization. The margin ξ^r is updated to be

$$\xi^{r+1} = M(\check{\underset{\sim}{\phi}}^r, \underset{\sim}{\alpha}^r) . \tag{118}$$

4.12 Minimax approximation via linear programming

Ishizaki and Watanabe [116] had the objective function M as (5), where the error function is as defined by (1) and (2). They transformed the problem to a nonlinear program of the form of (20) and (21), with the difference that the upper and lower specifications coincide, and an additional constraint

$$\phi_j/\phi_j^0 \geq 0 , \qquad j = 1, 2, \ldots, k . \tag{119}$$

The last constraint is to prevent ϕ_j from changing sign during the iteration process. By taking the first-order approximation to the constraints at a point $\underset{\sim}{\phi}^r$, the problem is reduced to a linear program, which is given by

$$\text{minimize } \chi_{k+1}$$

subject to

$$w_i \sum_{j=1}^{k} \phi_j \frac{\partial F_i(\underset{\sim}{\phi}^r)}{\partial \phi_j} \chi_j - \chi_{k+1} + e_i(\underset{\sim}{\phi}^r) \leq 0, \; i \in I , \qquad (120)$$

$$-w_i \sum_{j=1}^{k} \phi_j \frac{\partial F_i(\underset{\sim}{\phi}^r)}{\partial \phi_j} \chi_j - \chi_{k+1} - e_i(\underset{\sim}{\phi}^r) \leq 0, \; i \in I , \qquad (121)$$

$$- \chi_j \leq 1, \quad j = 1, 2, \ldots, k , \qquad (122)$$

where

$$\chi_j = \frac{\Delta\phi_j}{\phi_j} , \quad \chi_{k+1} = \phi_{k+1}$$

and $F_i(\underset{\sim}{\phi})$ is the approximating function (or the response function).

The superscript r denotes the iteration number of a sequence of linear programming problems. The linear program is solved by the simplex method. Some examples which include the design of attenuation and group delay equalizers have been presented. A discussion of this method is also presented by Temes and Calahan [161].

Bandler et al. [43] developed the grazor search method for nonlinear minimax optimization. The method is based on a linear programming problem which uses gradient information of one or more near maximum functions to produce a downhill direction followed by a linear search to find a minimum in that direction. They first define a subset $J \subset I$ such that

$$J(\underset{\sim}{\phi}^j, \varepsilon^j) = \{i \mid M_f(\underset{\sim}{\phi}^j) - f_i(\underset{\sim}{\phi}^j) \leq \varepsilon^j, \ i \in I\} , \tag{123}$$

$$\varepsilon^j \geq 0 , \tag{124}$$

where $\underset{\sim}{\phi}^j$ denotes a feasible point at the beginning of the jth iteration and ε^j is the tolerance with respect to the current $M_f(\underset{\sim}{\phi}^j)$ within which the f_i for $i \in J$ lie. Linearizing f_i at $\underset{\sim}{\phi}^j$

$$\delta \ f_i(\underset{\sim}{\phi}^j) = \underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^j) \ \Delta \ \underset{\sim}{\phi}^j \ , \quad i \in J(\underset{\sim}{\phi}^j, \varepsilon^j) \ . \tag{125}$$

To get $\Delta \ \underset{\sim}{\phi}^j$ in the descent direction for $M(\underset{\sim}{\phi}^j)$

$$\underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^j) \ \Delta \ \underset{\sim}{\phi}^j < 0, \qquad i \in J(\underset{\sim}{\phi}^j, \varepsilon^j) \ . \tag{126}$$

Considering

$$\Delta \ \underset{\sim}{\phi}^j = - \sum_{i \in J} \alpha_i^j \ \underset{\sim}{\nabla} \ f_i(\underset{\sim}{\phi}^j), \tag{127}$$

$$\sum_{i \in J} \alpha_i^j = 1, \tag{128}$$

$$\alpha_i^j \geq 0, \tag{129}$$

(126) can be written as

$$- \underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^j) \sum_{i \in J} \alpha_i^j \ \underset{\sim}{\nabla} \ f_i(\underset{\sim}{\phi}^j) < 0. \tag{130}$$

This inequality suggests the linear programming problem

$$\text{maximize } \alpha_{k+1}^j \ (\underset{\sim}{\phi}^j, \varepsilon^j) \geq 0$$

subject to

$$- \underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^j) \sum_{i \in J} \alpha_i^j \ \underset{\sim}{\nabla} \ f_i(\underset{\sim}{\phi}^j) \leq - \alpha_{k_r+1}^j \tag{131}$$

and subject to (128) and (129). K_r denotes the number of elements of $J(\underset{\sim}{\phi}^j, \varepsilon^j)$. A golden section search follows each linear program to obtain $\underset{\sim}{\phi}^{j+1}$.

Madsen et al. [127,129], developed two minimax algorithms based on successive linearizations of the nonlinear functions and the resulting linear systems are solved in the minimax sense. At the rth stage of the first algorithm a minimax solution $\Delta \underset{\sim}{\phi}^r$ to the linearized system is found subject to the constraints

$$||\Delta \underset{\sim}{\phi}^r|| = \max_j |\Delta \phi_j^r| \leq \lambda^r , \tag{132}$$

where λ^r is automatically adjusted during the process to satisfy the inequality

$$M_f(\underset{\sim}{\phi}^r + \Delta \underset{\sim}{\phi}^r) < M_f(\underset{\sim}{\phi}^r) , \tag{133}$$

so that the new point becomes

$$\underset{\sim}{\phi}^{r+1} = \underset{\sim}{\phi}^r + \Delta \underset{\sim}{\phi}^r . \tag{134}$$

The choice of λ^r gives the flexibility of taking a large step if the linear approximations represent the nonlinear functions well enough. If the decrease in the maximum function (the nonlinear one) exceeds a small multiple of the decrease predicted by the linear approximations (the maximum of the linearized functions) then $\underset{\sim}{\phi}^{r+1}$ remains $\underset{\sim}{\phi}^r$.

The second algorithm is similar to the first one but does not require derivatives. It uses the Broyden updating formula [58]

$$\underset{\sim}{B}^{r+1} = \underset{\sim}{B}^r + \frac{(f(\underset{\sim}{\phi}^r+\Delta\underset{\sim}{\phi}^r) - f(\underset{\sim}{\phi}^r) - \underset{\sim}{B}^r\Delta\underset{\sim}{\phi}^r)\,\Delta \underset{\sim}{\phi}^{r^T}}{\Delta\underset{\sim}{\phi}^{r^T}\,\Delta\underset{\sim}{\phi}^r} \tag{135}$$

to approximate the derivatives, where the initial approximation $\underset{\sim}{B}^0$ is obtained by perturbation.

Comparison by Madsen et al. of the new algorithms with existing ones has been reported. Design of microwave reflection amplifiers was also carried out. Madsen and Schjaer-Jacobsen [128] treated common singularities in nonlinear minimax problems by modifying the first algorithm. They developed an automatic procedure to detect ill-conditioning and singularities in a given problem which slows convergence. Intuitively, the reason for slow convergence is that the upper bound on the step taken in each iteration is very small when a narrow valley is reached. However, a common feature of these algorithms is the claim that they have a quadratic final convergence.

4.13 Minimax optimization of constrained problems

Bandler and Srinivasan [42,155] suggested two unconstrained minimax objectives for a constrained minimax problem. The constrained problem is to minimize M_f of (86) subject to

$$g_j(\underset{\sim}{\phi}) \geq 0, \qquad j = 1, 2, \ldots, n_g \,. \tag{136}$$

The problem is reduced to

$$\text{minimize } \phi_{k+1}$$

subject to

$$\phi_{k+1} - f_i(\underset{\sim}{\phi}) \geq 0, \quad i \in I \tag{137}$$

and (136). The problem is then reformulated as an unconstrained minimax problem. One approach is to minimize M of (82) w.r.t. $\underset{\sim}{\phi}$ and ϕ_{k+1}, where

$$\Phi_1 = \phi_{k+1} \tag{138}$$

$$\Phi_{i+1} = \Phi_1 - \alpha_1(\Phi_1 - f_i(\underset{\sim}{\phi})) , \quad i = 1, 2, \ldots, n, \tag{139}$$

$$\Phi_{n+i+1} = \Phi_1 - \alpha_{i+1}\, g_i(\underset{\sim}{\phi}) , \quad i = 1, 2, \ldots, n_g, \tag{140}$$

where

$$\alpha_i > 0 , \quad i = 1, 2, \ldots, n_g + 1 . \tag{141}$$

and sufficiently large.

An alternative is to minimize M w.r.t. $\underset{\sim}{\phi}$ and to let

$$\Phi_i = f_i(\underset{\sim}{\phi}) , \quad i = 1, 2, \ldots, n , \tag{142}$$

$$\Phi_{n+i} = -w_i\, g_i(\underset{\sim}{\phi}), \quad i = 1, 2, \ldots, n_g , \tag{143}$$

where

$$w_i > 0 , \quad i = 1, 2, \ldots, n_g . \tag{144}$$

Zero values can be assigned to the weights w_i associated with satisfied constraints if $M > 0$. The objective M can be minimized using a suitable minimax algorithm. The generalized least pth objective can also be used as discussed in Sections 4.9 and 4.10. We have to note that these formulations are very well related to the discussion in Section 4.10 on Fig. 14(a) and Fig. 14(b).

Dutta and Vidyasagar [92] developed two algorithms for solving the nonlinear constrained minimax problem. They are principally a generalization of Morrison's least squares algorithm [132] and are quite similar to the ξ algorithm by Bandler and Charalambous with different updating formulas.

4.14 Other methods

Charalambous and Conn [74] proposed a minimax optimization algorithm which overcomes the difficulty of discontinuities in the objective's first derivatives. The direction of search at each iteration has two components. The first, the horizontal component, tries to keep locally the same set of the functions near active and at the same time reduce the objective function. The second, the vertical component, attempts to satisfy the near active functions exactly by means of linearization. A linear search follows the horizontal component calculation.

Einarsson [93] employed the modified Lagrangians [147] (augmented Lagrangians) in solving minimax problems. In his formulation an assumed active function is to be minimized w.r.t. $\underset{\sim}{\phi}$ subject to n-1 nonlinear constraints. If this function is, for example, $f_1(\underset{\sim}{\phi})$ the constraints will be

$$f_i(\underset{\sim}{\phi}) - f_1(\underset{\sim}{\phi}) \leq 0, \qquad i = 2, \ldots, n. \tag{145}$$

The Hestenes-Powell [111,142] method is used for updating the multipliers. This method requires the constraints to be equalities. The algorithms developed are thus based on knowing the active set of constraints in advance.

Rabiner et al. [144] used the weighted Chebyshev approximation for the design of finite-duration impulse response digital filters. The specification function for this problem is a sum of independent cosine functions. The problem is formulated so as to find the set of filter coefficients which minimize the maximum absolute value of the error function over a given set of frequencies. Another method for the design is the Remez exchange algorithm which solves the Chebyshev approximation problem by searching for the extremal frequencies of the best approximation. A detailed discussion of the approach and the design examples are given by Rabiner et al. [144]. These methods have been considered and applied to the design of recursive digital filters by

Deczky [82].

4.15 Discrete optimization

Steiglitz [158] used discrete optimization for the design of a short-word recursive digital filter. The response function is the absolute value of the transfer function of the filter (54), with the coefficients b_i having the value 1, while the error functions are of the form of the absolute value of (1). The constraints on the coefficients are

$$
\begin{aligned}
&|a_i| < 2 \, , \\
&d_i < 1 \, , \\
&1 + d_i + c_i > 0 \, , \\
&1 + d_i - c_i > 0 \, ,
\end{aligned}
\tag{146}
$$

and

$$2^Q a_i, \; 2^Q c_i \text{ and } 2^Q d_i \text{ are integers,}$$

where Q+1 is the word length of the filter.

Steiglitz formulated the objective in a minimax sense and used the Hooke and Jeeves [114] pattern search for the optimization in conjunction with a random numbering of the coordinates. He compared his results with the ones obtained by Avenhaus and Schuessler [6], where they designed the filter in the minimax sense, rounded the filter's coefficients and then applied a discrete optimization method to obtain the short-word filter.

Charalambous and Best [73] used the branch and bound technique due to Dakin [80] in the design of recursive digital filters with finite word length. An optimization algorithm by Best and Ritter [47,48] was used to solve the continuous subproblems involving a nonlinear objective function and linear constraints.

Bandler et al. [18] also applied discrete optimization to the design of digital filters with the aim of optimizing the word length. The continuous nonlinear programming problem is formulated as a minimax problem using the approach discussed in Section 4.11. The branch and bound technique is used to discretize the continuous solution. An older version of the user-oriented package DISOPT3 [41], namely DISOPT [27], where approaches in Section 4.10 and 4.11 are implemented, was used for optimization.

Charalambous [66] proposed an algorithm for discrete optimization which involves finding a hypercube in which the discrete optimum will lie. If the number of variables is k, then 2k unconstrained optimizations (at most) are performed to find the extreme values (maximum and minimum) of the variables which bound the feasible region. The extreme points provide bounds on the variables and are used along with the original constraints.

5. Centering, tolerancing and tuning

In the classical design problem we are interested in finding one single point in the feasible region. This kind of solution is impractical from the manufacturing point of view. Many other points (design outcomes) can also meet the required specifications. The designer can take advantage of this fact and assign tolerances on component values [10,110,120,153] so as to minimize production cost. The cost of a component may be assumed, for example, to be inversely proportional to the tolerance associated with it.

The formulation of the design problem considering manufacturing tolerances, post-production tuning and model uncertainties, besides the objective of reducing the cost, renders the design more practical and tends to alleviate realization problems. Phenomena which can be considered are:

a) model uncertainties

Equivalent circuits are used to model actual circuits. The parameters of equivalent circuits usually have uncertainties associated with them.

b) parasitic effects

These parasitics can substantially alter the ideal circuit performance and should be taken into consideration where possible. They are marked in many analog electrical circuits (active, high frequency, etc.).

c) environmental uncertainties

Some circuits have to meet stringent specifications for a variety of different environmental conditions. Military and telephone equipment, for example, often has to be designed for extreme temperatures.

d) mismatched terminations

Network terminations or loads may be substantially different from ideal.

e) material uncertainty

In practice, during circuit fabrication components are either specially made, chosen randomly or selectively from stock. Figure 18 [139] shows some typical distributions either actually encountered or considered during the design process for electrical circuit components. The ultimate aim of tolerance assignment is, consequently, to define a region, in which every point represents an outcome, taking into consideration the aforementioned concepts. All the outcomes, or at least a large percentage, have to meet the specifications, after tuning if necessary.

5.1 The tolerance orthotope [30,33]

Consider the vector of nominal design parameters

$$\underset{\sim}{\phi}^0 \triangleq \begin{bmatrix} \phi_1^0 \\ \phi_2^0 \\ \vdots \\ \phi_k^0 \end{bmatrix} , \tag{147}$$

defining a nominal point and a vector of associated manufacturing tolerances

$$\underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_k \end{bmatrix} , \tag{148}$$

described as the tolerance vector and let

$$I_\phi \triangleq \{1, 2, \ldots, k\} \quad , \tag{149}$$

where k represents the number of network design parameters, assumed independent for simplicity in the ensuing presentation.

A nominal point $\underset{\sim}{\phi}^0$ will have a tolerance region R_ε associated with it defined, under the assumption of independent variables, as

$$R_\varepsilon \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi}^0 - \underset{\sim}{\varepsilon} \leq \underset{\sim}{\phi} \leq \underset{\sim}{\phi}^0 + \underset{\sim}{\varepsilon}\} \quad . \tag{150}$$

This region is a convex regular polytope of k dimensions with sides of length $2\varepsilon_i$, $i \in I_\phi$, and centered at $\underset{\sim}{\phi}^0$. The extreme points of the tolerance region, the vertices, are

$$R_v \triangleq \{\underset{\sim}{\phi} \mid \phi_i = \phi_i^0 + \varepsilon_i \mu_i,\ \mu_i\{-1, 1\},\ i \in I_\phi\} \quad , \tag{151}$$

and the index set of the vertices

$$I_v \triangleq \{1, 2, \ldots, 2^k\} \quad . \tag{152}$$

Any point in the tolerance region is a possible outcome given by a point $\underset{\sim}{\phi}$, which is

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu} \quad , \tag{153}$$

where

$$\underset{\sim}{E} \triangleq \begin{bmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \ddots & \\ & & & \varepsilon_k \end{bmatrix} \tag{154}$$

and $\underset{\sim}{\mu} \in R_\mu$, where

$$R_\mu = \{\underset{\sim}{\mu} \mid -1 \leq \mu_i \leq 1,\ i \in I_\phi\} \quad . \tag{155}$$

Figure 19 depicts a tolerance region inscribed in the constraint region for a two-dimensional case. In general,

$$R_c \triangleq \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) \geq 0,\ i \in I_c\} \quad , \tag{156}$$

where

$$I_c \triangleq \{1, 2, \ldots, m_c\} \quad , \tag{157}$$

is the index set for the performance specifications (response constraints) and other parameter constraints, m_c being the total number of constraints.

5.2 Worst-case design

In worst-case design the whole tolerance region has to lie in the constraint region, i.e., it is required that

$$R_\varepsilon \subset R_c \quad . \tag{158}$$

This is design with 100% yield, where the yield is defined by

$$Y \triangleq \frac{\text{number of outcomes which meet specifications}}{\text{total number of outcomes}} \quad .$$

The 2^k vertices of the tolerance region are usually the points considered as candidates for worst case. There are two main reasons. The first is that it is impractical, or even impossible, to consider explicitly the infinite number of points contained in the tolerance region. The second is that assuming one-dimensional convexity of the constraint region, Bandler [10] proved that it is sufficient for worst-case design to require that

$$R_v \subset R_c \quad . \tag{159}$$

Bandler and Liu [31] showed that the one-dimensional convexity assumption often holds for linear circuits whose frequency response is biquadratic (bilinear complex) in each component. Brayton et al. [57]

later proved the same concept for the special D.C. case. See Section 3.1.

5.3 Fixed tolerance problem

In this problem we want to find $\underset{\sim}{\phi}^0$, the center of the tolerance region, where the manufacturing tolerances on the components are held fixed. The problem is basically a centering problem.

Let us consider a problem with upper and lower performance specifications. The error functions in this case are

$$e_{ui}(\underset{\sim}{\phi}^j) \triangleq w_{ui}(F_i(\underset{\sim}{\phi}^j) - S_{ui}), \; i \in I_u, \; j \in I_v, \tag{160}$$

$$e_{\ell i}(\underset{\sim}{\phi}^j) \triangleq w_{\ell i}(F_i(\underset{\sim}{\phi}^j) - S_{\ell i}), \; i \in I_\ell, \; j \in I_v, \tag{161}$$

where j denotes the jth vertex contained in I_v, and $\underset{\sim}{\phi}^j$ is this vertex. According to a specified vertex numbering scheme, each j will have a corresponding $\underset{\sim}{\mu}$. Any of the objective functions discussed in Section 2 can be reformulated to incorporate these error functions and then minimized to obtain the optimal $\underset{\sim}{\phi}^0$. We have to note that a worst-case design, in this case, is not necessarily achievable since we might not be able to inscribe the whole tolerance region, with preselected fixed edges, in the constraint region.

5.4 Variable tolerance problem

In many cases the manufacturing tolerances are considered as variables instead of fixed. The larger they are the cheaper the circuit components will be. The design problem is reformulated as a nonlinear program [10,11,140] as follows:

$$\text{minimize } C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon})$$

w.r.t. $\underset{\sim}{\phi}^0$ and $\underset{\sim}{\varepsilon}$ subject to

$$\underset{\sim}{\phi} \in R_c \text{ for all } \underset{\sim}{\mu} \in R_\mu \ , \tag{162}$$

where $\underset{\sim}{\phi}$ is as given in (153), and

$$\underset{\sim}{\phi}^0, \ \underset{\sim}{\varepsilon} \geq \underset{\sim}{0} \ . \tag{163}$$

The objective function C is directly related to the component cost, and generally possesses the properties

$$C\ (\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}) \rightarrow \text{constant} \qquad \text{as } \underset{\sim}{\varepsilon} \rightarrow \underset{\sim}{\infty} \ , \tag{164}$$

$$C\ (\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}) \rightarrow \infty \qquad \text{as } \varepsilon_i \rightarrow 0 \ . \tag{165}$$

A common form of this objective is

$$\sum_{i=1}^{k} c_i \frac{\phi_i^0}{\varepsilon_i} \ , \tag{166}$$

where the c_i are constant weights. The number of variables for the optimization is 2k, namely, k independent nominal variables and k associated tolerances.

For large problems, with a large number of variables, the number of vertices of the tolerance region becomes enormous. Selection schemes which include purging as well as addition of vertices of the tolerance region during the optimization process alleviate the need for considering the 2^k vertices [32,34]. One of these schemes [34] is based on the iterative solution of necessary conditions for the worst vertex derived from the Kuhn-Tucker conditions. Efficient selection schemes relevant to the tolerance problem are still not well developed.

The tolerance problem described here implicitly solves the centering problem, in which we are interested in finding a "center" of the constraint region. Another approach is one recently developed by Director and Hachtel [86]. It is concerned with finding the center of the largest hypersphere inscribed in the constraint region. In the process, an internal approximation to the region is obtained. The problem of finding the largest hypersphere is solved by a sequence of linear programming problems. Butler [59,60] handled the design centering problem by considering constraint regions in two-dimensional subspaces corresponding to various pairs of design parameters. The boundaries of these regions are described by performance contours. These contours are supposed to provide the designer with insight directed at moving the nominal design towards the center of R_c.

5.5 Tolerancing and tuning

Tuning some of the components after production is quite common in electrical circuit fabrication. Considering independent tuning in the design procedure, a tuned design will imply $\underset{\sim}{\phi}$ such that

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu} + \underset{\sim}{T}\,\underset{\sim}{\rho} \quad , \tag{167}$$

for some $\underset{\sim}{\rho} \in R_\rho$, with

$$\underset{\sim}{T} \triangleq \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_k \end{bmatrix} . \tag{168}$$

An example of R_ρ is

$$R_\rho = \{\underset{\sim}{\rho} \mid -1 \leq \rho_i \leq 1,\ i \in I_\phi\} \quad . \tag{169}$$

The corresponding tuning region is defined as

$$R_t(\underset{\sim}{\mu}) = \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu} - \underset{\sim}{t} \leq \underset{\sim}{\phi} \leq \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu} + \underset{\sim}{t}\} , \qquad (170)$$

which is centered at $\underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}$. Figure 20 illustrates the constraint, tolerance and the tuning regions.

The design problem in this case is

$$\text{minimize } C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t})$$

subject to (162) where $\underset{\sim}{\phi}$ is as given in (167), and the constraints $\underset{\sim}{\phi}^0$, $\underset{\sim}{\varepsilon}$, $\underset{\sim}{t} \geq \underset{\sim}{0}$ for all $\underset{\sim}{\mu} \in R_\mu$ and some $\underset{\sim}{\rho} \in R_\rho$. C is a function which represents the component cost, for example,

$$\sum_{i=1}^{k} c_i \frac{\phi_i^0}{\varepsilon_i} + \sum_{i=1}^{k} c_i' \frac{t_i}{\phi_i^0} , \qquad (171)$$

where the c_i and c_i' are constants. These may be set to zero if the corresponding element is not to be toleranced or tuned, respectively. The worst-case solution of the problem must satisfy

$$R_t(\underset{\sim}{\mu}) \cap R_c \neq \emptyset \qquad (172)$$

for all $\underset{\sim}{\mu} \in R_\mu$, where $\emptyset$ denotes a null set.

The problem can be reduced by separating the components into effectively tuned and effectively toleranced parameters. Bandler et al. [33,125] proved that the solution of the reduced problem is the solution of the original one under certain conditions.

The tuning problem can be treated in a more general way by distinguishing between manufacturer and customer tuning. The ith component of the vector $\underset{\sim}{\phi}$ denoting a point is given by

$$\phi_i = \phi_i^0 + \varepsilon_i \mu_i + t_{mi} \rho_{mi} + t_{ci} \rho_{ci} , \qquad (173)$$

where subscripts m and c stand for manufacturer and customer, respectively. The components tuned by the manufacturer may be sealed after the product is tested and meets the specifications. Other tunable components may be left to the customer. The product performance, for example, may deteriorate due to aging. In other cases the customer tunable parameter might be for providing the flexibility of meeting various sets of specifications.

A tunable constraint region is denoted by $R_c(\underset{\sim}{\psi})$. Figure 21 depicts three different regions of an example of $R_c(\underset{\sim}{\psi})$. Overlapping of these regions is possible. The number of these regions may be infinite if the independent variable is continuous. This can occur, for example, if the constraint region is a function of temperature, which can take any value between certain upper and lower limits. In other cases, the values of $\underset{\sim}{\psi}$ can be discrete leading to distinct constraint regions.

Tuning, in practice, of more than one parameter does not have to be independently carried out. It is possible to have ganged parameters tuned simultaneously.

A highpass filter problem originally considered by Pinel and Roberts [140] is shown in Fig. 22. This provides an example to illustrate centering, tolerancing and tuning [33]. The insertion loss relative to the loss at 990 Hz is the response function $F(\underset{\sim}{\phi}, \psi)$ to meet given specifications where ψ is frequency in Hz. Constraints on resistances R_5 and R_7 relate them to the inductances L_5^0 and L_7^0 with a constant quality factor Q = 1456, where $Q = 2\pi \times 990 \times L^0/R$. The objective function considered is

$$C = \sum_{i=1}^{7} \frac{\phi_i^0}{\varepsilon_i} , \tag{174}$$

where

$$\underset{\sim}{\phi}^{0} = \begin{bmatrix} C_1^0 \\ C_2^0 \\ C_3^0 \\ C_4^0 \\ L_5^0 \\ C_6^0 \\ L_7^0 \end{bmatrix}, \quad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{C_1} \\ \varepsilon_{C_2} \\ \varepsilon_{C_3} \\ \varepsilon_{C_4} \\ \varepsilon_{L_5} \\ \varepsilon_{C_6} \\ \varepsilon_{L_7} \end{bmatrix} . \tag{175}$$

Fifty sample points ψ_i were considered along the ψ axis. The package DISOPT [27] was used for optimization. Note that the problem has 2^7 vertices to be considered at each frequency point. The total number of variables, including the tolerances and seven elements, is 14. The problem was run first without any tuning, considering 31 response constraints and 14 constraint bounds on the variables. It needed 3 trial runs to derive insight into the problem and to choose the 31 response constraints, and the solution was reached in 15 min. of CPU time on a CDC 6400.

The problem was also run after introducing 3 percent tuning for L_5. The number of response constraints considered was 37, and the results were achieved in 5 min. All the vertices were checked plus the nominal point, followed by Monte Carlo simulations uniformly distributed in the effective tolerance region, and no violations were detected. Figures 23(a) and 23(b) show the nominal and the worst-case responses in the passband and stopband of the filter.

5.6 Uncertainties

The values of $\underset{\sim}{\phi}$ sufficient to give an acceptable design depend on other uncertainties influencing design performance. In the simulation of actual circuits models or equivalent circuits are used, where uncertainties are associated with the model parameters. In microwave circuit design, for example, parasitic effects exist due to electromagnetic coupling. Models available for common parasitic elements normally include empirical uncertainties on the values of the model parameters. These uncertainties are due to the fact that the model itself is necessarily approximate and that further approximations often have to be made in the implementation of existing model formulas. Non-ideal terminations also alter the performance, i.e., mismatches at the source and the load of the circuit.

In modeling a physical circuit the vector of nominal model parameters $\underset{\sim}{p}^0$ will have a vector of model uncertainties associated with it, such that the model parameters are described by

$$\underset{\sim}{p} = \underset{\sim}{p}^0(\underset{\sim}{\phi}) + \underset{\sim}{\Delta}(\underset{\sim}{\phi})\, \underset{\sim}{\mu}_\delta \quad , \tag{176}$$

where

$$\underset{\sim}{\Delta} \triangleq \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \ddots & \\ & & & \delta_n \end{bmatrix} \tag{177}$$

and for example

$$-\underset{\sim}{1} \leq \underset{\sim}{\mu}_\delta \leq \underset{\sim}{1} \quad ,$$

where n is the number of model parameters with uncertainties and in general $n \neq k$. Although the model parameters and the uncertainties are explicit functions of the physical parameters $\underset{\sim}{\phi}$, it is difficult to map the tolerance region from the $\underset{\sim}{\phi}$ space to the $\underset{\sim}{p}$ space in selecting candidates for the worst-case design. Recently, Tromp [165] attempted to generalize the formulation of the tolerance problem to take into consideration model uncertainties.

Let $\underset{\sim}{g}(\underset{\sim}{\psi})$ denote a set of nonlinear constraint functions such that

$$\underset{\sim}{g}(\underset{\sim}{\psi}) \geq \underset{\sim}{0} \quad , \tag{178}$$

represents an acceptable situation for a particular setting of $\underset{\sim}{\psi}$. The nominal performance of the design under ideal environmental effects will be denoted by $\underset{\sim}{g}^0(\underset{\sim}{\psi})$. The actual performance might be described by

$$g_i = g_i^0 (\underset{\sim}{p}, \underset{\sim}{\psi}) + \mu_{g_i} (\underset{\sim}{p}, \underset{\sim}{q}, \underset{\sim}{\psi}), \; i=1,2,\ldots,m(\underset{\sim}{\psi}) \quad , \tag{179}$$

where

$\underset{\sim}{\mu}_g$ is the deviation from the ideal performance ,

$\underset{\sim}{q}$ is a vector of external parameters.

The worst-case problem considering tolerancing, manufacturer and customer tuning and uncertainties can be stated as

$$\text{minimize } C (\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t}_m, \underset{\sim}{t}_c, \underset{\sim}{q}) \; ,$$

where C is an appropriate cost function, subject to

$$\underset{\sim}{\phi} \in R_c(\underset{\sim}{\psi}) \tag{180}$$

for all permissible $\underset{\sim}{\mu}$ and $\underset{\sim}{\psi}$ and some permissible $\underset{\sim}{\rho}_m$ and $\underset{\sim}{\rho}_c$, where the constraint region is given by

$$R_c(\underset{\sim}{\psi}) \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{g}(\underset{\sim}{\phi}, \underset{\sim}{\psi}) \geq \underset{\sim}{0} \text{ for all permissible } \underset{\sim}{\mu}_g, \underset{\sim}{\mu}_\delta\} \tag{181}$$

A realistic example of a one-section transformer on stripline was considered by Bandler et al. [35]. The physical circuit and its equivalent (model) are shown in Fig. 24. Thirteen physical parameters, implying 2^{13} vertices, are

$$\underset{\sim}{\phi} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \ell \\ \sqrt{\varepsilon_{r_1}} \\ \sqrt{\varepsilon_{r_2}} \\ \sqrt{\varepsilon_{r_3}} \\ b_1 \\ b_2 \\ b_3 \\ t_{s_1} \\ t_{s_2} \\ t_{s_3} \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{variable nominal and variable tolerances} \\ \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} \text{fixed nominal and fixed tolerances} \end{matrix} \tag{182}$$

where

w is the strip width,

ℓ is the length of the middle section,

ε_r is the dielectric constant,

t_s is the strip thickness,

b is the substrate thickness.

Tolerances on ε_r, b and t_s were imposed independently for the three lines allowing independent outcomes. Nominal values for corresponding parameters were the same throughout. Six model parameters implying 2^6

extreme points were

$$p = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ L_1 \\ L_2 \\ \ell_t \end{bmatrix} , \qquad (183)$$

where

D is the effective linewidth,

L is the junction parasitic inductance,

ℓ_t is the effective section length.

The model parameters are functions of the physical parameters with uncertainties on these functions.

Mismatches at the source and load, for example due to poor connectors, were also taken into consideration. A maximum mismatch reflection coefficient (response function of interest) of 0.025 and arbitrary phase was chosen w.r.t. the given source and load resistances.

A worst case study was made to select a reasonable number of constraints from the possible $2^{19} = 2^{13} \times 2^6$. The vertex selection procedure for the 13 physical parameters followed Bandler et al. [32]. From each of the selected vertices the worst values of the modeling parameters were chosen. Only the band edges were used during optimization. After each optimization the selection procedure was repeated, new constraints being added, if necessary. Figure 25 shows the final result of this example.

5.7 Design with yield less than 100 percent

In worst-case design the yield is restricted to 100%. This may render the circuit very expensive due to tight tolerances. The restriction of 100% yield may be relaxed in order to increase the tolerances and reduce the cost of the elements. The overall cost, in this case, although failing circuits are discarded, will have to be lower than the one obtained by worst-case design.

The design problem with a restricted yield can be set up as

$$\text{minimize } C\,(\underset{\sim}{\phi}^0,\ \underset{\sim}{\varepsilon})$$

subject to

$$Y \geq X \quad , \tag{184}$$

where X is the specified percentage. For unrestricted yield the problem might, for example, have the objective function

$$C = \sum_{i=1}^{k} \frac{1}{\varepsilon_i} \,/\, Y \quad . \tag{185}$$

In both formulations, the yield has to be estimated [1,2,14-16].

The yield when the parameters are statistically distributed is defined by

$$Y = \int_{R_c} P(\underset{\sim}{\phi})\; d\phi_1\, d\phi_2 \ldots d\phi_k \quad , \tag{186}$$

where $P(\underset{\sim}{\phi})$ is the probability distribution function of the variable parameters. This k-fold integration is not very attractive, especially when the yield estimation is incorporated in an optimization process. The yield estimation problem itself has been treated largely by the Monte Carlo analysis.

Becker and Jensen [45] used pattern search for maximizing the yield by finding a set of nominal variables which is optimal for specified tolerances. A feasible solution search precedes the yield optimization.

Elias [94] developed a program which maximizes the yield when statistics on the components are given. The sensitivity of the yield w.r.t. ϕ_i is computed by

$$S_i \triangleq \int_{-\infty}^{\infty} |P_{pass}(\phi_i) - P_{fail}(\phi_i)| \, d\phi_i \quad , \tag{187}$$

where

$P_{pass}(\phi_i)$ is the probability density function of circuits meeting specifications,

$P_{fail}(\phi_i)$ is the probability density function of circuits violating specifications.

The program adjusts the component specifications iteratively to maximize the yield. The improvement in the yield is calculated by repeating the Monte Carlo analysis.

In the simplicial approximation approach [87] while finding the center of the constraint region an approximation to this region is also obtained. A crude estimate of the yield can be obtained by performing the Monte Carlo analysis directly in the parameter space. The yield estimation procedure can be improved by testing a sample point (of the Monte Carlo analysis) which lies outside the approximate region to determine whether or not it lies outside the actual region. If it lies inside the actual region it can be used to improve the approximate region.

Karafin [120] approximated the yield by computing upper and lower bounds on Y using truncated Taylor series approximations for the constraints. He assumed that each constraint is normally distributed for all choices of component tolerances.

Bandler and Abdel-Malek [16] derived exact formulas for the yield and its sensitivities w.r.t. design parameters. The formulas are based upon multidimensional linear cuts of the tolerance orthotope and uniform distributions of outcomes between tolerance extremes in the orthotope. The yield is given by

$$Y = 1 - \frac{\sum_{\ell=1}^{m_c} V^{\ell}}{2^k \prod_{i=1}^{k} \varepsilon_i} , \qquad (188)$$

where the V^{ℓ} are nonfeasible nonoverlapping hypervolumes in the tolerance region. Each is chosen according to an ℓth constraint. The V^{ℓ} is obtained, for example, through intersections between the hypersurface $g_{\ell}(\tilde{\phi}) = 0$ and the orthotope edges. These intersections lead to a linear constraint approximation used to provide a multi-dimensional linear cut of the tolerance orthotope.

This approach has been generalized to estimate the yield when components have arbitrary statistical distributions. In this case, the tolerance orthotope is partitioned into a collection of orthotopic cells (orthocell) and a weight is assigned to each orthocell. A uniform distribution is assumed inside it. The weights are obtained from tabulated values for known distributions or according to sampling the components used. A formula for the weighted nonfeasible hypervolume V^{ℓ} with respect to the ℓth constraint is derived. Assuming nonoverlapping of nonfeasible regions defined by m_c different constraints inside the tolerance orthotope, the yield is expressed as

$$Y = 1 - \sum_{\ell=1}^{m_c} V^{\ell} \quad . \qquad (189)$$

5.8 Discrete tolerance optimization

Karafin [119] used a modified branch and bound approach for discrete tolerance optimization applied to filter design. The possible tolerances on each element are assumed to be given by a set of discrete values. The first step in the design is to obtain the minimum intercept for each parameter, or in other words how far the parameter can be moved from its nominal value before violating the specifications (or going outside the constraint region). The tolerances which are larger than the minimum intercept are automatically excluded. Karafin considered $\binom{k}{2}$ performance contours for the k parameters, which are regions formed by constraints in two-dimensional subspaces, where (k-2) parameters are held fixed at their nominal values. For each pair of parameters a table of two columns is constructed. The left column contains the set of tolerances for the parameter with smaller minimum intercept and the right column is the corresponding largest allowable tolerance within the set of tolerances of the other parameter.

Once these tables are set up a modified approach of the branch and bound method is used to obtain the tolerances which minimize a cost function. Karafin considered the multi-root tree structure of Fig. 26, where the nodes of the first column are tolerances of the first parameter. Each of these nodes is connected to all the tolerances of the second parameter, and so on until we get $\prod_{i=1}^{k} m_i$ nodes at the last column, where k is the number of parameters and m_i is the number of allowable tolerances of the ith parameter. Paths from left to right through the tree are examined to choose the path associated with minimum cost. A very large number of paths do not have to be explored to the end, being eliminated by simple terminating rules.

Bandler et al. [32] applied discrete optimization to worst-case circuit tolerance optimization along with design centering. The branch and bound technique was used in the form of Dakin's tree search [80]. A user-oriented package called TOLOPT [28] incorporates the algorithms which produced the results.

6. Practical industrial problems

Some of the optimization and design approaches discussed in this paper have been used in industry and research organizations to solve practical problems. In this section, we review a few relevant examples which have been solved or suggested. We briefly indicate the method used for carrying out the design. The examples in previous sections also fall into the category of practical problems.

6.1 Example 1: channel bank filter [141]

This example concerns a tolerance assignment and tuning problem. The circuit, a channel bank filter, is shown in Fig. 27. It consists of 25 elements, 7 inductors and 18 capacitors. The response function of interest is the insertion loss. Tuning was introduced in this problem to counteract the prevailing uncontrollable parasitic effects. A ±3% tuning range was assumed for the inductors while tolerances were associated with all capacitors. The objective function, resembling (166), reflected the sum of capacitor costs. Having specifications on the relative loss as performance constraints, a worst-case design increased the tolerances of five capacitors. When the cost of the ith capacitor was taken as ϕ_i^0/ε_i in the objective function a capacitor cost reduction of 13% was obtained.

Alternative realistic formulas for the component costs can be defined. The same problem solved with different cost functions will, of course, have different solutions. This problem illustrates the difficulties facing a manufacturer in attempting to define a realistic cost of a product based only on costs of individual components and their tolerances.

The authors [141] suggest the use of the generalized least pth or the interior penalty function approach for the cost function minimization.

6.2 Example 2: planar transducer array [123]

This is an example of a planar transducer array for monitoring ocean traffic. The array has to listen only in a certain direction (to be highly directional) by meeting two criteria. First, the main lobe has to be as narrow as possible and secondly all the side (minor) lobes must be below a specified level. These are two contradicting criteria. An additional requirement is that the array beam must be steerable to any angle in the plane of the array, which is parallel to the water's surface.

In this problem the response function is the directivity of the system and the independent variable is the angle α. The response (or the pattern) is symmetrical about the line $\alpha = 0$ so that $\alpha = [0 - 180]$ only is considered. The variables are the transducer scale of the wave amplitude (shading parameter), the radial location and the angular location.

The problem was formulated as a nonlinear programming problem by introducing an additional variable (Section 2.3). The problem was tried with a different number of arms. One of the problems had 14 arms and 6 transducers per arm and had a total of 91 variables. The initial pattern failed to meet the specifications by about 2.5 dB and the optimum met the specifications and exceeded it by 2 dB.

An interior penalty method was used. The additional variable was chosen sufficiently large to generate an interior point.

6.3 Example 3: microwave amplifier [94]

The microwave amplifier shown in Fig. 28 has been designed for 100% yield. The program TOLERATE [94], according to statistics on components, adjusts the component tolerances to maximize the circuit yield. Thus it specifies a device yield for the components or device parameters. The three capacitors C_1, C_2 and C_3 were optimized to let the circuit meet a specification on the insertion gain in the range 26 - 28 dB with a ripple less than 0.5 dB (peak to peak) in the frequency range 1.2 - 1.6 GHz. This was performed by the Fletcher-Powell algorithm incorporated in SLIC [115] (a general analysis program). TOLERATE identified the device parameters of the transistor with respect to which the circuit yield is most sensitive. Three transistor parameters were identified, namely, the current gain beta, the diffusion capacitance and the output resistance. At the beginning of yield maximization the circuit yield was 83% and reached 100% after two iterations, and the device yield was 79%. The procedure for yield maximization is discussed in Section 5.7.

6.4 Example 4: current switch emitter follower [113]

This is an example of the worst-case design of a CSEF (current switch emitter follower) in the time domain. The circuit is shown in Fig. 29. The variable parameters for the design were E_4, the d.c. voltage source, Z_0, the characteristic impedance of the transmission line, the resistance R_4 and the output capacitance C_0. The analysis was performed using the state equation approach. Gear's algorithm [102] was used for integration.

To prevent an unrealistic nominal value the output capacitor C_0 had to be constrained by $C_0^0 - \varepsilon_4 \geq 1.0$. The optimal worst case [1] nominal parameters E_4^0, Z_0^0, R_4^0, and C_0^0 are 1.655 V, 92.004 Ω, 45.533 Ω and 1.248 pF, respectively. The corresponding relative tolerances ε_i/ϕ_i^0 are 4.46, 8.29, 13.77 and 14.00 percent. The objective function is (166) with $c_i = 1$.

The approach for the design is discussed in Section 5.4. The multidimensional approximation approach was used in the optimization process. This avoided the evaluation of actual response sensitivities. The approximation region had to be updated once (one of the parameters drifted from the original approximation region) and 30 response evaluations were needed. The total approximation time including the response evaluations on a CDC computer was 48s, while the time needed for optimization was 55s. The nominal design response and responses for the active vertices are shown in Fig. 30.

6.5 Example 5: design of FIR digital filter

This example indicates one of the techniques used in the design of FIR filters [143]. The filter is lowpass and is assumed to have a symmetrical impulse response. Its N samples of duration are odd and equal to 41. The filter frequency response is

$$H(e^{j\omega T}) = h_0 + \sum_{n=1}^{(N-1)/2} 2h_n \cos(\omega nT) . \qquad (190)$$

The specifications are 1 in the passband with a ripple of 0.1 and 0.0 in the stopband with a ripple of 10^{-4}. The unknowns of the design problem are (N+1)/2 coefficients in the impulse response and (N-3)/2 frequencies at which extrema of the deviation error δ occur. (Since the response has ripples it can not be approximated by 1.0 or zero exactly and a deviation error is specified.) (N-1) nonlinear equations are obtained from the equations representing the response and its derivatives with respect to ω (zero at the extrema). These equations can be solved by any suitable nonlinear optimization technique.

7. Conclusions and directions for future research

Most available algorithms appear to be based on the assumption that the solution is to be obtained in one run of an optimization program. This goal is unrealistic in the case of engineering design problems. If the optimization process is stopped and we want to restart it, for example, information from the previous run is generally lost. Programs which can exploit available information and can efficiently restart the optimization process are highly desirable. The designer should have the possibility of supplying input information other than just the starting point.

All the algorithms which have been referred to in this paper are sequential algorithms. Besides the human mind, which has been taught to think sequentially, the computers which we now use are sequential machines. It will not be very long before parallel machines will be widely available. The parallel processors will be much faster than present machines. New optimization algorithms suitable for the new computing machines have to emerge or the existing ones have to be modified. In some of the ill-conditioned problems, for example, it is required to run the problem from different starting points, which might be considered as running different problems in parallel. The tolerance-tuning problem can also be formulated to fit a parallel algorithm because it is inherently a parallel problem.

We mentioned the approximation of nonlinear functions, for example, by multidimensional polynomials. Such approximations can be used in optimization, replacing the actual functions. The approach facilitates the design of circuits by exploiting large and general simulation programs to conduct the analyses. These programs, because they are general, cannot be efficient for every problem. The multidimensional approximation approach avoids calling the simulation program many times during optimization and, furthermore, is able to exploit sparsity features in the simulation program. In fact, since several simulations at fixed points might be required simultaneously, a parallel approach would be most effective.

Automated optimal design of circuits where the topology is not fixed has often been suggested. Presently, the location and type of components to be added to the given network have to be specified a priori. Fully automated design, where the topology can change arbitrarily, and criteria for augmenting or shrinking the circuit are not well established. This will need optimization methods where the number of variables can be automatically increased or decreased in an effective manner during the process.

Other design aids would be

(a) Algorithms which indicate and act upon the existence of symmetry.

(b) Algorithms which stack the constraints in the order of complexity to avoid unnecessary calculations, starting with sets of crude but not necessarily linear approximations.

(c) Algorithms which permit the flexibility of examining the effects of alternative objectives and weights without rerunning the whole problem each time a change is made.

In centering, tolerancing, and tuning problems several concepts need further development. Efficient vertex selection schemes will lead to an enormous reduction of the tolerance problem as well as the possibility of full automation of the whole process. This concept is related to determining active constraints.

Optimization in the presence of uncertainty can be applied to power system problems, where environmental fluctuations can cause failure of the system. The reliability problem is clearly an extension to the tolerance assignement problem. The main difference, we feel, is the redundancy which enhances system reliability and the observation that not every component fails simultaneously. Parameter changes to be considered might be much larger than in tolerance assignment.

Recent developments in optimization methods have helped electrical circuit and system designers to perform efficient optimal design and operation of some electrical networks. Reasonably large and practical problems have been solved. Nevertheless, improvements of problem formulations and further development of optimization methods are of great importance.

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Table 1

Types of Electrical Networks and Numerical Methods of Solution

Network Type		Analysis				Network Equations						Methods of Solution								Features
Broad electrical network classificiations are indicated in this column		D.C. (direct current)	A.C. (alternating current)	Transient	Steady state	Linear algebraic	Nonlinear algebraic	Linear O.D.E.	Nonlinear O.D.E.	Difference equations	Partial differential equations	Gauss, LU factorization	State equations	Newton-Raphson	Piecewise-linear	Numerical integration	Lumped model	Fast-Fourier transform	Special purpose techniques	Comments are made relating to the physical, mathematical and numerical nature of the analysis problem
	linear resistive	X				X						X								no energy storage
analog	linear dynamic		X			X						X								energy storage, time and complex frequency domains are related by Laplace transform
				X				X					X							
lumped	nonlinear resistive	X					X							X	X					no energy storage, contains nonlinear resistors
	nonlinear dynamic			X					X				X			X				energy storage, contains nonlinear capacitors and/or nonlinear inductors
					X				X										X	
analog distributed	linear commensurate and non-commensurate		X			X						X	X				X			time and complex frequency domains are related by Laplace transform
				X							X		X				X	X	X	
	nonlinear		X				X										X		X	
				X			X										X	X	X	
digital			X			X				X		X								time and frequency domains can be related by the z transform
				X						X									X	
power system					X		X							X					X	the system of equations is mildly nonlinear and very sparse

Table 2

Lower Bounds for the Seven-section Filter

Stopband specification (dB)	First maximum error (dB)	Predicted lower bound (dB)	Next maximum error (dB)
50	-0.0256	-0.0283	-0.0282
55	0.1430	0.1154	0.1160
60	0.6211	0.4954	0.4986
65	1.5486	1.3148	1.3195

Figure Captions

Fig. 1 A common approximation problem illustrating (a) a single response function, specification and weighting function, (b) the corresponding error function as defined by (1).

Fig. 2 An amplifier design problem indicating (a) an applied voltage $V_1(j\omega)$ and output voltage $V_2(j\omega)$, where ω is the frequency and $j = \sqrt{-1}$, (b) a possible gain specification for the amplifier.

Fig. 3 Possible upper and lower specifications showing (a) the response function of a bandpass filter violating upper and lower specifications, (b) an arbitrary response function satisfying the specifications.

Fig. 4 An example of multiple objectives in filter design, (a) the insertion loss specification in the frequency domain of a lowpass filter, (b) an impulse response specification in the time domain of the lowpass filter.

Fig. 5 Multidimensional specifications, (a) a possible specification for a two-dimensional digital filter, (b) upper and lower specifications for an amplifier to be designed to operate over a specified temperature range.

Fig. 6 A simple linear, time-invariant RLC circuit.

Fig. 7 A cascaded network, consisting of two-port subnetworks connected in cascade, with conventional directions of currents and voltages.

Fig. 8

$$\begin{bmatrix} G + \frac{1}{j\omega L_1} & -\frac{1}{j\omega L_1} & 0 \\ -\frac{1}{j\omega L_1} & j\omega C + \frac{1}{j\omega L_1} + \frac{1}{j\omega L_2} & -\frac{1}{j\omega L_2} \\ 0 & -\frac{1}{j\omega L_2} & G + \frac{1}{j\omega L_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_s \\ 0 \\ 0 \end{bmatrix}$$

A linear circuit (lowpass filter) and its nodal admittance equations at frequency ω.

Fig. 9

$$f_1(v_1,v_2,v_3) = -\frac{E-v_1}{R_1} + I_s(e^{\lambda(v_1-v_3)}-1) - I_s(e^{\lambda(v_2-v_1)}-1) = 0$$

$$f_2(v_1,v_2,v_3) = \frac{v_2-v_3}{R_2} + I_s(e^{\lambda(v_2-v_1)}-1) + I_s(e^{\lambda v_2}-1) = 0$$

$$f_3(v_1,v_2,v_3) = \frac{v_3-v_2}{R_2} - I_s(e^{\lambda(v_1-v_3)}-1) - I_s(e^{-\lambda v_3}-1) = 0$$

A nonlinear resistor-diode circuit and the nonlinear equations which have to be solved when the diodes are identical [79]. The variables v_1, v_2 and v_3 are the node voltages.

Fig. 10 Two possible realizations for the digital transfer function $\frac{a + b z^{-1}}{1+cz^{-1}+dz^{-2}}$.

Fig. 11 A recursive digital filter in parallel form.

Fig. 12 An example of a small power network [52].

Fig. 13 Response of a filter designed by the interior penalty approach of Lasdon and Waren [122].

Fig. 14 Contours of a least pth objective function (81) (a) without inclusion of constraints, (b) with one parameter constraint.

Fig. 15 A three-section 10:1 lossless transmission-line transformer. Z_i is the characteristic impedance and ℓ_i is the length of the ith section.

Fig. 16 A seven-section microwave filter consisting of lossless transmission lines, short-circuit and open-circuit transmission lines.

Fig. 17(a) Response of the filter optimized by the p algorithm with 21 uniformly spaced passband points (only 10 were actually employed because of symmetry).

Fig. 17(b) Responses of the filter after two optimizations with the ξ algorithm for p=2 using the ripple maxima from Fig. 17(a). Four sets of specifications were considered.

Fig. 18 Possible statistical distributions for electrical circuit components which are often assumed at the design stage [139].

Fig. 19 A tolerance region R_ε inscribed in the constraint region R_c. If $\underset{\sim}{\varepsilon} = \underset{\sim}{0}$ the conventional nominal design problem is implied.

Fig. 20 An illustration of the constraint, tolerance and tuning regions and a possible outcome $\underset{\sim}{\phi}$. If $\underset{\sim}{t} = \underset{\sim}{0}$ we recover the essential features of Fig. 19.

Fig. 21 An example of a tunable constraint region $R_c(\underset{\sim}{\psi})$ indicating three distinct possibilities.

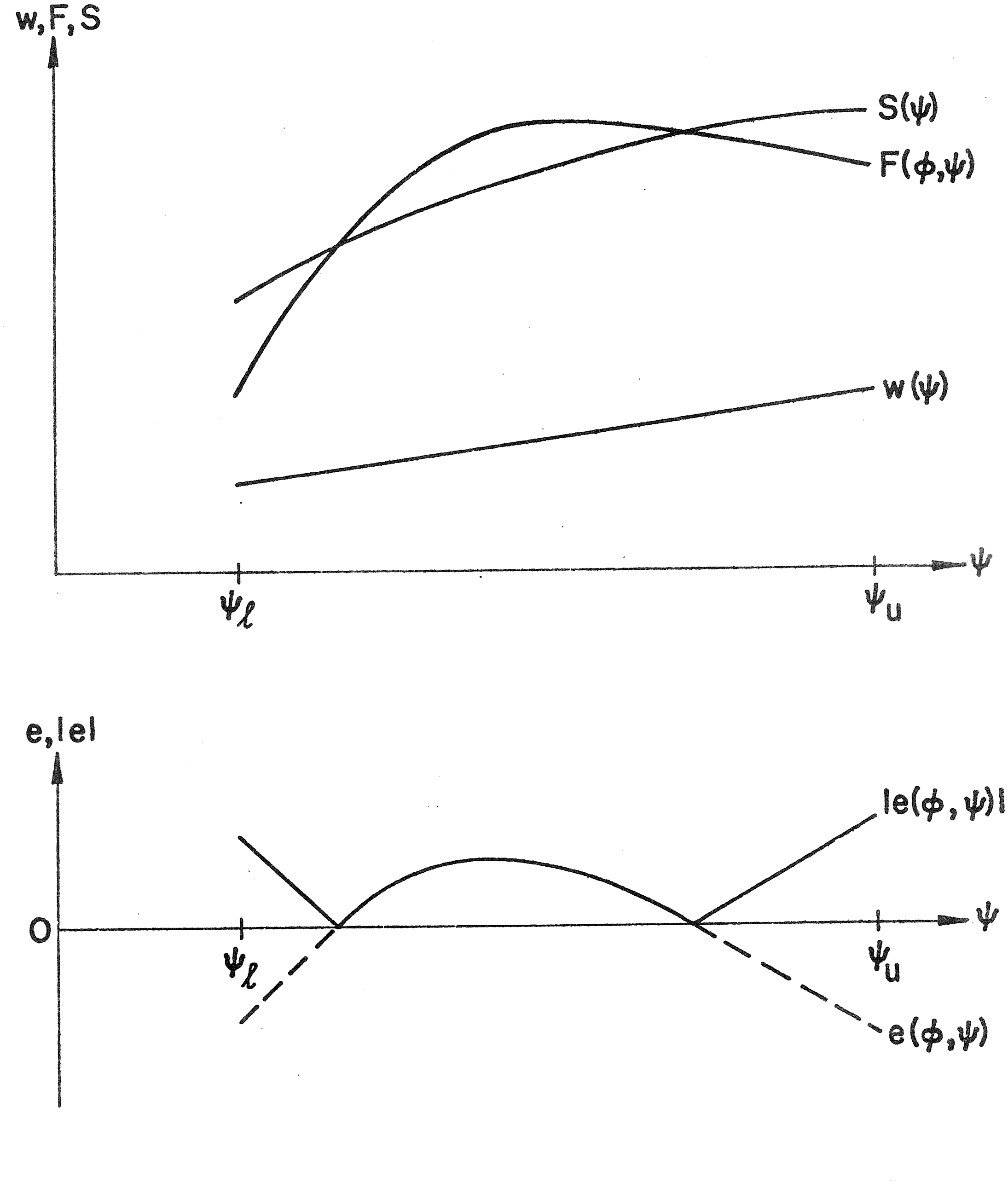
Fig. 22 Highpass filter due to Pinel and Roberts [140].

Fig. 23 The nominal response function (relative insertion loss) of the optimized highpass filter of Fig. 22 and the worst-case response in (a) the passband and (b) the stopband.

Fig. 24 A microwave stripline transformer showing (a) physical parameters and (b) the equivalent electrical circuit [35].

Fig. 25 Final results for the stripline transformer. The letters a, b, ..., f indicate different vertices (designs) determining the worst case in different frequency bands.

(a)



(b)

Fig. 1.

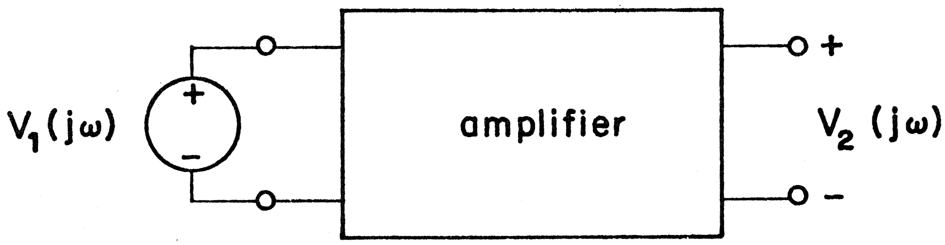


Fig. 2(a).

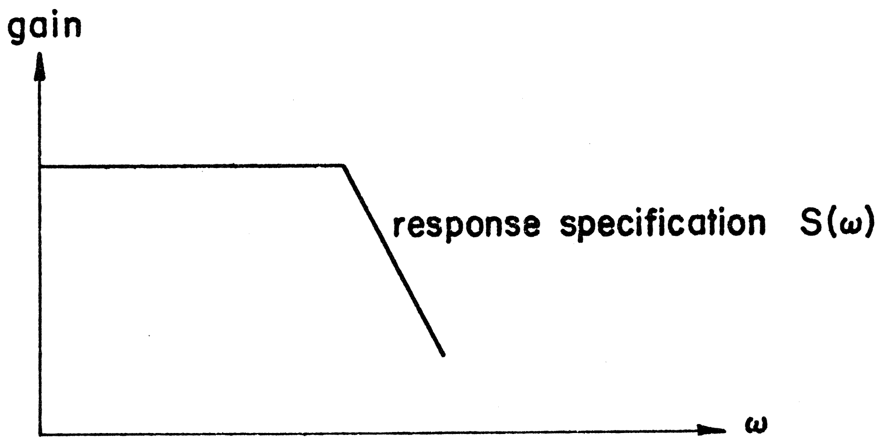
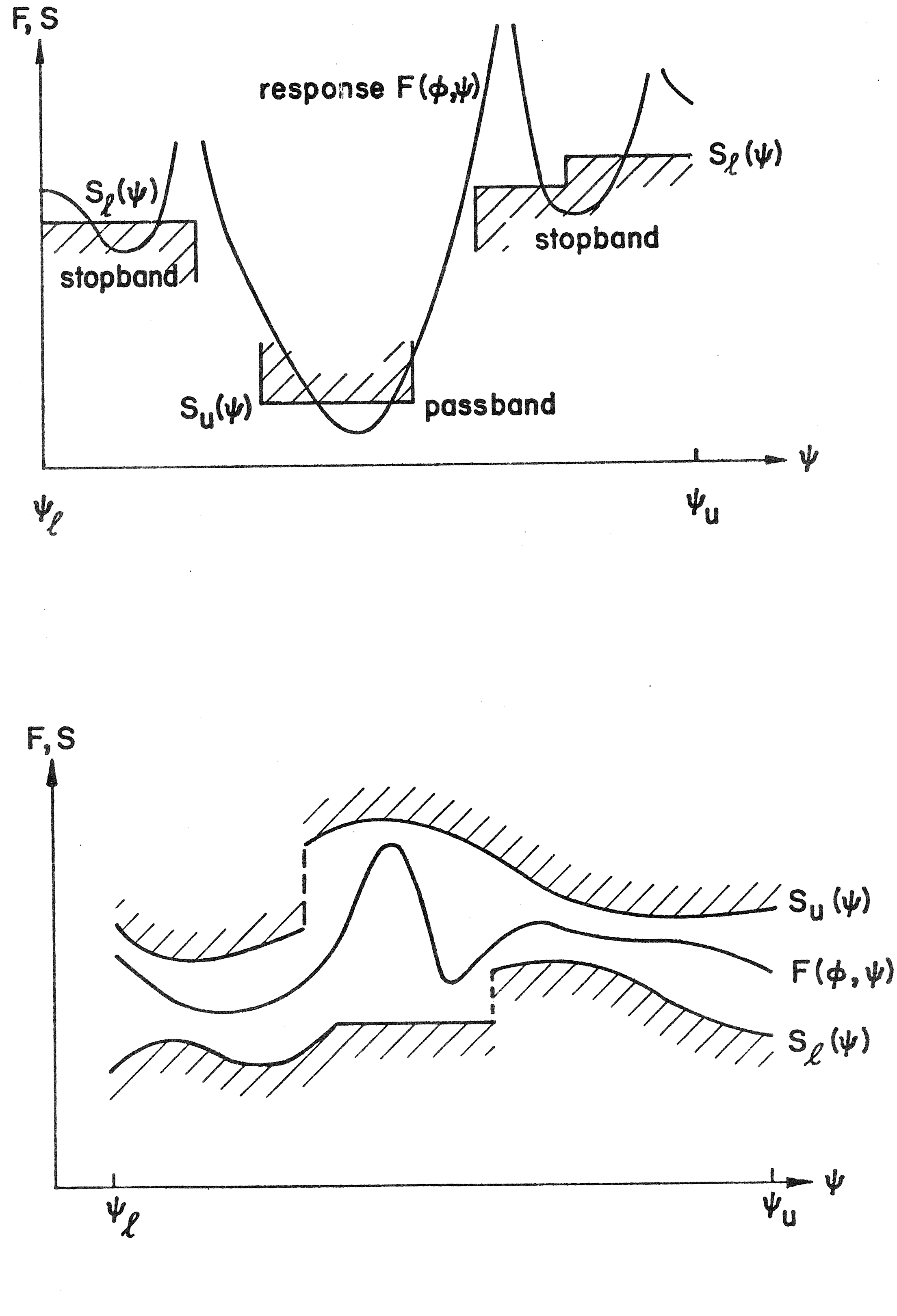


Fig. 2(b).

(a)

(b)



F, S
response F(ϕ,ψ)
Sℓ(ψ)
Sℓ(ψ)
stopband
stopband
Su(ψ)
passband
ψ
ψℓ
ψu
F, S
Su (ψ)
F(ϕ, ψ)
Sℓ(ψ)
ψ
ψℓ
ψu

Fig. 3.

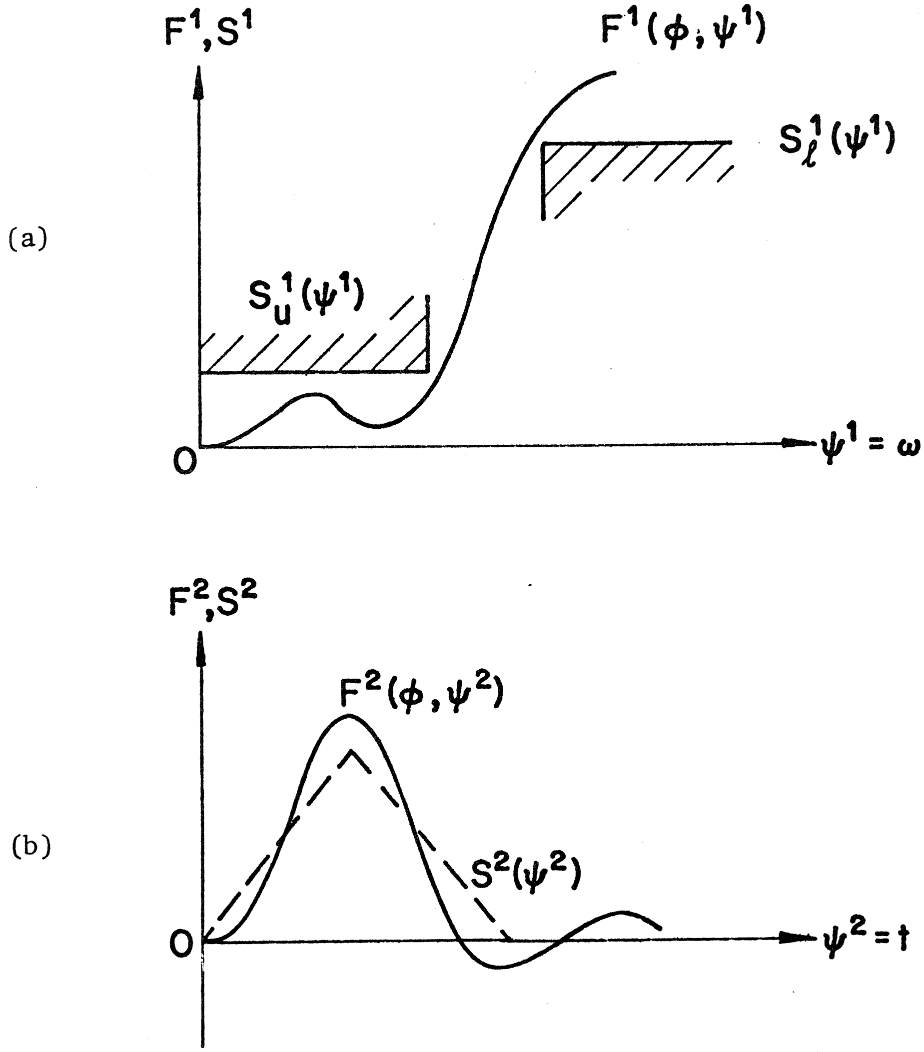
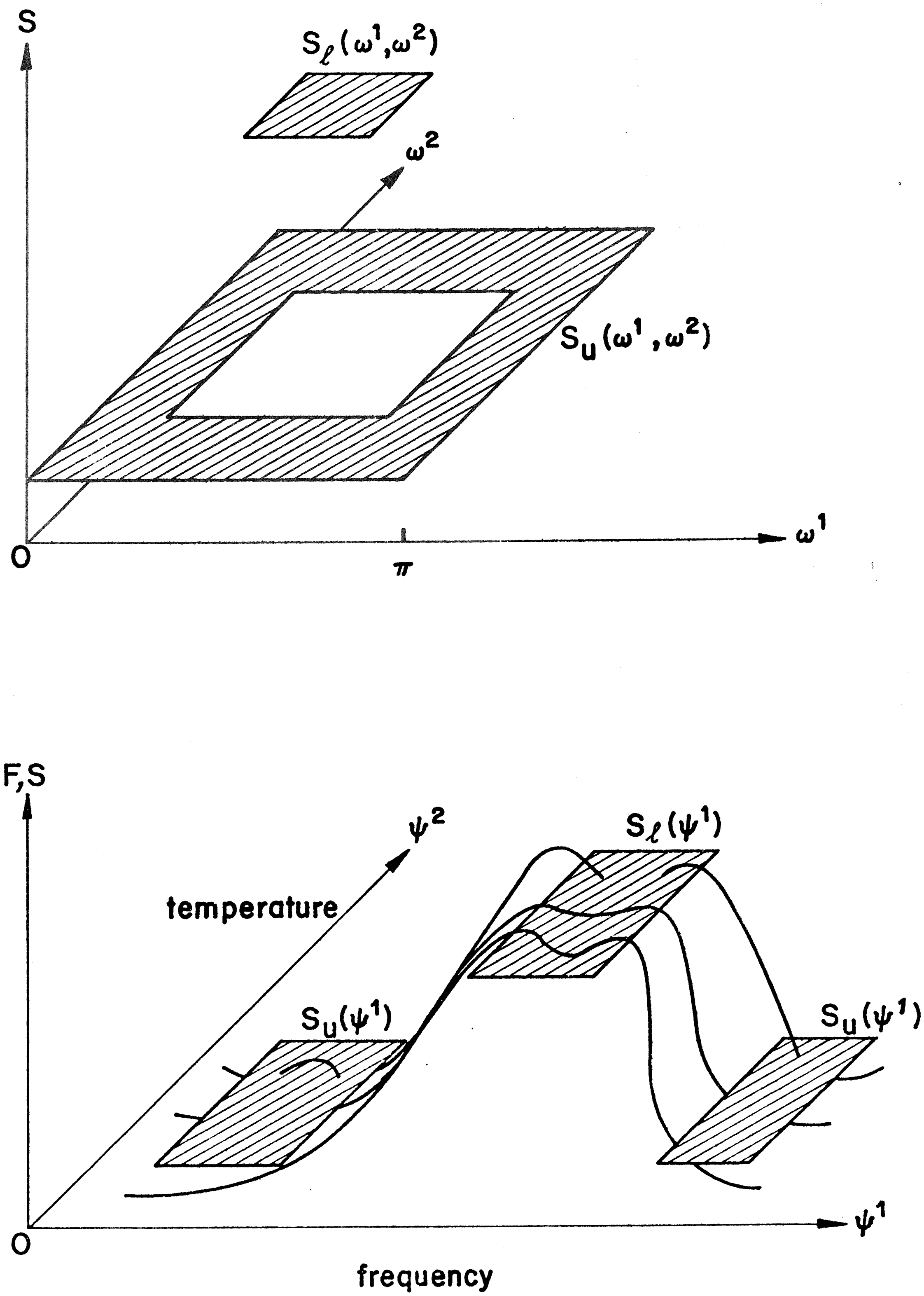


Fig. 4.

(a)

(b)



S
$S_\ell(\omega^1,\omega^2)$
ω^2
$S_u(\omega^1,\omega^2)$
O
π
ω^1
F,S
ψ^2
temperature
$S_\ell(\psi^1)$
$S_u(\psi^1)$
$S_u(\psi^1)$
O
ψ^1
frequency

Fig. 5.

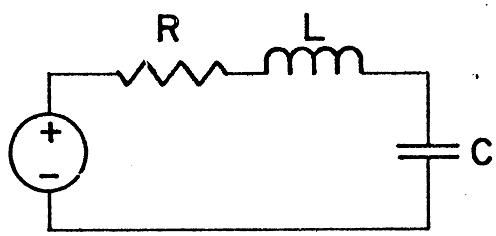


Fig. 6.

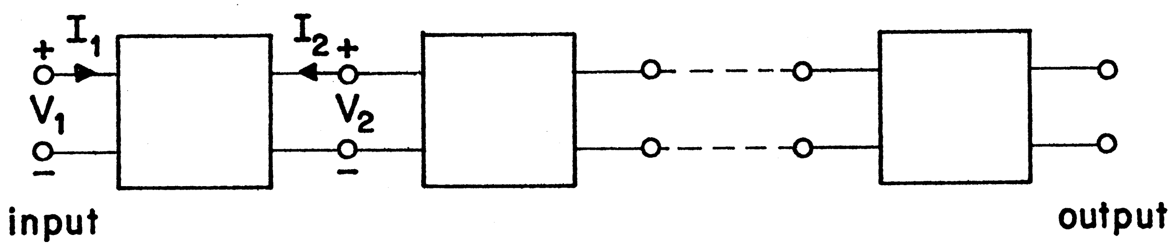


Fig. 7.

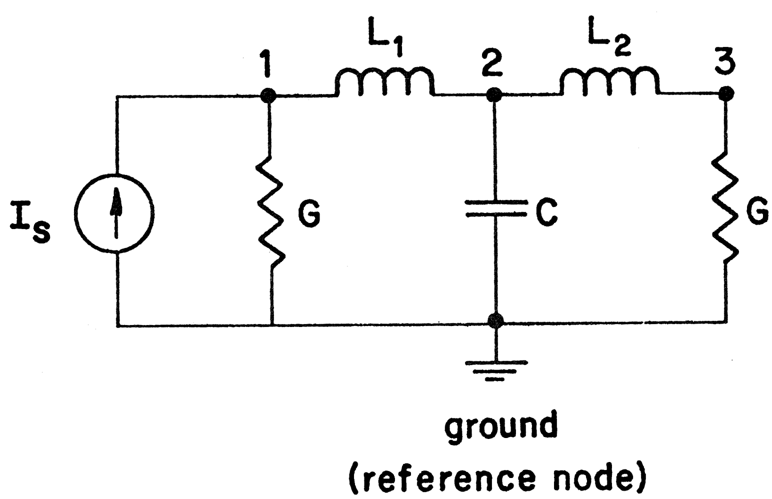


Fig. 8.

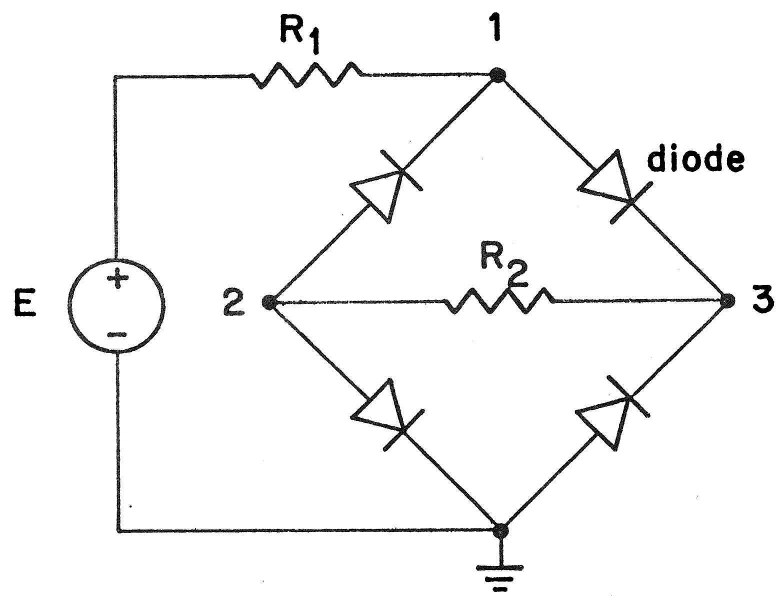
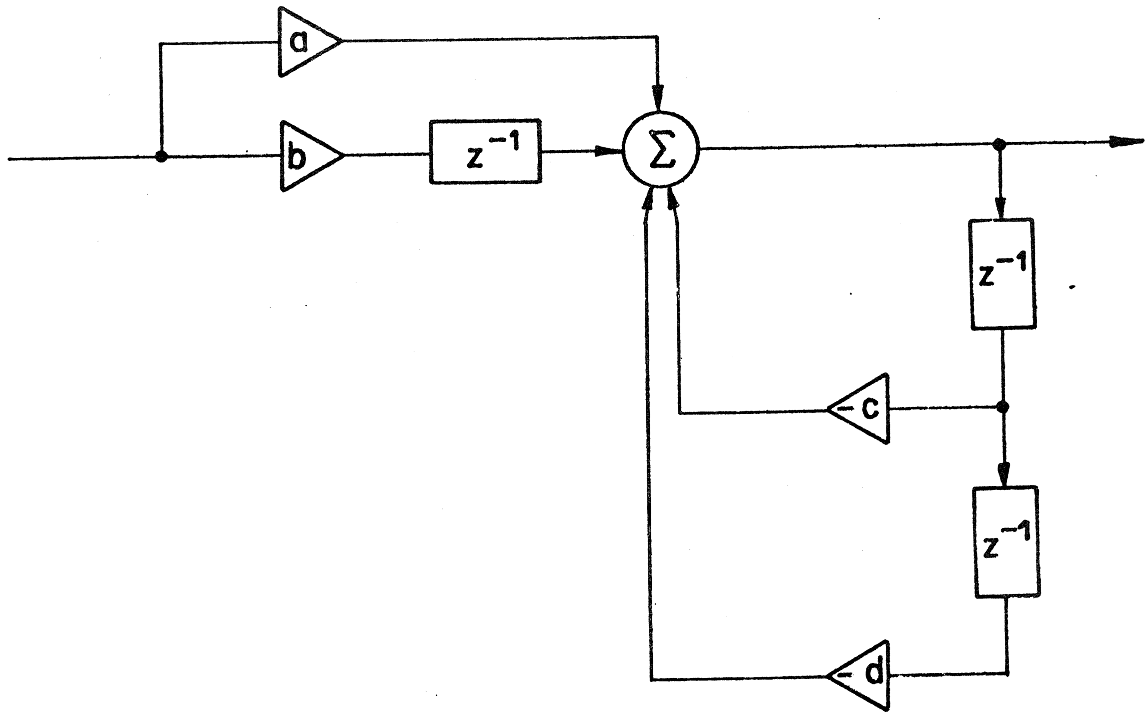
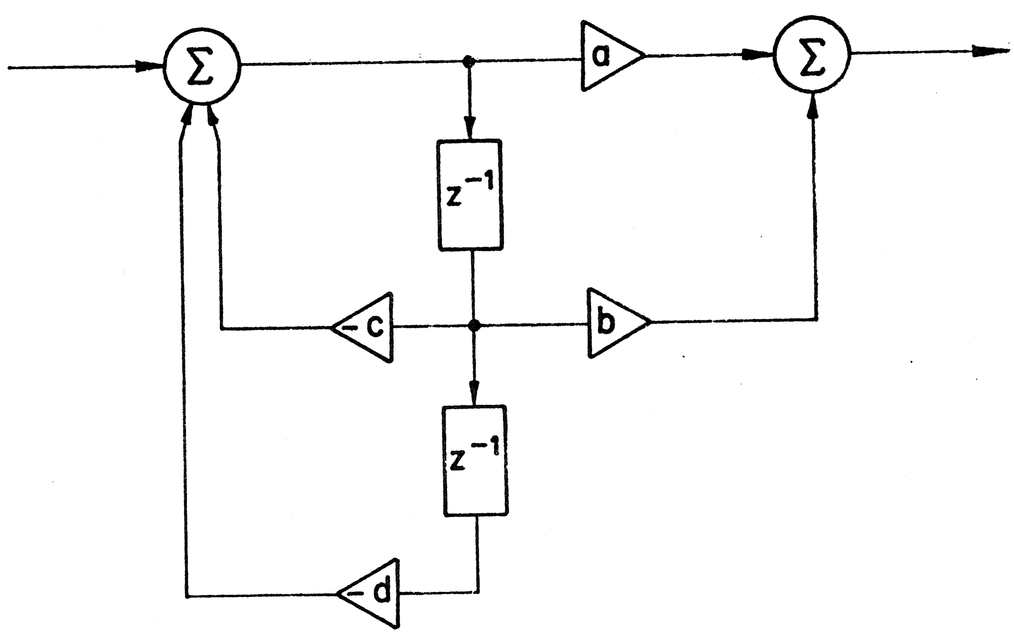


Fig. 9.



(a)



(b)

Fig. 10.

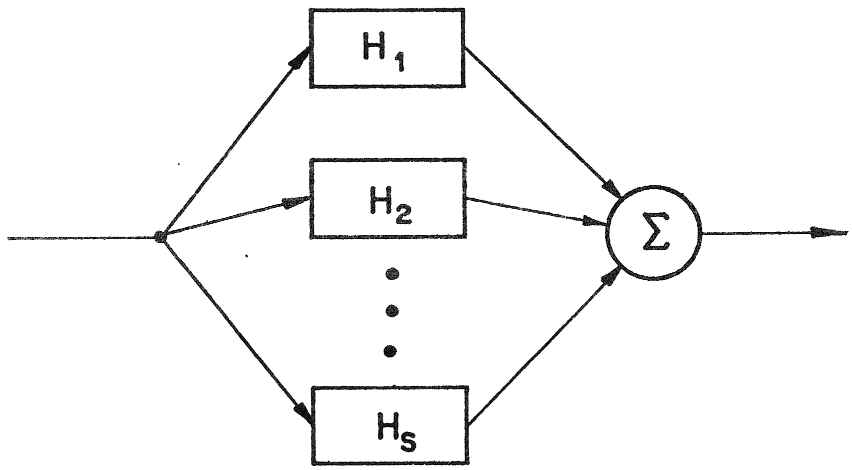


Fig. 11.

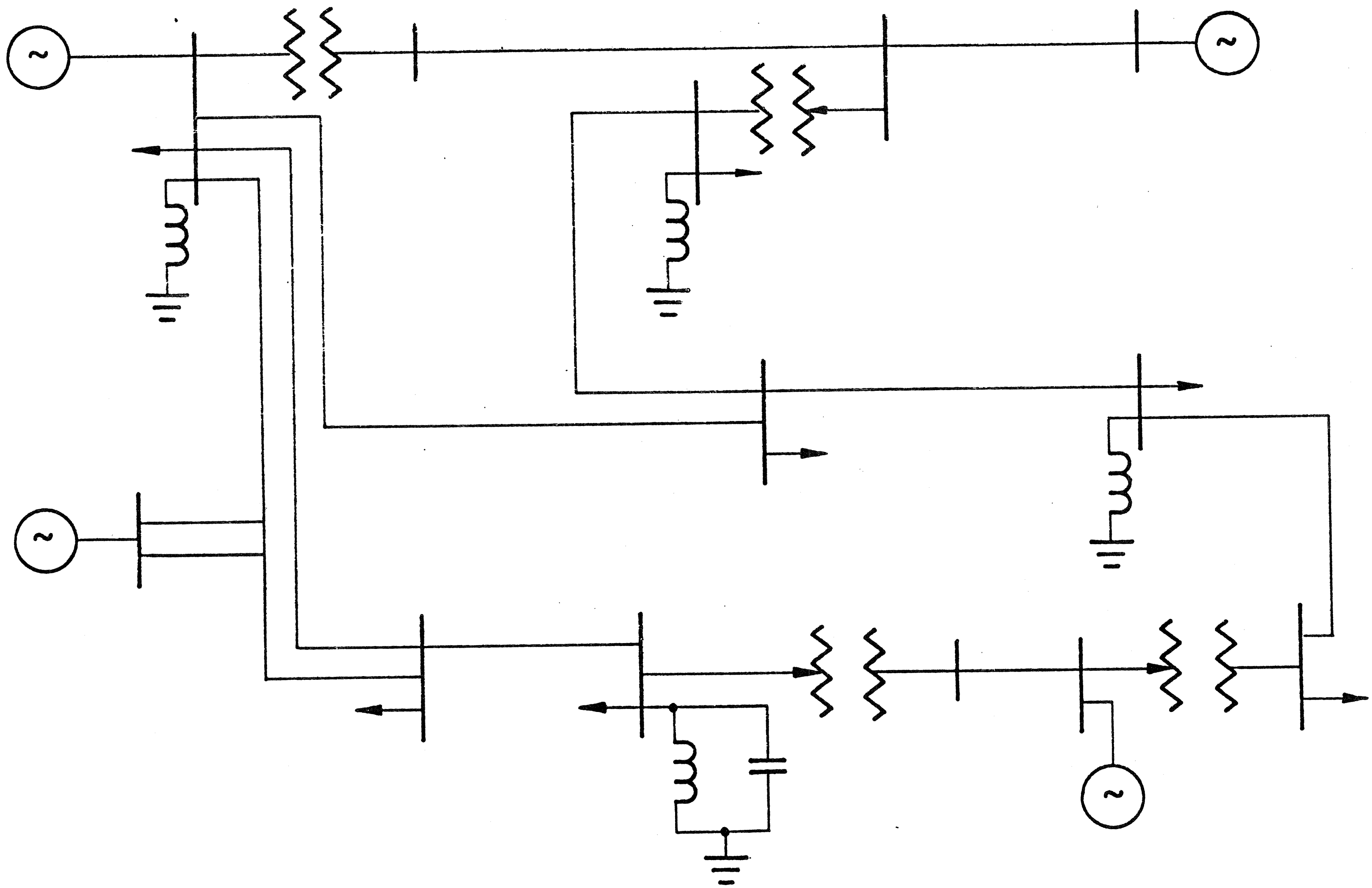
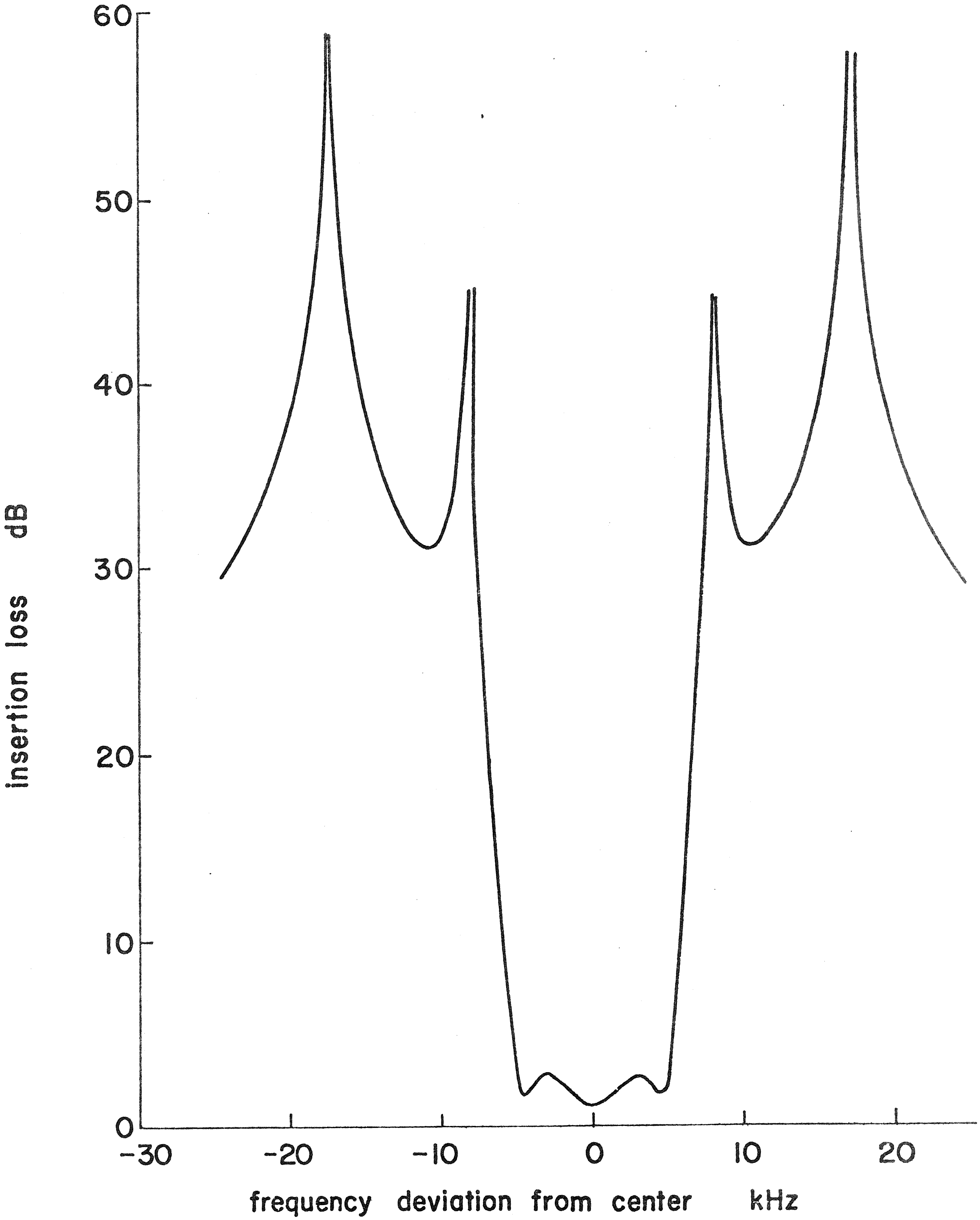


Fig. 12.



60
50
40
30
20
10
0
insertion loss dB
-30
-20
-10
0
10
20
frequency deviation from center kHz

Fig. 13.

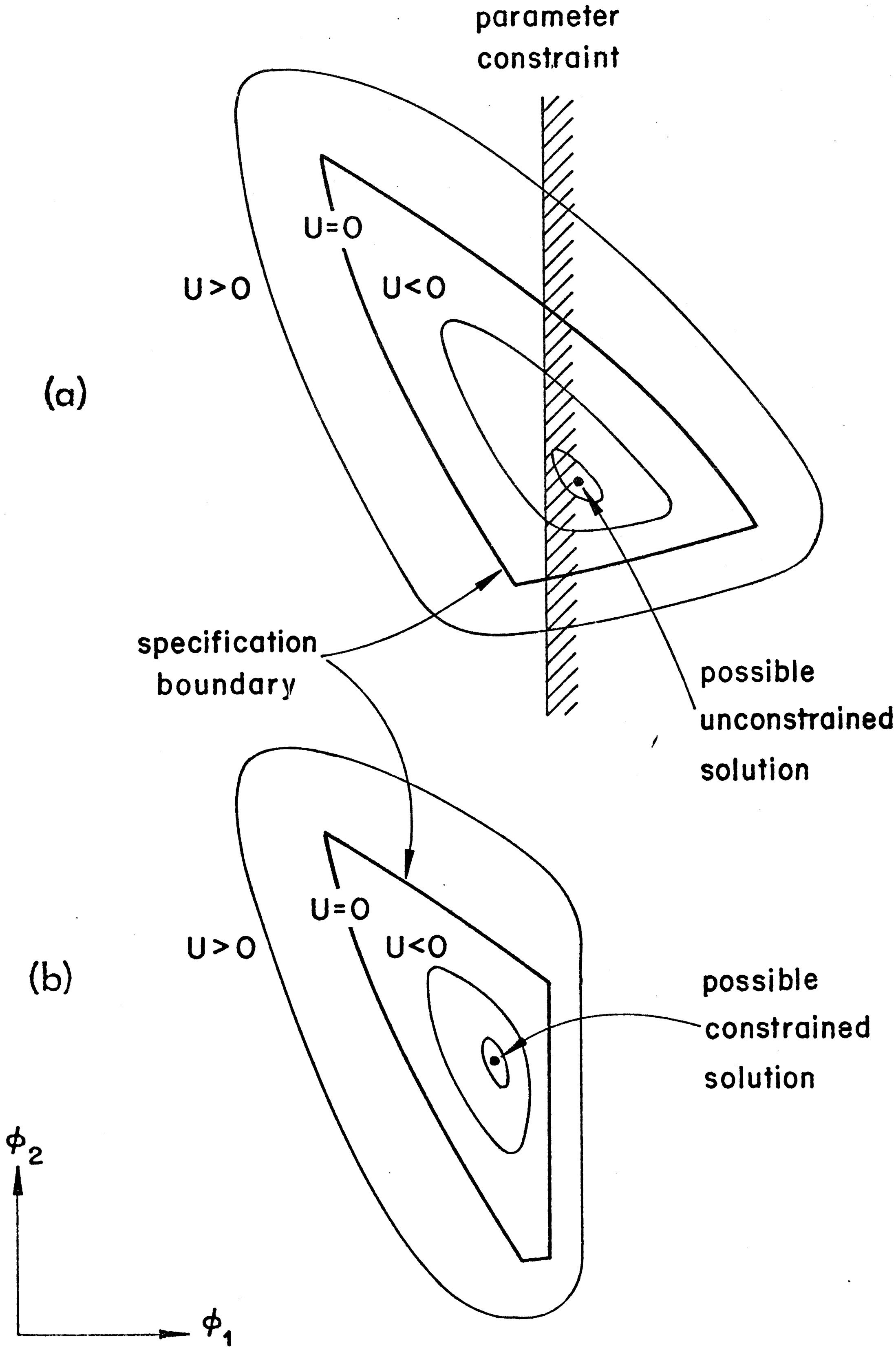

parameter
constraint
U=0
U>0
U<0
(a)
specification
boundary
possible
unconstrained
solution
U=0
U>0
U<0
(b)
possible
constrained
solution
ϕ_2
ϕ_1

Fig. 14.

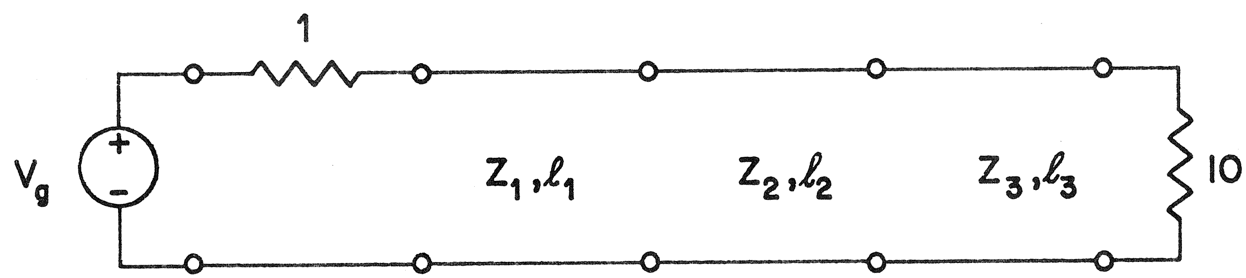


Fig. 15.

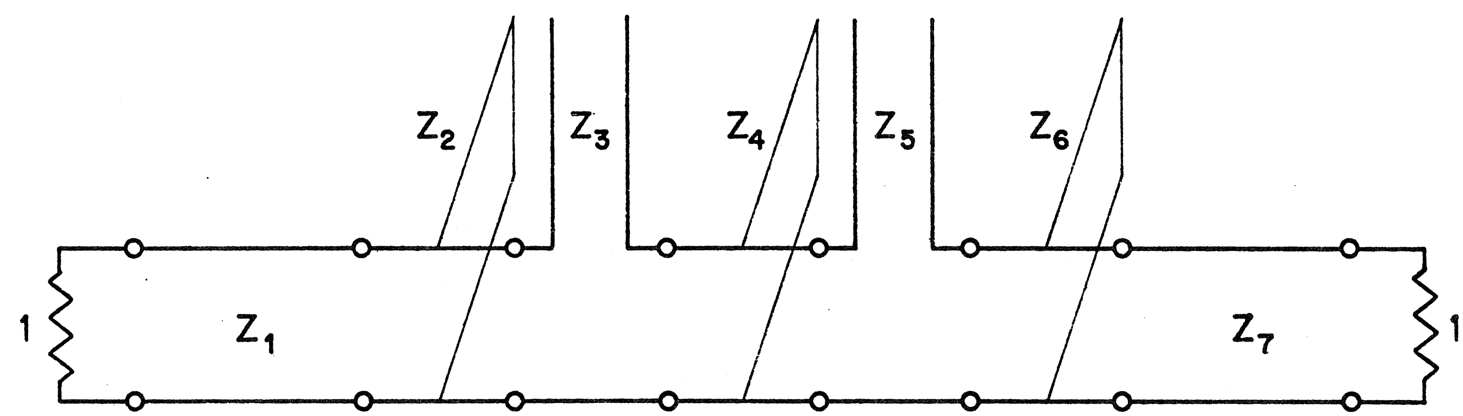


Fig. 16.

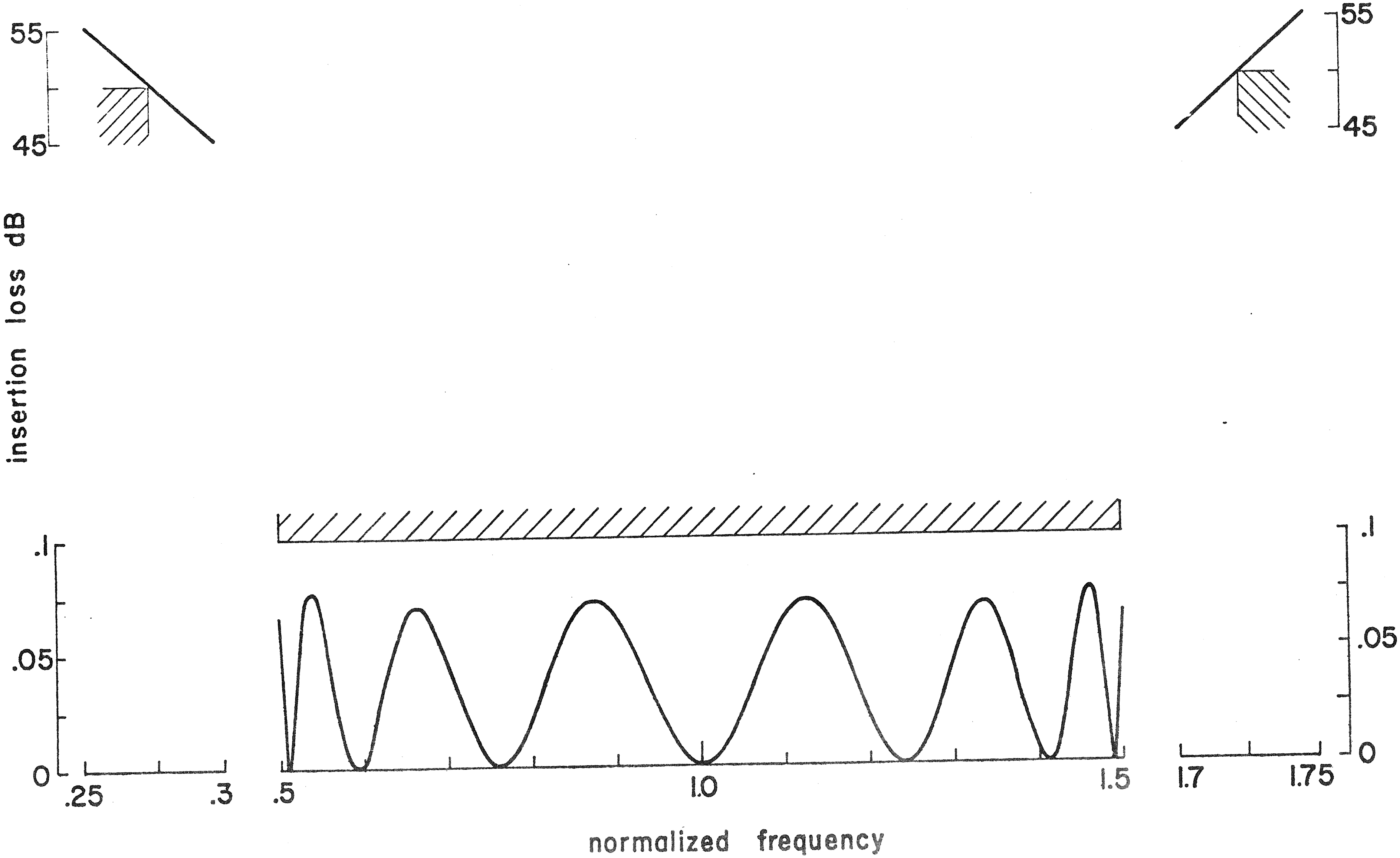

55
45
55
45
insertion loss dB
.1
.05
0
.1
.05
0
.25
.3
.5
1.0
1.5
1.7
1.75
normalized frequency

Fig. 17(a).

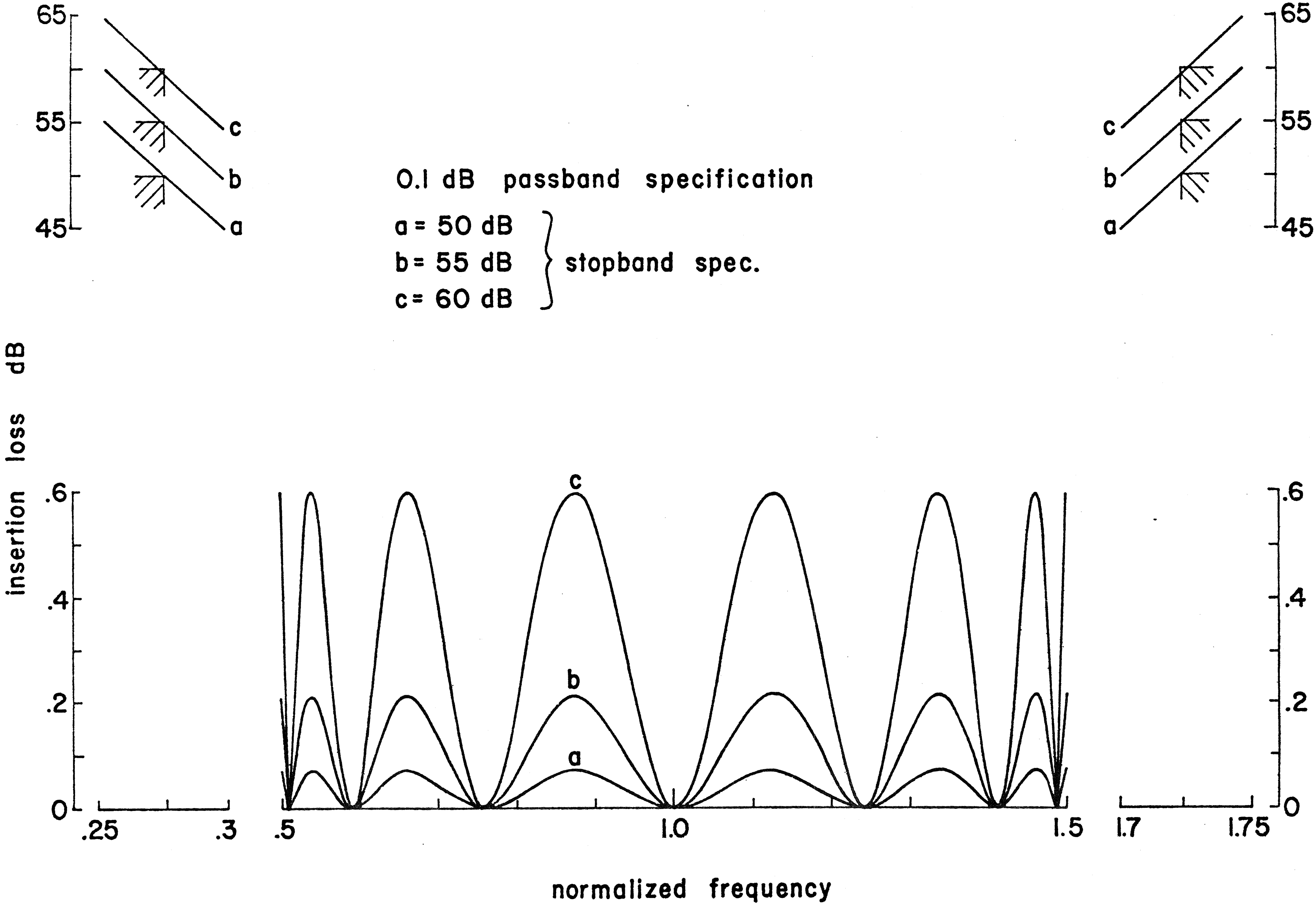

65
55
45
c
b
a
0.1 dB passband specification
a = 50 dB
b = 55 dB
c = 60 dB
stopband spec.
insertion loss dB
.6
.4
.2
0
.25
.3
.5
1.0
1.5
1.7
1.75
normalized frequency

Fig. 17(b).

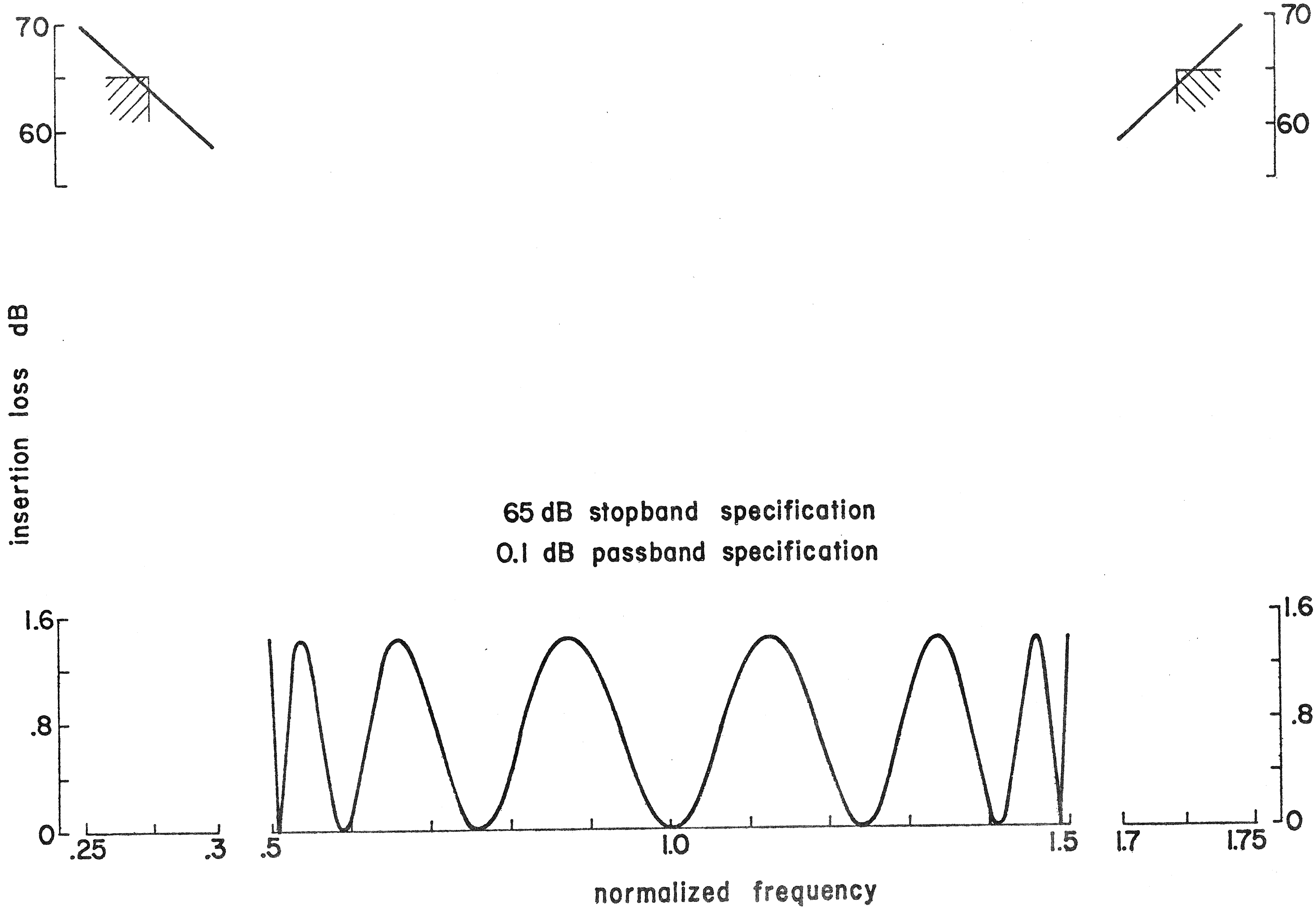


Fig. 17(b)[continued].

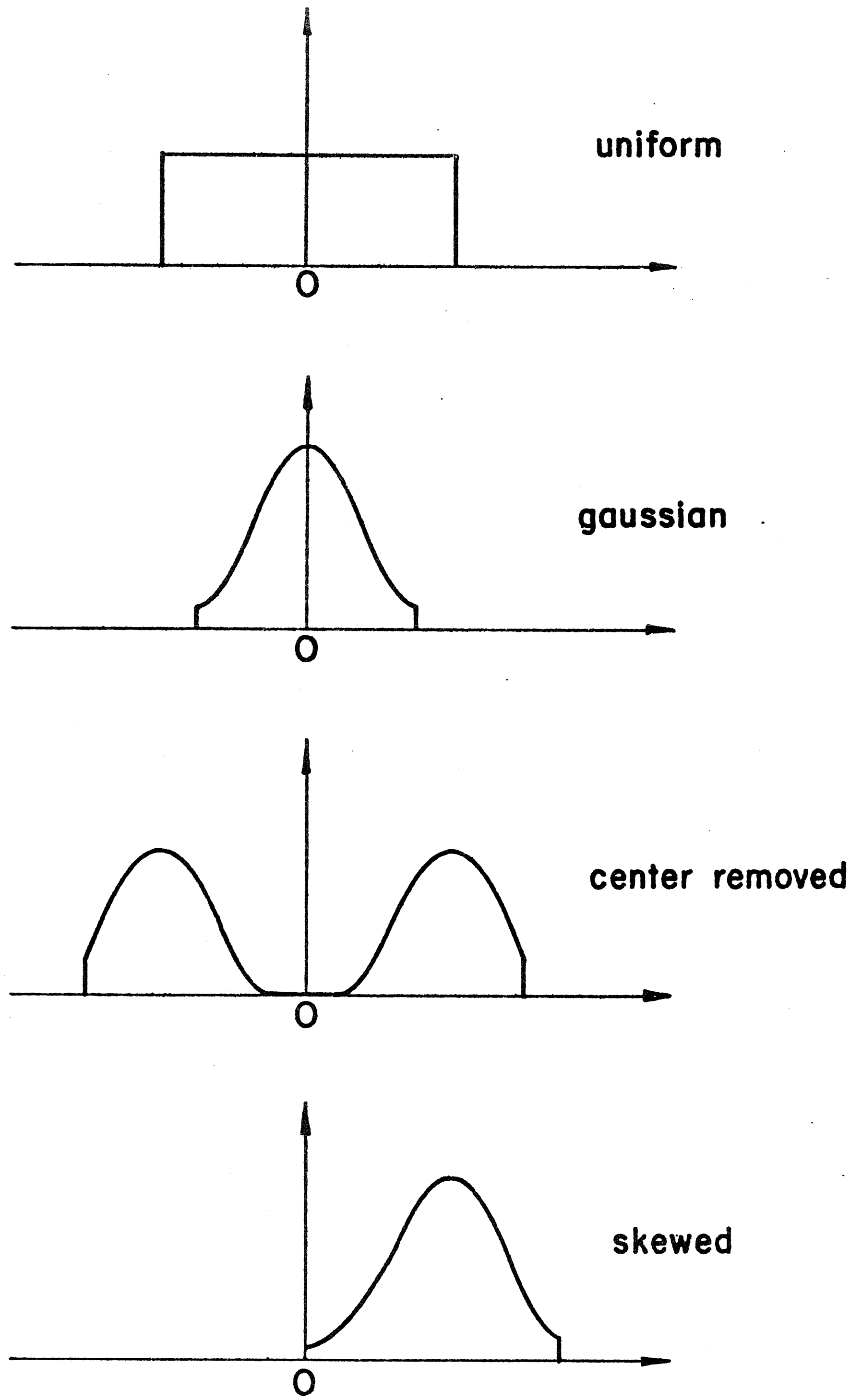
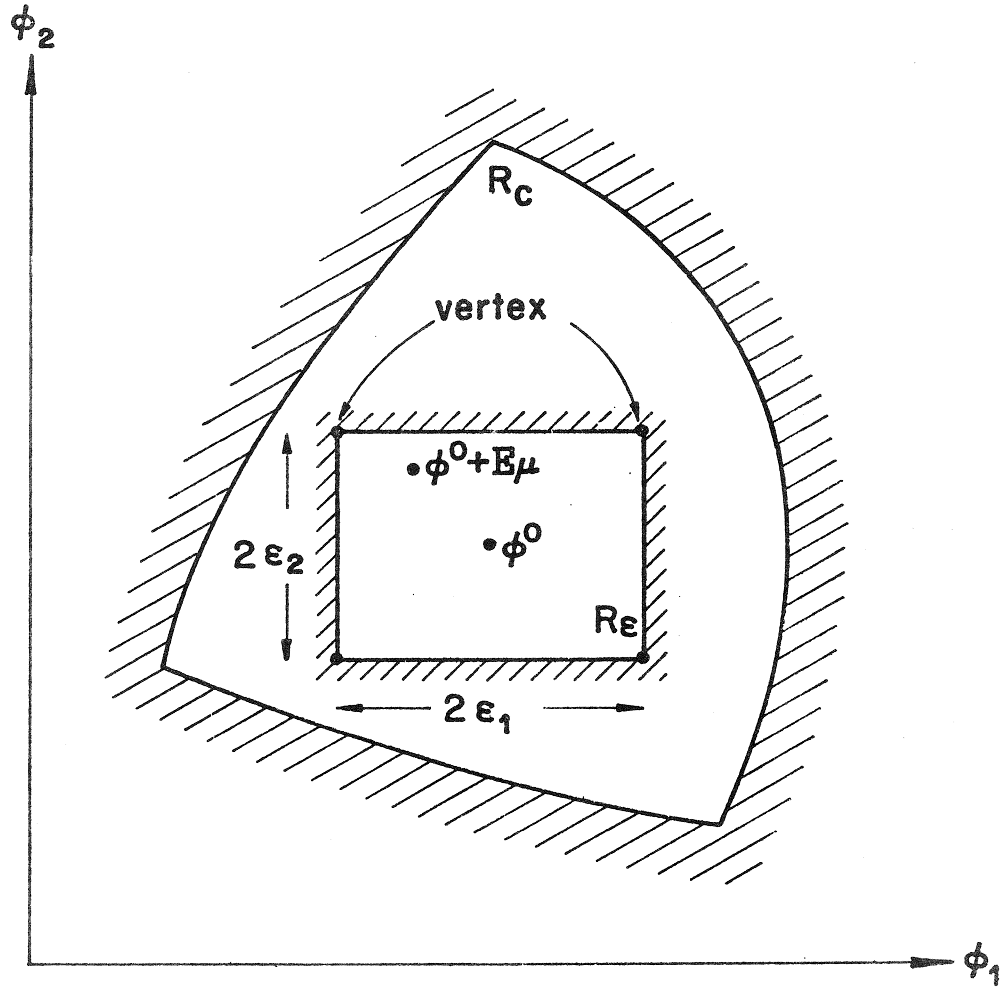
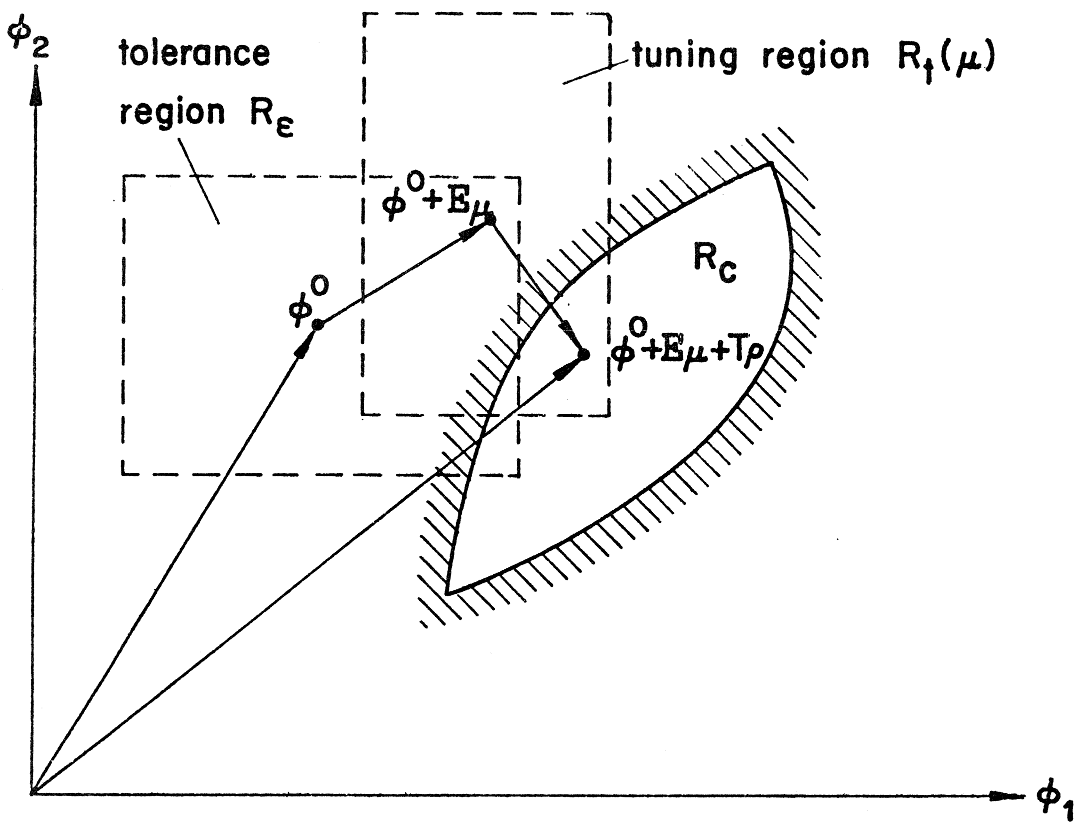


Fig. 18.



ϕ_2
R_c
vertex
$\phi^o + E\mu$
$2\varepsilon_2$
ϕ^o
R_ε
$2\varepsilon_1$
ϕ_1

Fig. 19.



ϕ_2
tolerance
region R_ε
tuning region $R_t(\mu)$
$\phi^0+E\mu$
ϕ^0
R_c
$\phi^0+E\mu+T\rho$
ϕ_1

Fig. 20.

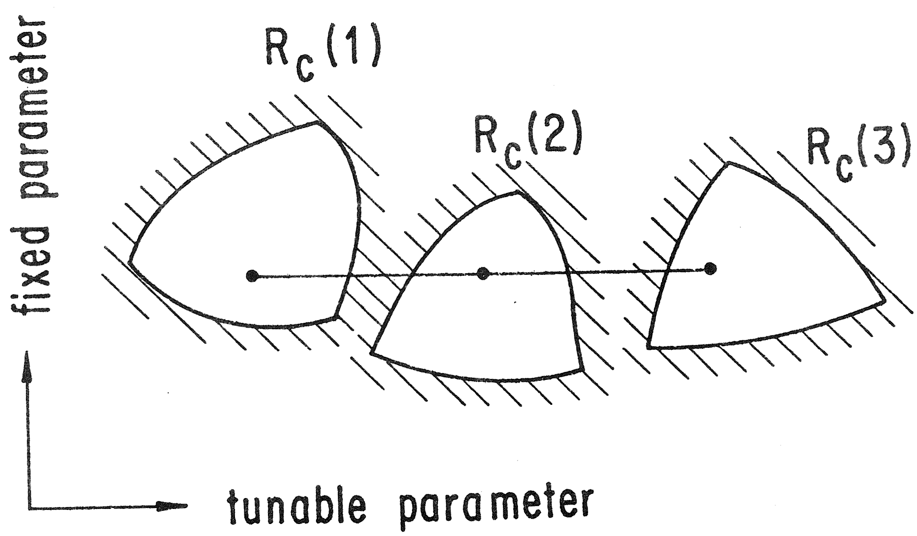


Fig. 21.

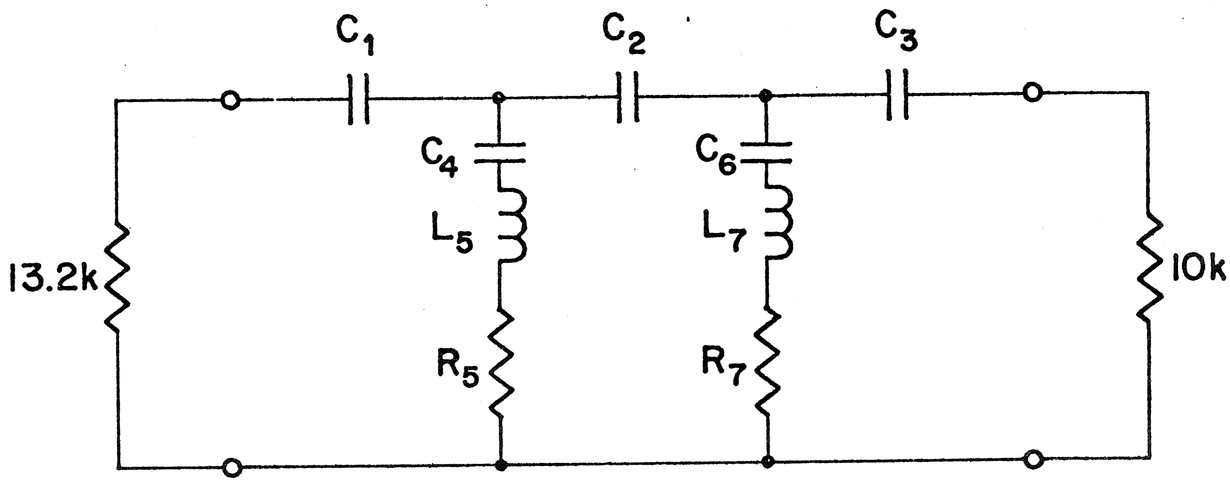
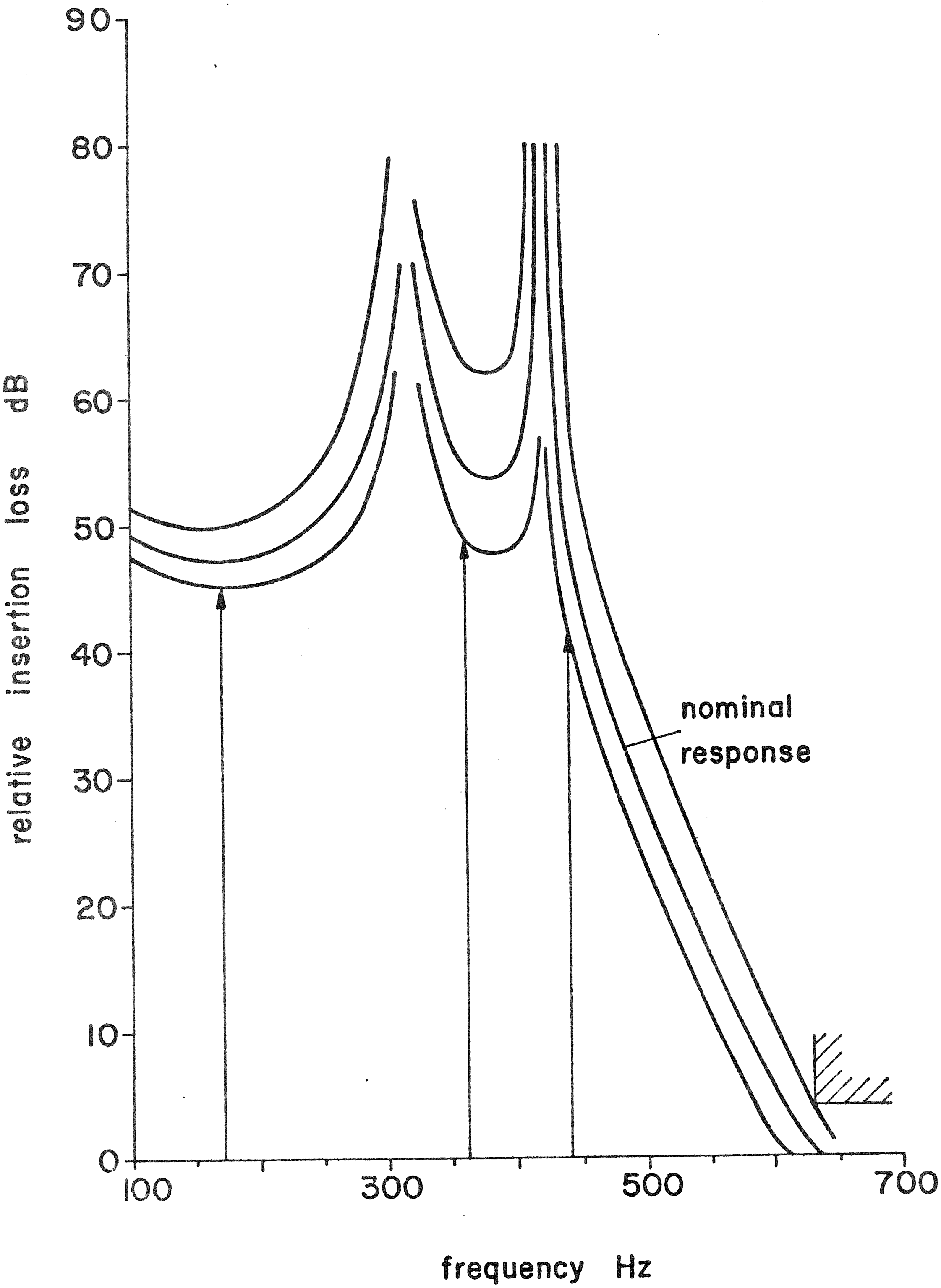
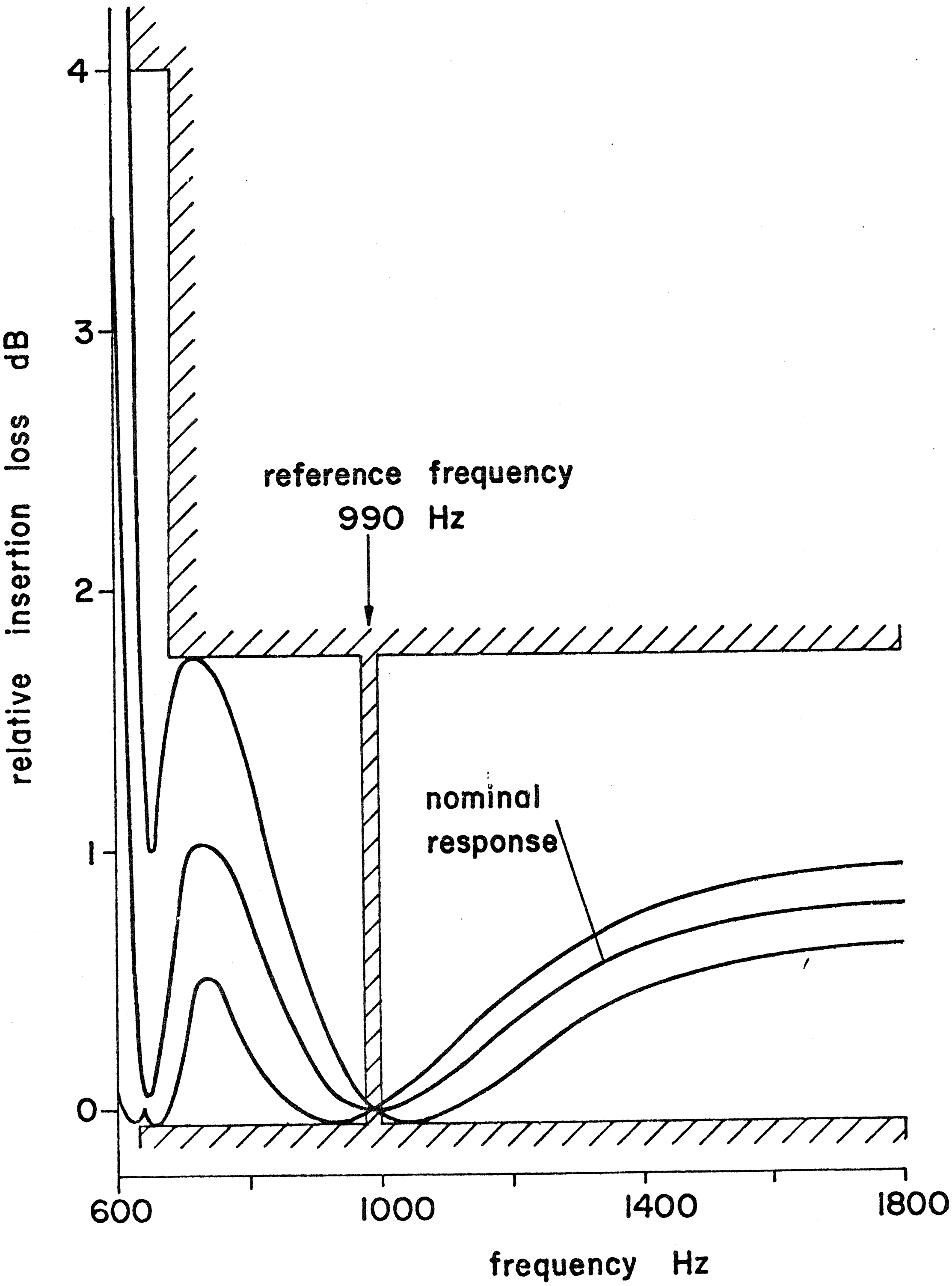


Fig. 22.



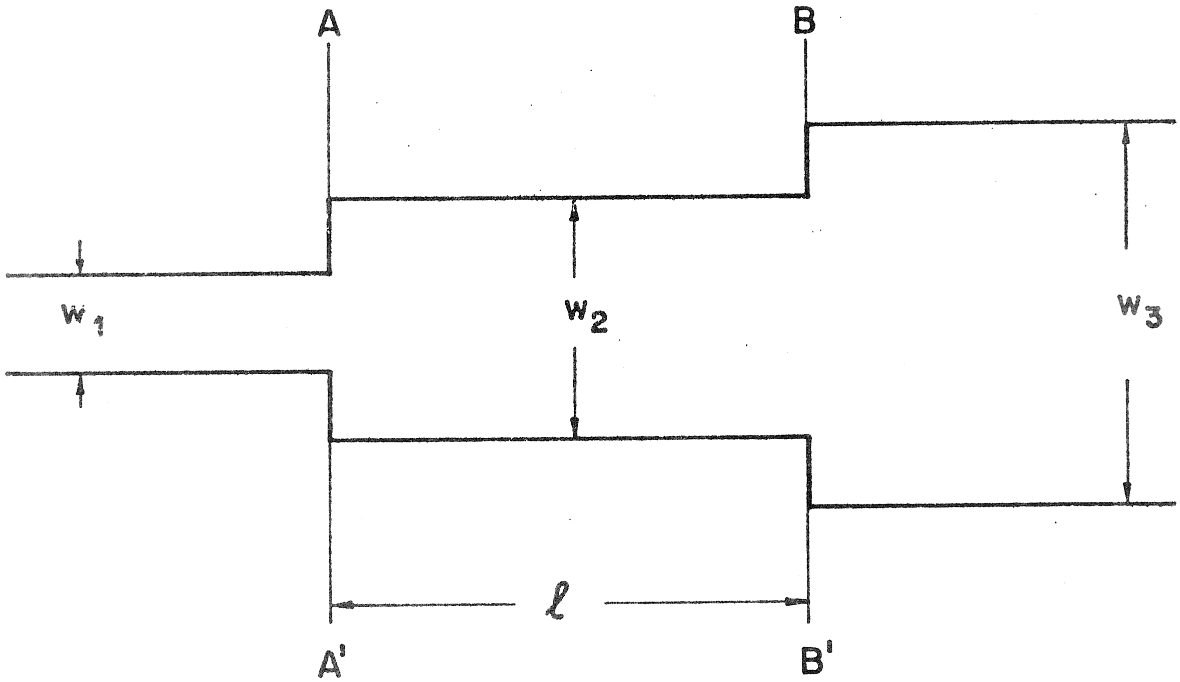
90
80
70
60
50
40
30
20
10
0
relative insertion loss dB
100
300
500
700
frequency Hz
nominal
response

Fig. 23(b).

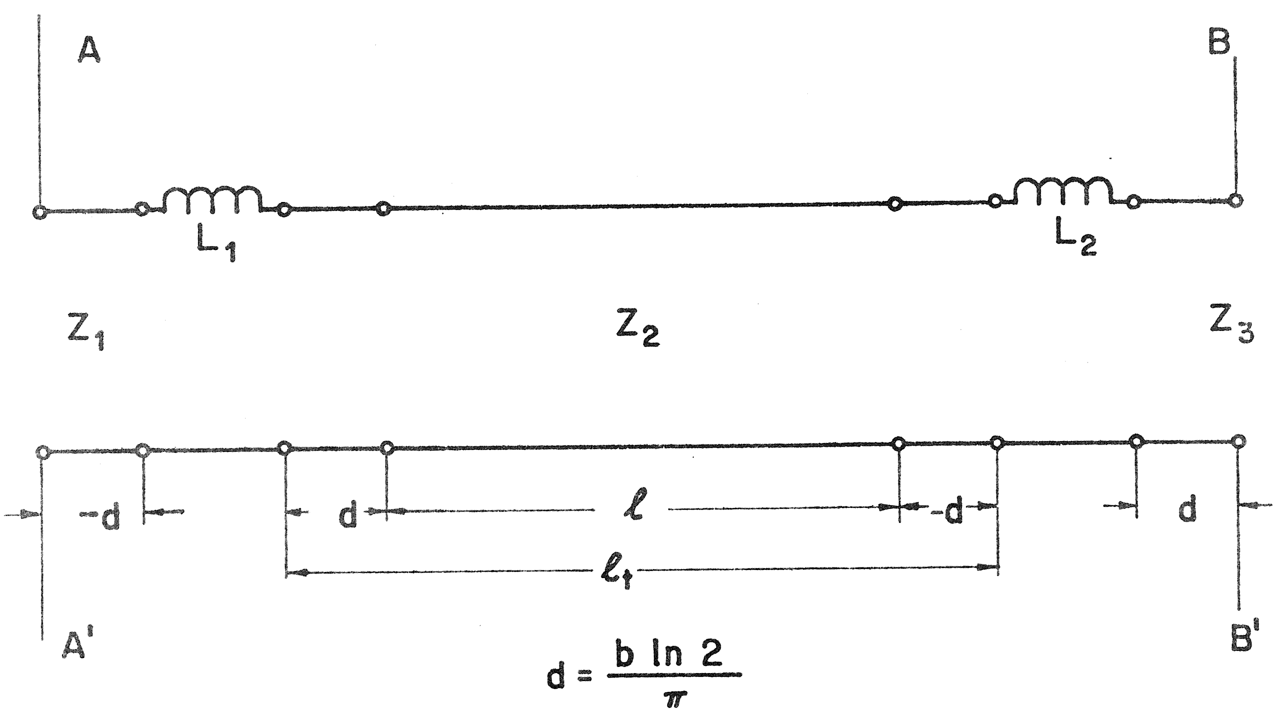


4
3
2
1
0
relative insertion loss dB
reference frequency
990 Hz
nominal
response
600
1000
1400
1800
frequency Hz

Fig. 23(a).



(a)



(b)

Fig. 24.

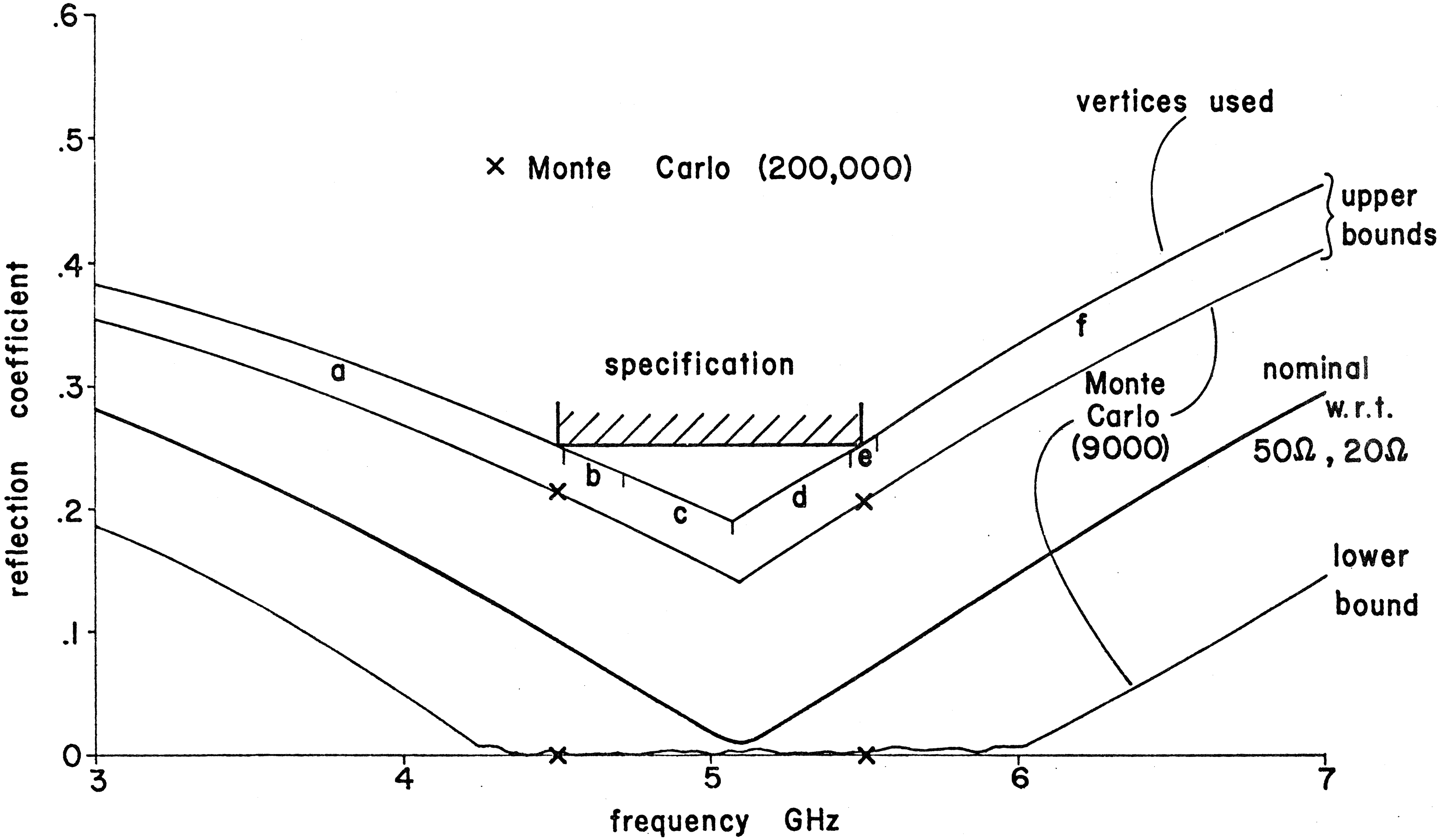
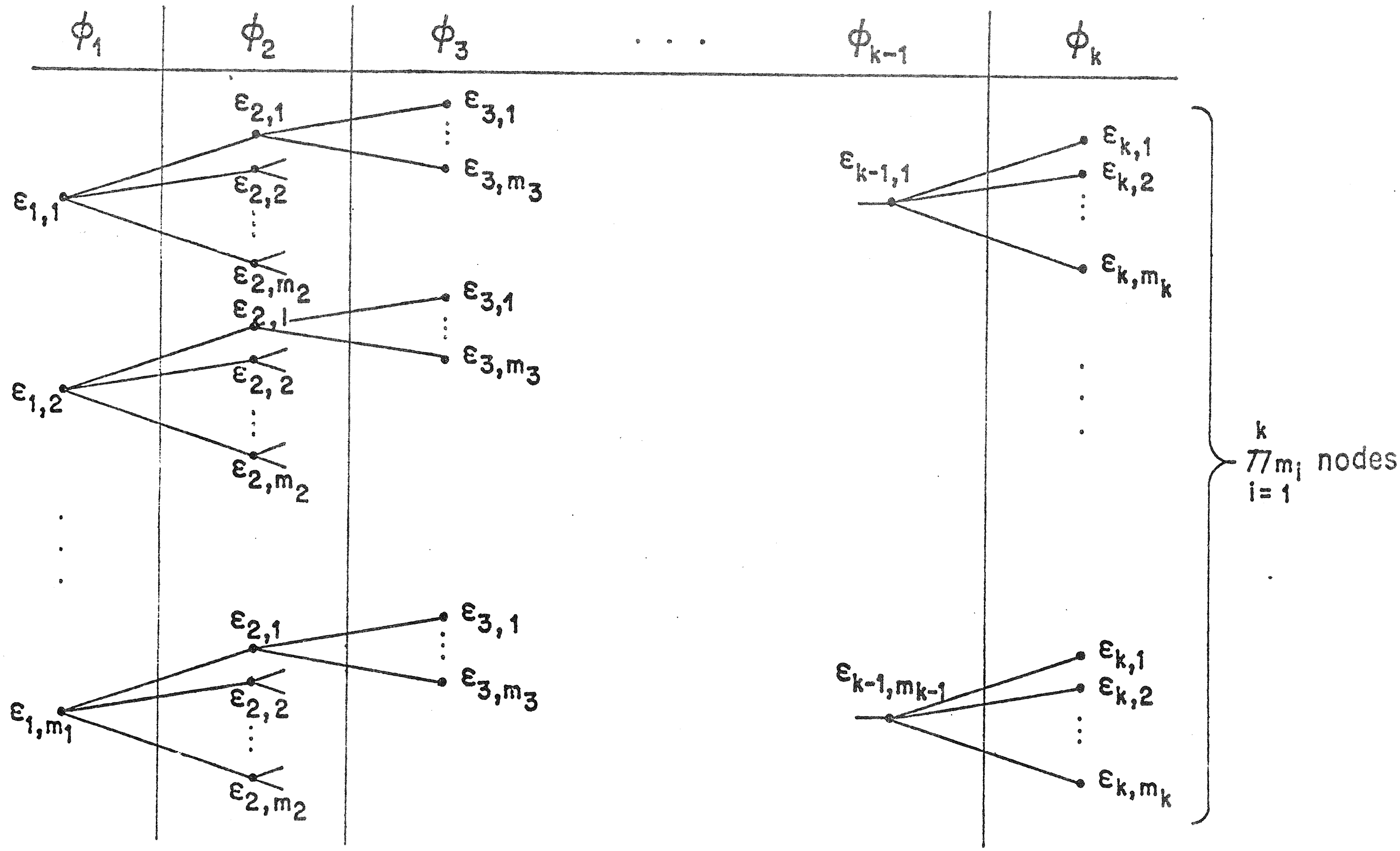

.6
.5
.4
.3
.2
.1
0
reflection coefficient
3
4
5
6
7
frequency GHz
× Monte Carlo (200,000)
specification
a
b
c
d
e
f
vertices used
upper bounds
Monte Carlo (9000)
nominal w.r.t. 50Ω, 20Ω
lower bound

Fig. 25.



φ1
φ2
φ3
. . .
φk-1
φk
ε1,1
ε1,2
ε1,m1
ε2,1
ε2,2
ε2,m2
ε3,1
ε3,m3
εk-1,1
εk-1,mk-1
εk,1
εk,2
εk,mk
k
77mi nodes
i= 1

Fig. 26.

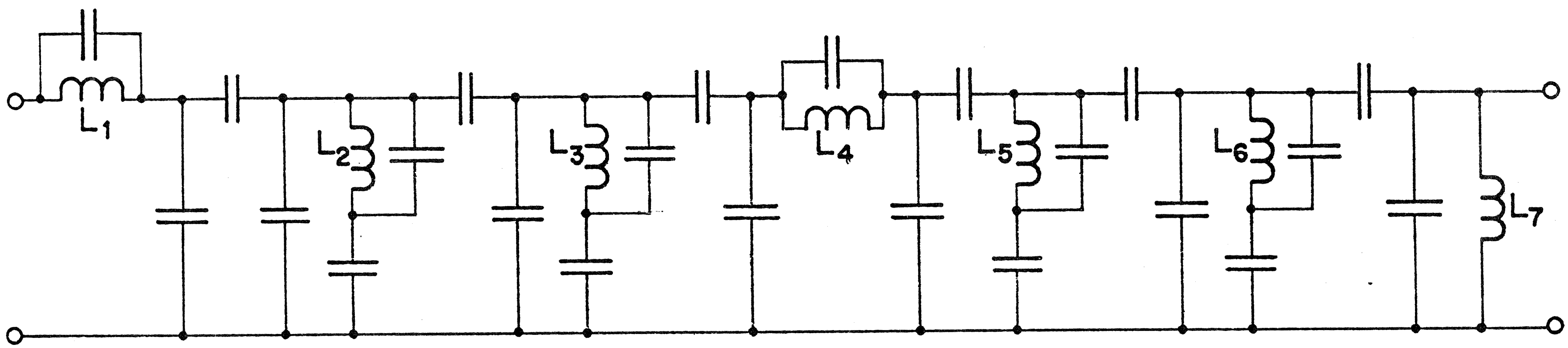


Fig. 27.

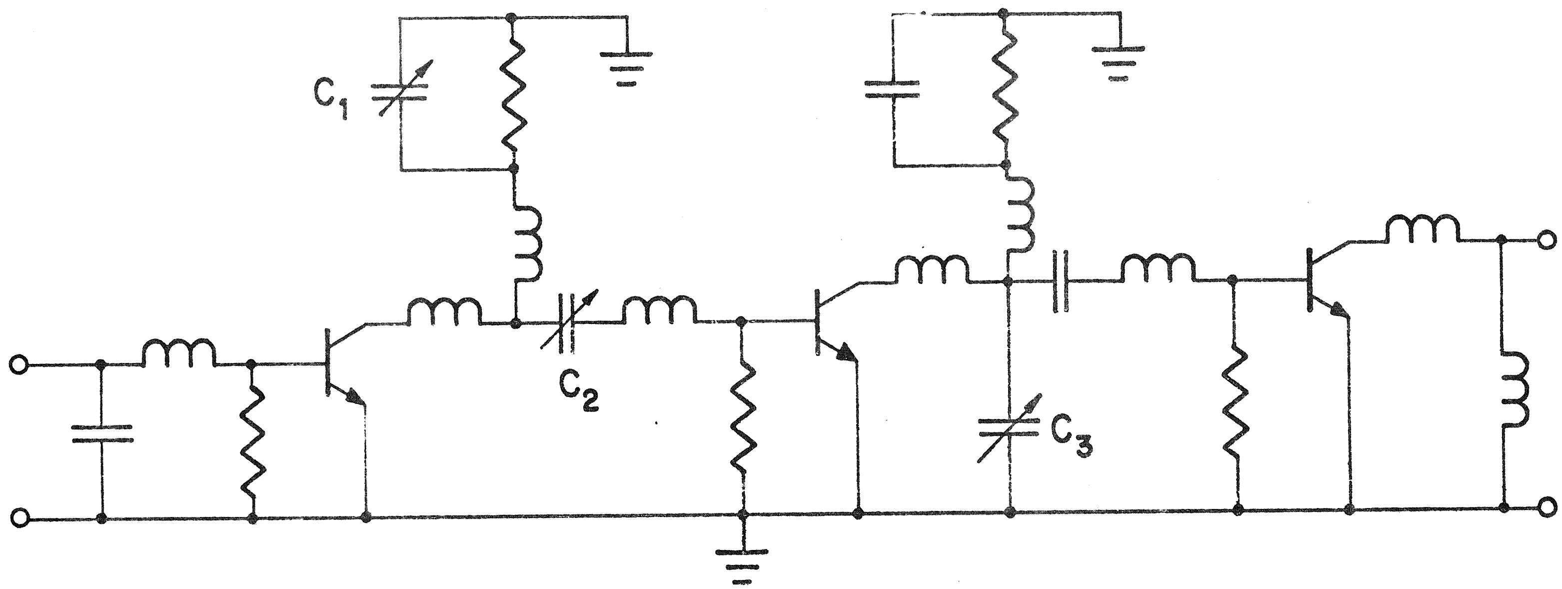


Fig. 28.

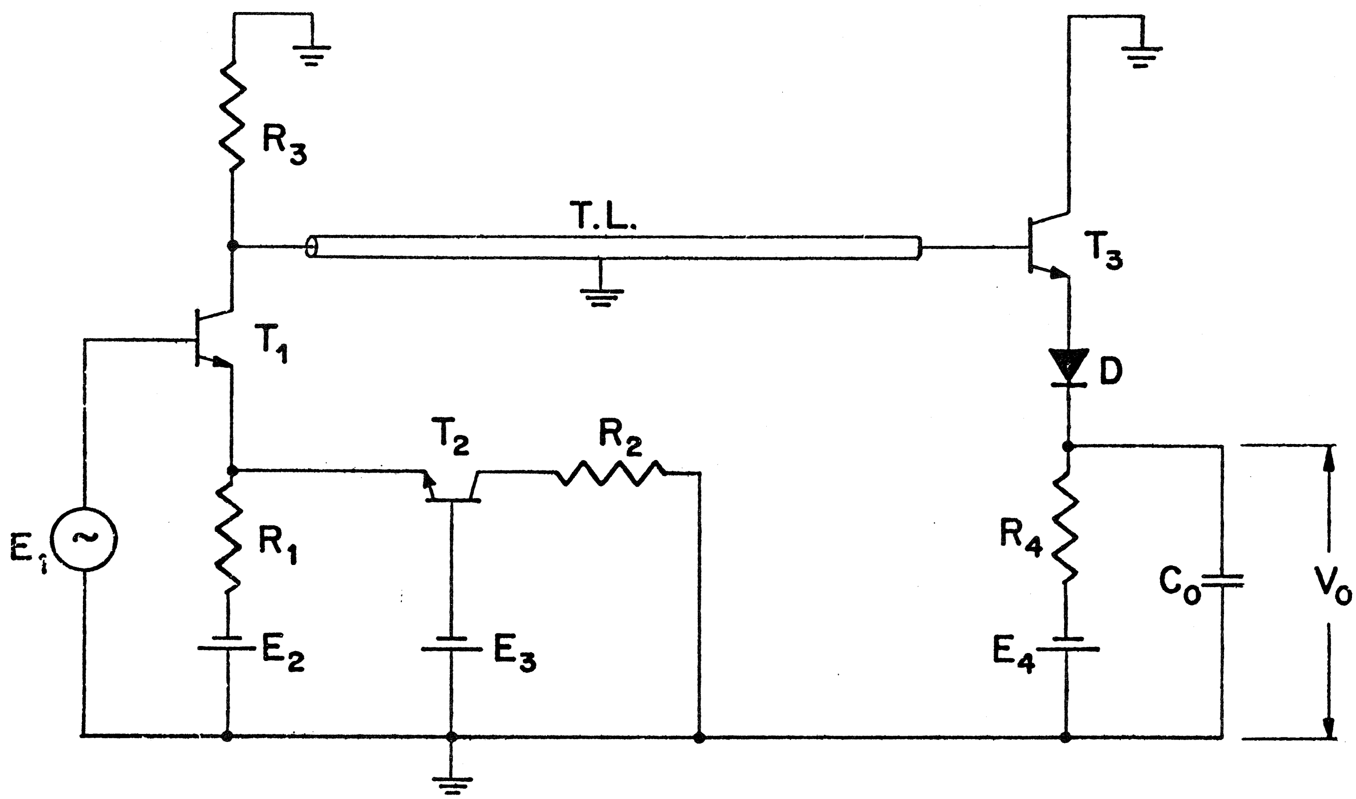
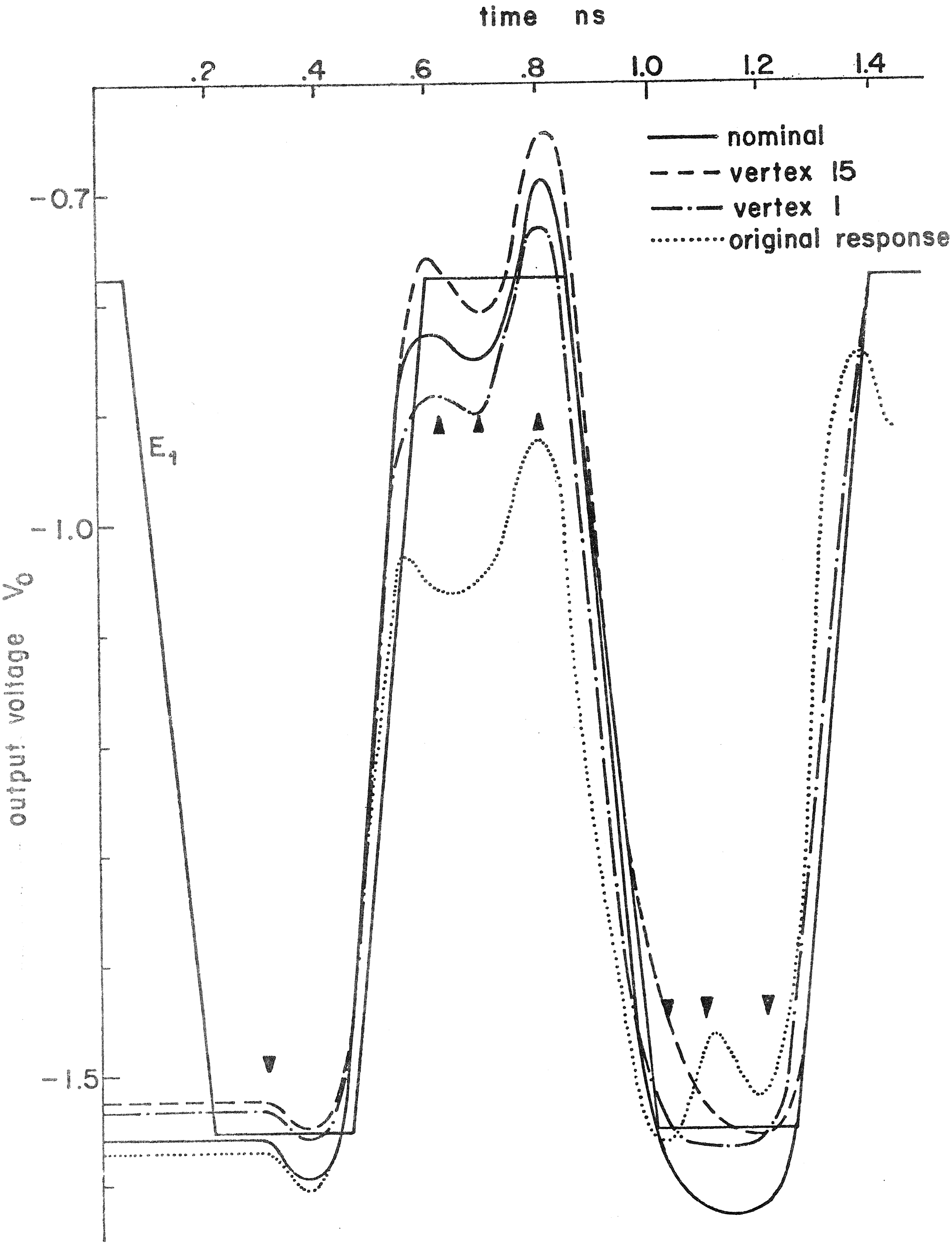


Fig. 29.



time ns
.2
.4
.6
.8
1.0
1.2
1.4
nominal
vertex 15
vertex 1
original response
-0.7
-1.0
-1.5
E1
output voltage V0

Fig. 30.

SOC-183

OPTIMIZATION OF ELECTRICAL CIRCUITS

J.W. Bandler and M.R.M. Rizk

November 1977, No. of Pages: 134

Revised:

Key Words: Mathematical programming, electrical circuit optimization, filter design, power system optimization, tolerance assignment

Abstract: This paper reviews applications of optimization methods in the area of electrical circuit design. It is addressed to engineers in general as well as mathematical programmers. As a consequence, a brief introduction to electrical circuits is presented, including analog, digital and power concepts. Network analysis techniques along with response evaluation and the determination of partial derivatives (useful in gradient methods of optimization) provide the nonelectrical reader with some necessary background. Objective functions aimed at improving network performance are presented, including least pth and minimax criteria. The approaches by many contributors to optimal circuit design are outlined, concentrating on general methods within the domain of nonlinear programming, nonlinear approximation and nonlinear discrete optimization techniques. A complete section is devoted to recent work in design centering, optimal assignment of manufacturing tolerances and postproduction tuning. The inclusion of model and environmental uncertainties is discussed. Practical examples illustrate the current state of the art. difficulties facing the design optimizer as well as directions of possible future research are elaborated on. A long but by no means exhaustive list of references is appended.

Description: Subject review.

Related Work: SOC-113, SOC-155, SOC-177.

Price: $ 15.00.