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A ONE-DIMENSIONAL MINIMAX ALGORITHM BASED ON BIQUADRATIC MODELS AND ITS APPLICATION IN CIRCUIT DESIGN

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Abstract This paper exploits the biquadratic behaviour w.r.t. a variable exhibited in the frequency domain by certain lumped, linear circuits. A globally convergent and extremely efficient minimax algorithm is developed and tested to optimize the frequency response w.r.t. any circuit parameter. It is shown that the algorithm converges to the global minimax optimum and that the rate of convergence is at least of second order. The algorithm is based on the linearization of error functions at boundary points of valid intervals. Boundary points of the region of acceptable designs are explicitly calculated and an algorithm to exactly determine the region itself for the general nonconvex case is presented and illustrated.

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1. INTRODUCTION

A number of researchers have considered properties of response or constraint functions w.r.t. one designable variable at a time in the contexts of sensitivity evaluation of linear circuits [1-9] and the prediction of worst cases in design centering and tolerance assignment [9-15]. The bilinear behaviour of certain linear circuits has been used to derive relationships between, e.g., first-order and large change sensitivities. In the tolerance problem, attempts have been made to find conditions which satisfy the common assumption that worst cases occur at extremes of parameter uncertainty intervals.

In this paper we exploit the resulting biquadratic function obtained from the modulus squared of the bilinear function to produce some new results. In particular, at any point in the frequency domain we can explicitly calculate boundary points of the constraint region of acceptable designs. These boundary points are further utilized to exactly determine the constraint region itself for the general nonconvex case.

The same approach is used to determine valid parameter intervals which are utilized in a globally convergent and extremely efficient minimax algorithm. The algorithm is based on the linearization of suitably chosen error functions in extreme points of all valid intervals. Examples employing a realistic tunable active filter demonstrate the optimization of the frequency response w.r.t. a circuit parameter. Our analysis leads to the explicit determination of circuit tunability. Furthermore, design centering and tolerance assignment w.r.t. each parameter at a time is facilitated.

II. THE ONE-DIMENSIONAL MINIMAX ALGORITHM

We begin by describing the one-dimensional algorithm for the minimax problem

$$\underset{\phi}{\text{minimize}} \quad \max_{1 \leq i \leq m} \; e_i(\phi), \tag{1}$$

where the $e_i(\phi)$ are biquadratic functions of the form (see Appendix A)

$$e_i(\phi) = \frac{A_i + 2B_i\phi + C_i\phi^2}{1 + 2D_i\phi + E_i\phi^2} \; . \tag{2}$$

We assume that the functions (2) have no real poles and are irreducible. The problem (1) is an unconstrained minimax problem. However, simple but important from a practical point of view, the constraints on the range of ϕ can easily be taken into account. The algorithm for solving this problem is based on the linearization of suitably chosen functions (2) at extreme points of all valid intervals. As a valid interval we mean a continuous interval

$$I \triangleq [\check{\phi}, \hat{\phi}] \tag{3}$$

such that, for all i and all $\phi \in I$,

$$e_i(\phi) \leq \delta, \tag{4}$$

where δ is a given number, and there exist indices $\check{i}$, $\hat{i}$, such that

$$e_{\check{i}}(\check{\phi}) = e_{\hat{i}}(\hat{\phi}) = \delta. \tag{5}$$

In the case when $\check{\phi}$ and/or $\hat{\phi}$ is at $-\infty$ or $+\infty$, respectively, the corresponding equality (5) is not required.

In order to find all valid intervals $I_\ell \triangleq [\check{\phi}_\ell, \hat{\phi}_\ell]$, $\ell = 1, 2, \ldots, k$, and the functions $e_{\check{i}_\ell}(\phi)$ and $e_{\hat{i}_\ell}(\phi)$ which define the extreme points of the intervals we will use Algorithm 2 described in Section III.

The algorithm for solving problem (1) is illustrated in Fig. 1. The following steps set it out in sufficient detail, with appropriate definitions to be used subsequently in the convergence proof (Appendix B).

Algorithm 1

Data ϕ_L, ϕ_U which determine the constraint $\phi_L \leq \phi \leq \phi_U$ or, alternatively, they are suitably taken as large negative and positive real numbers.

Step 1 Find $\tilde{I}$ such that

$$C_{\tilde{I}}/E_{\tilde{I}} \geq C_i/E_i \text{ for } i = 1, 2, \ldots, m.$$

Let $\phi^* \leftarrow \pm\infty$ and stop if $B_{\tilde{I}}\, E_{\tilde{I}} = C_{\tilde{I}} D_{\tilde{I}}$ and $A_{\tilde{I}} E_{\tilde{I}} - C_{\tilde{I}} > 0$.

Comment This step is to eliminate the trivial case that $\phi = \pm\infty$ is the minimax optimum.

Step 2 Initialize δ.

Comment In order to make sure that the extreme points of valid intervals are finite it can be recommended to choose δ as

$$\min \{\max_i e_i(\phi_L), \max_i e_i(\phi_U)\}.$$

Step 3 Find valid intervals $I_\ell \triangleq [\check{\phi}_\ell, \hat{\phi}_\ell]$ and $\check{i}_\ell$, $\hat{i}_\ell$, $\ell = 1,2,\ldots,k$.

Comment This is carried out using Algorithm 2 presented in Section III.

Step 4 Find $\check{g}_\ell$ and $\hat{g}_\ell$, $\ell = 1, 2, \ldots, k$, given by

$$\check{g}_\ell = \left.\frac{de_{\check{i}_\ell}}{d\phi}\right|_{\check{\phi}_\ell}, \tag{6}$$

$$\hat{g}_\ell = \left.\frac{de_{\hat{i}_\ell}}{d\phi}\right|_{\hat{\phi}_\ell}. \tag{7}$$

Comment These are simply the sensitivities at the extreme points of each valid interval. It is to be noted that $\check{g}_\ell \leq 0$ and $\hat{g}_\ell \geq 0$.

Step 5 If $k = 1$, set $j \leftarrow 1$ and go to Step 7.

Step 6 Find j such that

$$\Delta_j \geq \Delta_\ell, \quad \ell = 1, 2, \ldots, k, \tag{8}$$

where

$$\Delta_\ell = \begin{cases} \hat{g}_\ell \check{g}_\ell (\hat{\phi}_\ell - \check{\phi}_\ell)/(\check{g}_\ell - \hat{g}_\ell). \\ 0 \text{ if } \check{g}_\ell = \hat{g}_\ell = 0. \end{cases} \tag{9}$$

Comment In this step we select the jth interval which appears to be the most promising one in terms of the expected improvement in the minimax optimum based on linearization. Δ_ℓ will always be positive unless either $\check{g}_\ell = 0$, $\hat{g}_\ell = 0$ or $\check{\phi}_\ell = \hat{\phi}_\ell$.

Step 7 Set

$$\phi^* \leftarrow (\check{g}_j\check{\phi}_j - \hat{g}_j\hat{\phi}_j)/(\check{g}_j - \hat{g}_j) \text{ if } \check{i}_j \neq \hat{i}_j \text{ and } \Delta_j \neq 0. \tag{10}$$

Comment If the extremes of the jth interval are defined by two different functions, the new value of ϕ is determined by the intersection of the linear approximation to the two functions.

Step 8 Set ϕ^* to the minimizing point of the function $e_{\hat{i}_j}$ if $\check{i}_j = \hat{i}_j$.

Comment Since the function $e_{\hat{i}_j}(\phi)$ is a biquadratic function it is easy to find the minimizing point as a root of equation

$$(B_{\hat{i}_j} E_{\hat{i}_j} - C_{\hat{i}_j} D_{\hat{i}_j})\phi^2 + (A_{\hat{i}_j} E_{\hat{i}_j} - C_{\hat{i}_j})\phi + (A_{\hat{i}_j} D_{\hat{i}_j} - B_{\hat{i}_j}) = 0$$

which belongs to the interval I_j.

Step 9 Set

$$\phi^* \leftarrow (\check{\phi}_j + \hat{\phi}_j)/2 \text{ if } \phi^* \notin (\check{\phi}_j, \hat{\phi}_j) \text{ or } \Delta_j = 0.$$

Comment This is a default value to obviate any numerical problem which may arise in Step 6 or Step 7, for example, $\hat{g}_j = 0$.

Step 10 Find

$$\delta = \max_i e_i(\phi^*). \tag{11}$$

Step 11 Stop if $k = 1$ and if $(\hat{\phi}_1 - \check{\phi}_1)$ is sufficiently small.

Step 12 Go to Step 3.

In the following, superscript n will denote the index of iteration of the algorithm. The convergence properties of the algorithm are stated by the following theorem.

<u>Theorem</u>

If $I^n \triangleq [\check{\phi}^n, \hat{\phi}^n]$ is a unique interval such that $e_i(\phi) \leq \delta^n$ for $i = 1, 2, \ldots, m$, then $|\hat{\phi}^n - \check{\phi}^n| \to 0$ as $n \to \infty$. The rate of convergence is at least of second order.

Proof of the theorem is presented in Appendix B.

Now, we will show that the algorithm is guaranteed to converge to the global minimax optimum. This is due to switching from one valid interval to another one in Step 6.

According to the comment after Step 6, Δ_ℓ^n is always positive if $|\hat{\phi}_\ell^n - \check{\phi}_\ell^n| > 0$. We can omit the cases when $\hat{g}_\ell^n = 0$ and/or $\check{g}_\ell^n = 0$ since $\check{g}_\ell^n < 0$ and $\hat{g}_\ell^n > 0$ almost everywhere and Step 9 secures us against these situations. Moreover, it is easy to notice that $\Delta_\ell^n \to 0$ if $|\hat{\phi}_\ell^n - \check{\phi}_\ell^n| \to 0$.

Let us consider two intervals I_1^n and I_2^n which are found by the algorithm in the nth iteration. Let us assume that $\bar{\phi} \in I_2^n$ is a unique global minimax optimum. According to (9) and using the following notation

$$a_i^n = \min\,(-\check{g}_i^n, \hat{g}_i^n);\; b_i^n = \max\,(-\check{g}_i^n, \hat{g}_i^n), \tag{12}$$

where i = 1, 2 is the index of the interval, we have

$$\Delta_1^n = \frac{a_1^n\, b_1^n}{a_1^n + b_1^n}\,(\hat{\phi}_1^n - \check{\phi}_1^n) \leq \frac{b_1^n}{2}\,(\hat{\phi}_1^n - \check{\phi}_1^n) \tag{13}$$

and

$$\Delta_2^n = \frac{a_2^n\, b_2^n}{a_2^n + b_2^n}\,(\hat{\phi}_2^n - \check{\phi}_2^n) \geq \frac{a_2^n}{2}\,(\hat{\phi}_2^n - \check{\phi}_2^n). \tag{14}$$

Thus,

$$\frac{\Delta_1^n}{\Delta_2^n} \le \frac{b_1^n}{a_2^n} \cdot \frac{\hat{\phi}_1^n - \check{\phi}_1^n}{\hat{\phi}_2^n - \check{\phi}_2^n} . \tag{15}$$

Since $\bar{\phi} \in I_2^n$ is a unique global minimax optimum $|\hat{\phi}_2^n - \check{\phi}_2^n| \to \text{const} \neq 0$ if $|\hat{\phi}_1^n - \check{\phi}_1^n| \to 0$ so that $(\hat{\phi}_1^n - \check{\phi}_1^n)/(\hat{\phi}_2^n - \check{\phi}_2^n) \dot{\to} 0$. The left hand side of (15) can converge to a value different from zero only if $a_2^n \to 0$ if $|\hat{\phi}_1^n - \check{\phi}_1^n| \to 0$.[1] But this means that there is a local minimum of at least one of the functions $e_{\check{i}_2}(\phi)$ or $e_{\hat{i}_2}(\phi)$ of value equal to the local minimum value at $\phi_{1min} \in I_1^n$ so that $\bar{\phi} \in I_2^n$ is not the unique global minimax optimum. Otherwise, since $\Delta_1^n/\Delta_2^n \to 0$ the algorithm will select the second interval according to (8).

As is seen from the proof of the theorem (Appendix B) and the foregoing discussion the algorithm, after minor modifications, can be used for a broader class of functions than biquadratic functions of the form of (2). Only mild assumptions on first and second order derivative behaviour are required. The efficiency of this approach, however, depends strongly on the computational effort required to obtain valid parameter intervals as well as to calculate derivatives at all extreme points of the intervals. Since this effort is considerably small for biquadratic functions the algorithm proposed is not only globally convergent but also extremely efficient in this case.

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[1] The left hand side of (15) can converge to a value different from zero also when $b_1^n \to \infty$. But this would mean that the pole of every function $e_i(\phi)$, $i = 1, 2, \ldots, m$, existed at the point ϕ_{1min}, so $\bar{\phi}$ could not be the unique global optimum point. Moreover, since we assume error functions having no real poles (at least in the interval $[\phi_L, \phi_U]$) this case is of no interest.

III. VALID PARAMETER INTERVALS

Consider a single error function $e_i(\phi)$ of the form of (2). Given a value δ we can easily determine valid interval(s) for this function (see Appendix A for details). According to the results summarized in Table I we notice that, due to particular properties of biquadratic functions, it is always possible to define a <u>continuous</u> interval $R_{i\delta}$ such that either

$$\begin{aligned} e_i(\phi) &\leq \delta \text{ for all } \phi \in R_{i\delta} \\ \text{and } e_i(\phi) &> \delta \text{ for all } \phi \notin R_{i\delta} \end{aligned} \tag{16}$$

or

$$\begin{aligned} e_i(\phi) &\geq \delta \text{ for all } \phi \in R_{i\delta} \\ \text{and } e_i(\phi) &< \delta \text{ for all } \phi \notin R_{i\delta}. \end{aligned} \tag{17}$$

Typically, the interval $R_{i\delta}$ is unique. For particular cases, however, we can find two continuous intervals such that one of them fits the situation of (16) and the other one satisfies (17). In such cases we decide to consider the interval which is underlined in Table I. Boundary points are included in $R_{i\delta}$. To indicate the type of the interval $R_{i\delta}$ we will use a logical variable $t_{i\delta}$ which is set to "true" or "false" if $R_{i\delta}$ satisfies (16) or (17), respectively.

Now, consider the set of error functions $e_i(\phi)$, $i = 1, 2, \ldots, m$. Then the valid region R_δ (intersection of valid regions for all error functions) can be obtained as

$$R_\delta = \bigcap_{t_{i\delta}=\text{True}} R_{i\delta} - \bigcup_{t_{i\delta}=\text{False}} (R_{i\delta} - Fr(R_{i\delta})), \tag{18}$$

where $Fr(R_{i\delta})$ denotes the boundary of $R_{i\delta}$.

It is to be noted that R_δ is not necessarily a continuous interval.

In general,

$$R_\delta = \bigcup_{\ell=1}^{k} [\check{\phi}_\ell, \hat{\phi}_\ell], \tag{19}$$

where k is the number of separate intervals. The algorithm below provides k and the intervals $[\check{\phi}_\ell, \hat{\phi}_\ell]$, $\ell = 1, 2, \ldots, k$, as well as the indices of the functions e_i which actually define the extreme points of each interval. These indices are denoted $\check{i}_\ell$ and $\hat{i}_\ell$ for the lower and upper extremes, respectively. In the following, subscript δ will be dropped for the sake of simplicity.

Algorithm 2

Data δ, $e_i(\phi)$, $i = 1, 2, \ldots, m$.

Step 1 Set $k = 1$, $P_L = 2$, $P_U = 0$, θ = True.

Comment These values of P_L and P_U indicate initial bounds on R as $-\infty$ and $+\infty$, respectively. θ = True signifies that R is nonempty. k denotes the number of separate intervals.

Step 2 For $i = 1, 2, \ldots, m$ calculate R_i(i.e., r_{1i} and/or r_{2i}, $r_{1i} \leq r_{2i}$) and determine t_i, P_{Li} and P_{Ui}.

Comment P_{Li} is set to 2 if the left bound of R_i is at $-\infty$; P_{Ui} is set to 0 if the right bound of R_i is at $+\infty$. Otherwise they are set to 1.

Step 3 For each $i = 1, 2, \ldots, m$ for which t_i = False set θ = False and stop if $P_{Li} - P_{Ui} = 2$; otherwise proceed.

Comment If there is a function $e_i(\phi)$ such that $e_i(\phi) > \delta$ for all ϕ then, obviously, the valid region R is empty and the algorithm stops at this step.

Step 4 For $i = 1, 2, \ldots, m$, such that t_i = True, set

(1) $\check{\phi}_1 \leftarrow r_{1i}$ and $\check{i}_1 \leftarrow i$ if ($P_{Li} = 1$ and $P_L = 2$) or ($P_{Li} = 1$, $P_L = 1$ and $r_{1i} > \check{\phi}_1$),

(2) $\hat{\phi}_1 \leftarrow r_{2i}$ and $\hat{i}_1 \leftarrow i$ if ($P_{Ui} = 1$ and $P_U = 0$) or ($P_{Ui} = 1$, $P_U = 1$ and $r_{2i} < \hat{\phi}_1$),

(3) $P_L \leftarrow \min\{P_L, P_{Li}\}$,

(4) $P_U \leftarrow \max\{P_U, P_{Ui}\}$.

Comment The first term of (18) is calculated in this step as the intersection of intervals R_i for which t_i = True.

Step 5 If $P_L = P_U$ and $\check{\phi}_1 > \hat{\phi}_1$ set θ = False and stop.

Comment If the intersection calculated in Step 4 is empty the algorithm stops at this stage.

Step 6 Set $i \leftarrow 1$.

Comment We now start to remove from R the intervals for which t_i = False.

Step 7 If t_i = True go to Step 21.

Comment The intervals for which t_i = True have already been exploited.

Step 8 If $P_U = 0$ set $j_L = 0$; otherwise find $j_L = \max\{0, 1, \ldots, P_U\}$ such that $r_{1i} \geq \hat{\phi}_\ell$ for all $\ell = 1, \ldots, j_L$.

Comment j_L denotes the greatest index of intervals located to the left of R_i, thus the intervals with indices $1, 2, \ldots, j_L$ will not be affected by taking out R_i; if $j_L = 0$ there are no such intervals. P_U is now used as the number of finite upper bounds of valid intervals and is updated in Step 20.

Step 9 If $k+1 - P_L = 0$ set $j_U = k+1$; otherwise find $j_U = \min\{P_L, \ldots, k, k+1\}$ such that $r_{2i} \leq \check{\phi}_\ell$ for all $\ell = j_U, \ldots, k$.

Comment j_U assigned here indicates the lowest index of intervals located to the right of R_i, thus the intervals with indices j_U, j_U+1, ..., k will not be affected by taking out R_i; if $j_U = k+1$ there are no such intervals. The value of $k+1 - P_L$ indicates the number of finite lower bounds of intervals.

Step 10 If $j_U - j_L = 1$ go to Step 21.

Comment When $j_U = j_L+1$ the removal of R_i does not affect any of the existing intervals so we continue with the next i.

<u>Step 11</u> If $j_L+1 \geq P_L$ and $\check{\phi}_{j_L+1} > r_{1i}$ go to Step 15.

<u>Comment</u> This checks whether the lower bound of interval $j_L + 1$ is greater than the lower bound of R_i.

<u>Step 12</u> If $j_U-1 \leq P_U$ and $\hat{\phi}_{j_U-1} < r_{2i}$ go to Step 17.

<u>Comment</u> This checks whether the upper bound of interval $j_U - 1$ is lower than the upper bound of R_i.

<u>Step 13</u> Set $k_0 \leftarrow (j_U - j_L - 3)$; if $k_0 \geq 0$ go to Step 14; otherwise set $\check{\phi}_{\ell+1} \leftarrow \check{\phi}_\ell$ and $\check{i}_{\ell+1} \leftarrow \check{i}_\ell$ for $\ell=k, k-1, \ldots, j_U$ (only if $k \geq j_U$) and $\hat{\phi}_{\ell+1} \leftarrow \hat{\phi}_\ell$ and $\hat{i}_{\ell+1} \leftarrow \hat{i}_\ell$ for $\ell = P_U, P_U-1, \ldots, j_U-1$ (only if $P_U \geq j_U-1$).

<u>Comment</u> Steps 13 and 14 deal with the situation when the interval R_i intersects the (j_L+1)th as well as the $(j_U - 1)$th intervals and the k_0 intermediate intervals. This means that k_0 intervals are removed and the (j_L+1)th and (j_U-1)th intervals are reduced. If $k_0 = -1$ the (j_L+1)th interval is split into two intervals, in which case the above renumbering takes place.

<u>Step 14</u> For $\ell = j_L + 1$ set $\hat{\phi}_\ell \leftarrow r_{1i}$ and $\hat{i}_\ell \leftarrow i$.
For $\ell = j_L + 2$ set $\check{\phi}_\ell \leftarrow r_{2i}$ and $\check{i}_\ell \leftarrow i$; if $k_0 > 0$ and $j_U-1 \leq P_U$ set $\hat{\phi}_\ell \leftarrow \hat{\phi}_s$ and $\hat{i}_\ell \leftarrow \hat{i}_s$, where $\ell = j_L+2$ and $s = j_U-1$.
Go to Step 19.

<u>Comment</u> Interval reduction is executed here.

Step 15 If $j_U - 1 \le P_U$ and $\hat{\phi}_{j_U-1} < r_{2i}$ go to Step 18.

Comment This checks for the same phenomenon as Step 12.

Step 16 For $\ell = j_L + 1$ set $\check{\phi}_\ell \leftarrow r_{2i}$ and $\check{i}_\ell \leftarrow i$.
Set $k_0 \leftarrow (j_U - j_L - 2)$.
If $k_0 > 0$ and $j_U-1 \le P_U$ set $\hat{\phi}_\ell \leftarrow \hat{\phi}_s$ and $\hat{i}_\ell \leftarrow \hat{i}_s$, where $s = j_U-1$.
Go to Step 19.

Comment Only the interval j_U-1 is affected and consequently reduced.

Step 17 For $\ell = j_L + 1$ set $\hat{\phi}_\ell \leftarrow r_{1i}$ and $\hat{i}_\ell \leftarrow i$.
Set $k_0 \leftarrow (j_U - j_L - 2)$.
Go to Step 19.

Comment Only the interval j_L+1 is affected and consequently reduced.

Step 18 Set $k_0 \leftarrow (j_U - j_L - 1)$.

Comment All intermediate intervals (see Comment following Step 13) are to be completely removed.

Step 19 If $k_0 > 0$ set $\check{\phi}_{\ell-k_o} \leftarrow \check{\phi}_\ell$ and $\check{i}_{\ell-k_o} \leftarrow \check{i}_\ell$ for $\ell = j_U, j_U+1, \ldots, k$ (only if $j_U \le k$) and $\hat{\phi}_{\ell-k_o} \leftarrow \hat{\phi}_\ell$ and $\hat{i}_{\ell-k_o} \leftarrow \hat{i}_\ell$ for $\ell = j_U, j_U+1, \ldots, P_U$ (only if $j_U \le P_U$). Otherwise proceed.

Comment Renumbering takes place if necessary.

Step 20 Set $P_U \leftarrow (P_U - k_0)$ and $k \leftarrow (k - k_0)$.
If $k = 0$ set $\theta \leftarrow$ False and stop.

Step 21 If $i < m$ set $i \leftarrow i+1$ and go to Step 7; otherwise stop.

Algorithm 2 is used in Step 3 of Algorithm 1 for consecutive values of δ. According to Step 10 and the comment after Step 2 of Algorithm 1, the valid region will never be empty or unbounded so a little simpler version of Algorithm 2 can be utilized. Algorithm 2, however, can also be used independently of Algorithm 1. For instance, one run of it with $\delta = 0$ provides the feasible region for functions (A3) (Appendix A) to meet given specifications. This can be used in the case of single parameter tuning as is shown in the next section.

IV. EXAMPLE

A tunable active filter [15] has been chosen to implement the theory and algorithms. The filter is shown in Fig. 2 and its equivalent circuit in Fig. 3. The specifications w.r.t. frequency on the modulus squared of the transfer function $F = |V_2/V_g|^2$ are

$$F \leq 0.5 \text{ for } f/f_0 \leq 1-10/f_0,$$
$$F \leq 1.21 \text{ for } 1-10/f_0 \leq f/f_0 \leq 1+10/f_0,$$
$$F \leq 0.5 \text{ for } f/f_0 \geq 1+10/f_0,$$
$$F \geq 0.5 \text{ for } 1-8/f_0 \leq f/f_0 \leq 1+8/f_0,$$
$$F \geq 1 \text{ for } f = f_0 \text{ Hz},$$

where f_0 is the center frequency. Using the one pole roll-off model for the operational amplifiers, given by

$$A(s) = \frac{A_0 \omega_a}{s+\omega_a} ,$$

where s is the complex frequency, A_0 is the d.c. gain and ω_a the 3 dB radian bandwidth, the nodal equations are

$$\begin{bmatrix} G_1+G_g & 0 & -G_1 & 0 \\ 0 & G_2+G_3+sC_2+A_2G_3 & -sC_2 & -G_2+A_1A_2G_3 \\ -G_1 & -sC_2 & G_1+G_4+sC_1+sC_2 & -sC_1 \\ 0 & -G_2 & -sC_1 & G_2+sC_1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} G_gV_g \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Based on two consecutive analyses a biquadratic model in R_4 was obtained at each sample frequency. The normalized sample frequencies are taken as 1 and $1 \pm 10/f_0$ for the relevant upper specifications, 1 and $1 \pm 8/f_0$ for the relevant lower specifications. This leads to six error functions e_i, i = 1, 2, ..., 6. The range of R_4 for which the specifications are satisfied is that for which $e_i \leq 0$, i = 1, 2, ..., 6. The maximum of the error functions e_i, versus R_4 is shown in Fig. 4. A single run of a program implementing Algorithm 2 indicated that the filter is tunable for the specifications defined at a center frequency of 100 Hz. It meets these specifications if

$$R_4 \in [181.126, \quad 187.166]$$

and with other circuit parameters fixed at values given in Table II. It is also tunable around a center frequency of 700 Hz (see Fig. 5) and meets the specifications if

$$R_4 \in [3.4881, \quad 3.5012] .$$

To find $\min_{R_4} \max_i e_i$, we are faced with the local minima in Fig. 4. The convergence of other algorithms to the global minimum depends upon the starting point. For the proposed algorithm the results are shown in Table III for different starting points and at different center frequencies. Note how small is the number of iterations required.

When R_1 was altered to the value 14 kΩ the filter is not tunable as is determined by one run of the program. The optimum value of R_4, however, was obtained in only two iterations (see Table III). In fact, the algorithm converged in the first iteration since the optimum is defined by one function, however, the second iteration was performed to satisfy the stopping criterion.

V. CONCLUSIONS

Implications of the bilinear behaviour of certain linear circuits in the frequency domain have been investigated. The explicit determination of the points defining the boundary of the feasible region w.r.t. one parameter led to a considerably simple check on the tunability of an outcome of the manufacturing process by adjusting a single parameter at a time. Detection of worst cases within an interval for any circuit parameter, of course, is also facilitated.

The proposed minimax algorithm is not only extremely efficient but is also globally convergent. It has been shown how few iterations are required for convergence to the global minimax optimum from different starting points even when local minima exist.

APPENDIX A: BIQUADRATIC MODELS

For certain lumped, linear circuits, we can express the circuit response as a bilinear function in a variable parameter ϕ (see, for example, Fidler [2])

$$f(\phi) = \frac{u + a\phi}{1 + b\phi}, \tag{A1}$$

where f is the circuit response at a particular frequency s, while u, a and b are complex constants in general. The variable ϕ does not necessarily have the value of the parameter, but it may take the value of the parameter ρ referred to a reference value ρ^0. Hence, we take

$$\phi = \rho - \rho^0. \tag{A2}$$

It is to be noted that while u and a may assume zero value, b is never zero for all practical problems.

As is known, only two analyses with the same LU factors are required to obtain the complex constants in (A1) [3].

Since the magnitude of the response $|f|$ or functions of this magnitude are often of interest, we may write

$$|f(\phi)|^2 = \frac{|u|^2 + 2\,R(u^* a)\phi + |a|^2\phi^2}{1 + 2\,R(b)\phi + |b|^2\phi^2}, \tag{A3}$$

where u^* is the complex conjugate of u and $R(\cdot)$ denotes the real part of $(\cdot)$.

If (A3) is to be greater than a lower specification S_L and/or smaller than an upper specification S_U we consider error function(s)

$$e \stackrel{\Delta}{=} w\,(F - S) \tag{A4}$$

where

$$w = \begin{cases} -1 \text{ for lower specification} \\ \\ 1 \text{ for upper specification} \end{cases}$$

After substituting (A3) into (A4) the resulting function is also a biquadratic function

$$e(\phi) = \frac{A + 2B\phi + C\phi^2}{1 + 2D\phi + E\phi^2}\,. \tag{A5}$$

Since (A3) represents the magnitude squared of the response f we have $F \geq 0$ and, in particular, we can assume

$$1 + 2D\phi + E\phi^2 > 0 \tag{A6}$$

for any ϕ. The following discussion is also valid in the case when the inequality (A6) is weak. This is possible only if (A5) and (A3) have a double pole at the point $\phi = 1/b$ when b is real. Otherwise (A6) holds. We can also assume that the function (A1) essentially depends on ϕ and, as a consequence, (A5) is irreducible, i.e.,

$$E > 0 \tag{A7}$$

and

$$4(AD-B)(BE-DC) - (AE-C)^2 \neq 0. \tag{A8}$$

Now, suppose we are interested in finding the values of ϕ such that

$$e(\phi) \leq \delta, \tag{A9}$$

where δ is a given number (not necessarily greater than zero). Substituting δ on the left hand side of (A5) we obtain the equation

$$(C-\delta E)\phi^2 + 2(B-\delta D)\phi + (A-\delta) = 0, \tag{A10}$$

whose two, one or zero roots indicate the boundary points of valid intervals. It is easy to check that the Table I exploits all cases which are possible under the assumptions (A6)-(A8).

APPENDIX B: PROOF OF CONVERGENCE OF ALGORITHM 1

Let us consider two different functions $e_{\check{i}}(\phi)$ and $e_{\hat{i}}(\phi)$ which define the extreme points $\check{\phi}^n$, $\hat{\phi}^n$ of I^n. (The proof is obvious from Step 8 of the algorithm if only one function is considered.) Without loss of generality, we can assume that $e_i(\phi) \leq e_{\check{i}}(\phi)$, for $\phi \in \check{I}^n$ and $e_i(\phi) \leq e_{\hat{i}}(\phi)$ for $\phi \in \hat{I}^n$, $i = 1, 2, \ldots, m$, where $\check{I}^n \triangleq [\check{\phi}^n, \phi_{min}]$, $\hat{I}^n = [\phi_{min}, \hat{\phi}^n]$ and ϕ_{min} is the unique intersection point of $e_{\check{i}}(\phi)$ and $e_{\hat{i}}(\phi)$ in the interval I^n. There is also no loss of generality if we assume $\check{\phi}^{n+1} = \phi^{n*}$ for all n and that $\check{g}^n < 0$, $\hat{g}^n > 0$, since there is only a finite set of ϕ for which the derivative is zero. We will show that there exists a value $\gamma < 1$ such that

$$|\hat{\phi}^{n+1} - \check{\phi}^{n+1}| \leq \gamma \, |\hat{\phi}^{n} - \check{\phi}^{n}| \text{ for any } n. \qquad \text{(B1)}$$

Since $\hat{g}^n > 0$ the interval I^{n+1} can be estimated as follows. We have

$$\hat{\phi}^{n+1} - \check{\phi}^{n+1} = \hat{\phi}^{n+1} - \phi^{n*} < \hat{\phi}^{n} - \phi^{n*} = \frac{\check{g}^n}{\check{g}^n - \hat{g}^n} (\hat{\phi}^{n} - \check{\phi}^{n}). \qquad \text{(B2)}$$

If $e_{\hat{i}}(\phi)$ is such that $e'_{\hat{i}}(\phi) \geq \xi$ for any $\phi \in \hat{I}^n$, where ξ is a sufficiently small positive number, we will find γ as

$$\gamma = \frac{\eta}{\eta+\xi} < 1, \qquad \text{(B3)}$$

where

$$\eta = \max_{\phi \in \check{I}^n} -e'_{\check{i}}(\phi). \qquad \text{(B4)}$$

The above estimate is not possible only if $e'_{\hat{i}}(\phi) \to 0$ and $e'_{\check{i}}(\phi) \to c \neq 0$ for $\phi \to \phi_{min}$.[2] In this case the function $e_{\hat{i}}(\phi)$ becomes convex in the interval $[\phi_{min}, \hat{\phi}^n]$ for any $n \geq N$ when N is sufficiently large. Then the interval I^{n+1} can be estimated as follows

$$\hat{\phi}^{n+1} - \check{\phi}^{n+1} < \hat{\phi}^{nL} - \phi^{n*}, \qquad \text{(B5)}$$

---

[2] If both $e'_{\check{i}}(\phi) \to 0$ and $e'_{\hat{i}}(\phi) \to 0$ the rates of convergence of $\check{g}^n$ and $\hat{g}^n$ are of the same order and it is possible to find an estimate of $\hat{g}^n/\check{g}^n$ such that (B1) is satisfied. See [16] for details.

where $\hat{\phi}^{nL}$ is the intersection point of the linearization of $e_i^{\wedge}(\phi)$ at the point $\hat{\phi}^n$ and the line $\delta^{n+1} = e_i^{\vee}(\phi^{n*})$.

From the appropriate geometrical relations (see Fig. 6) we obtain

$$\frac{\hat{\phi}^{nL} - \phi^{n*}}{\hat{\phi}^{n} - \phi^{n*}} = \frac{\Delta^n + e_i^{\vee}(\phi^{n*}) - e_i^{\vee}(\check{\phi}^n)}{\Delta^n} . \tag{B6}$$

After some manipulations, we find that

$$\hat{\phi}^{nL} - \phi^{n*} = \frac{R^n}{\Delta^n}(\hat{\phi}^n - \phi^{n*}) = \frac{R^n}{\hat{g}^n} , \tag{B7}$$

where R^n is the second-order remainder of Taylor's formula for the function $e_i^{\vee}(\phi^{n*})$ at the point $\check{\phi}^n$. It can be written in the form

$$\hat{\phi}^{nL} - \phi^{n*} = \left[\frac{1}{2}\check{q}^n \hat{g}^n \frac{\hat{\phi}^n - \check{\phi}^n}{(\check{g}^n - \hat{g}^n)^2}\right](\hat{\phi}^n - \check{\phi}^n), \tag{B8}$$

where $\check{q}^n$ is the second derivative of $e_i^{\vee}(\phi)$ at some $\phi \in \check{I}^n$. Since $\check{q}^n$ is limited by a number ζ and $\hat{g}^n \to 0$, $\check{g}^n \to c \neq 0$, we can find a sufficiently large number $N_1 \geq N$ such that the number

$$\gamma = \frac{1}{2}\zeta\, \hat{g}^{N_1} \frac{\hat{\phi}^{N_1} - \check{\phi}^{N_1}}{c^2} < 1 \tag{B9}$$

satisfies the condition (B1) for all $n \geq N_1$. But according to (B2) the interval I^{N_1} can be reached after a finite number of steps since $\hat{g}^n$ and $-\check{g}^n$ are greater than sufficiently small positive numbers for any $n < N_1$. This proves that $|\hat{\phi}^n - \check{\phi}^n| \to 0$ as $n \to \infty$. Now, we shall investigate the rate of this convergence.

Because of the estimate (B5) and equality (B8) we have already proved that the convergence is at least of the second order in the foregoing case. This result can be generalized for any case when the function $e_i^{\wedge}(\phi)$ becomes convex in a neighbourhood of ϕ_{min}. This is because the neighbourhood in question can be reached after a finite number of steps, following which the estimate (B5) is valid. The only exception is for the case when ϕ_{min} is the minimizing point of both functions $e_i^{\vee}(\phi)$ and $e_i^{\wedge}(\phi)$ since the denominator of (B8) approaches zero if $n \to \infty$. Detailed proof for this case can be found in [16]. Now, we can assume that both $\check{g}^n$ and $\hat{g}^n$ do not approach zero as $n \to \infty$.

Using the second order Taylor formula we can write

$$e_i^{\vee}(\check{\phi}^{n+1}) = e_i^{\vee}(\check{\phi}^n) + \check{g}^n (\check{\phi}^{n+1} - \check{\phi}^n) + \frac{1}{2} \check{q}^n (\check{\phi}^{n+1} - \check{\phi}^n)^2, \quad \text{(B10)}$$

$$e_i^{\wedge}(\hat{\phi}^{n+1}) = e_i^{\wedge}(\hat{\phi}^n) + \hat{g}^n (\hat{\phi}^{n+1} - \hat{\phi}^n) + \frac{1}{2} \hat{q}^n (\hat{\phi}^{n+1} - \hat{\phi}^n)^2. \quad \text{(B11)}$$

Knowing that $e_i^{\vee}(\check{\phi}^n) = e_i^{\wedge}(\hat{\phi}^n)$ for any n, (B10) and (B11) give the relation

$$\hat{g}^n (\hat{\phi}^{n+1} - \hat{\phi}^n) - \check{g}^n (\check{\phi}^{n+1} - \check{\phi}^n) = \frac{1}{2}\left[\check{q}^n (\check{\phi}^{n+1} - \check{\phi}^n)^2 - \hat{q}^n (\hat{\phi}^{n+1} - \hat{\phi}^n)^2\right]. \quad \text{(B12)}$$

Using (10), the left hand side of (B12) can be written as

$$\phi^{n*} (\check{g}^n - \hat{g}^n) + \hat{g}^n \hat{\phi}^{n+1} - \check{g}^n \check{\phi}^{n+1}.$$

Now, if we assume as before, for example, that $\phi^{n*} = \check{\phi}^{n+1}$, (B12) can be rewritten as

$$\hat{g}^n (\hat{\phi}^{n+1} - \check{\phi}^{n+1}) = \frac{1}{2}\left[\check{q}^n (\check{\phi}^{n+1} - \check{\phi}^n)^2 - \hat{q}^n (\hat{\phi}^{n+1} - \hat{\phi}^n)^2\right]. \quad \text{(B13)}$$

From the above we have the estimate

$$|\hat{\phi}^{n+1} - \check{\phi}^{n+1}| \leq \frac{1}{2\hat{g}^n} \left[|\check{q}^n| \, (\check{\phi}^{n+1} - \check{\phi}^n)^2 + |\hat{q}^n| \, (\hat{\phi}^{n+1} - \hat{\phi}^n)^2 \right]$$

$$\leq \frac{1}{2\hat{g}^n} (|\check{q}^n| + |\hat{q}^n|) \, (\hat{\phi}^n - \check{\phi}^n)^2. \qquad \text{(B14)}$$

The final estimate is based on the fact that both $\check{\phi}^{n+1}$ and $\hat{\phi}^{n+1}$ are interior points of I^n so $(\check{\phi}^{n+1} - \check{\phi}^n)^2 < (\hat{\phi}^n - \check{\phi}^n)^2$ and $(\hat{\phi}^{n+1} - \hat{\phi}^n)^2 < (\hat{\phi}^n - \check{\phi}^n)^2$. Since second derivatives $\check{q}^n$ and $\hat{q}^n$ are bounded and $\hat{g}^n$ does not approach zero the factor on the right side of (B14) has a finite limit, so the convergence of the algorithm is at least of the second order.

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TABLE I

VALID INTERVALS FOR INEQUALITY (A9)

$sgn(\delta - \frac{C}{E})$	$sgn[(B-\delta D)^2 - (C-\delta E)(A-\delta)]$	sgn(BE-CD)	sgn(AE-C)	Intervals (points) of ϕ such that $e(\phi) < \delta$	$e(\phi) > \delta$	$e(\phi) = \delta$
1	1 or 0	*	*	$(-\infty, r_1) \vee (r_2, \infty)$	$\underline{(r_1, r_2)}$	r_1, r_2
1	-1	*	*	$\underline{(-\infty, \infty)}$	-	-
-1	1 or 0	*	*	$\underline{(r_1, r_2)}$	$(-\infty, r_1) \vee (r_2, \infty)$	r_1, r_2
-1	-1	*	*	-	$\underline{(-\infty, \infty)}$	-
0	*	1	*	$\underline{(-\infty, r)}$	(r, ∞)	r
0	*	-1	*	$\underline{(r, \infty)}$	$(-\infty, r)$	r
0	*	0	1	-	$\underline{(-\infty, \infty)}$	-
0	*	0	-1	$\underline{(-\infty, \infty)}$	-	-

* denotes values of no interest,

$r_1 \leq r_2$ denote the two real roots of equation (A10),

r denotes single real root of equation (A10) when $\delta = C/E$.

TABLE II

CIRCUIT PARAMETERS

R_g	=	50.000 Ω	C_1	=	0.728556 μF
R_1	=	12.446 kΩ	C_2	=	0.728556 μF
R_2	=	26.500 kΩ	A_0	=	2×10^5
R_3	=	75.000 Ω	ω_a	=	$12\,\pi$ rad/s

TABLE III

MINIMAX OPTIMUM OF R_4

Center Frequency (Hz)	R_4(Ω) Starting	R_4(Ω) Optimum	Optimum δ	N.O.I.*
100	100.0	184.3998	-0.0458	3
	300.0	184.3998	-0.0458	3
	∞	184.3998	-0.0458	3
700	10.0	3.4946	-0.0403	3
	200.0**	3.4946	-0.0403	3
	200.0	3.4940	0.1434	2

* N.O.I. = number of iterations

** R_1 was altered to 14.0 kΩ and the filter is not tunable since $\delta>0$.

Formula (11) was used to initialize δ at starting values of R_4. Running times per example on a CDC 6400 computer were about 0.1 s.

FIGURE CAPTIONS

Fig. 1 Illustration of the behaviour of the one-dimensional minimax algorithm. Note that the algorithm switches from interval 1 to interval 2, based on predictions of the decrease in the maximum.

Fig. 2 Tunable active filter.

Fig. 3 Equivalent circuit for nodal analysis.

Fig. 4 $\max_{1 \leq i \leq 6} e_i$ versus the tuning resistor R_4 for specifications defined around f_0 = 100 Hz indicating the active functions (and hence active frequency points).

Fig. 5 $\max_{1 \leq i \leq 6} e_i$ versus R_4 for specifications defined around f_0 = 700 Hz for two cases (a) R_1 = 12.446 kΩ, (b) R_1 = 14 kΩ.

Fig. 6 Two functions which define the minimax optimum. The point $\hat{\phi}^{nL}$ at which the linear approximation at $\hat{\phi}^n$ takes the value of $\check{e}_i(\phi^{n*})$ is indicated.

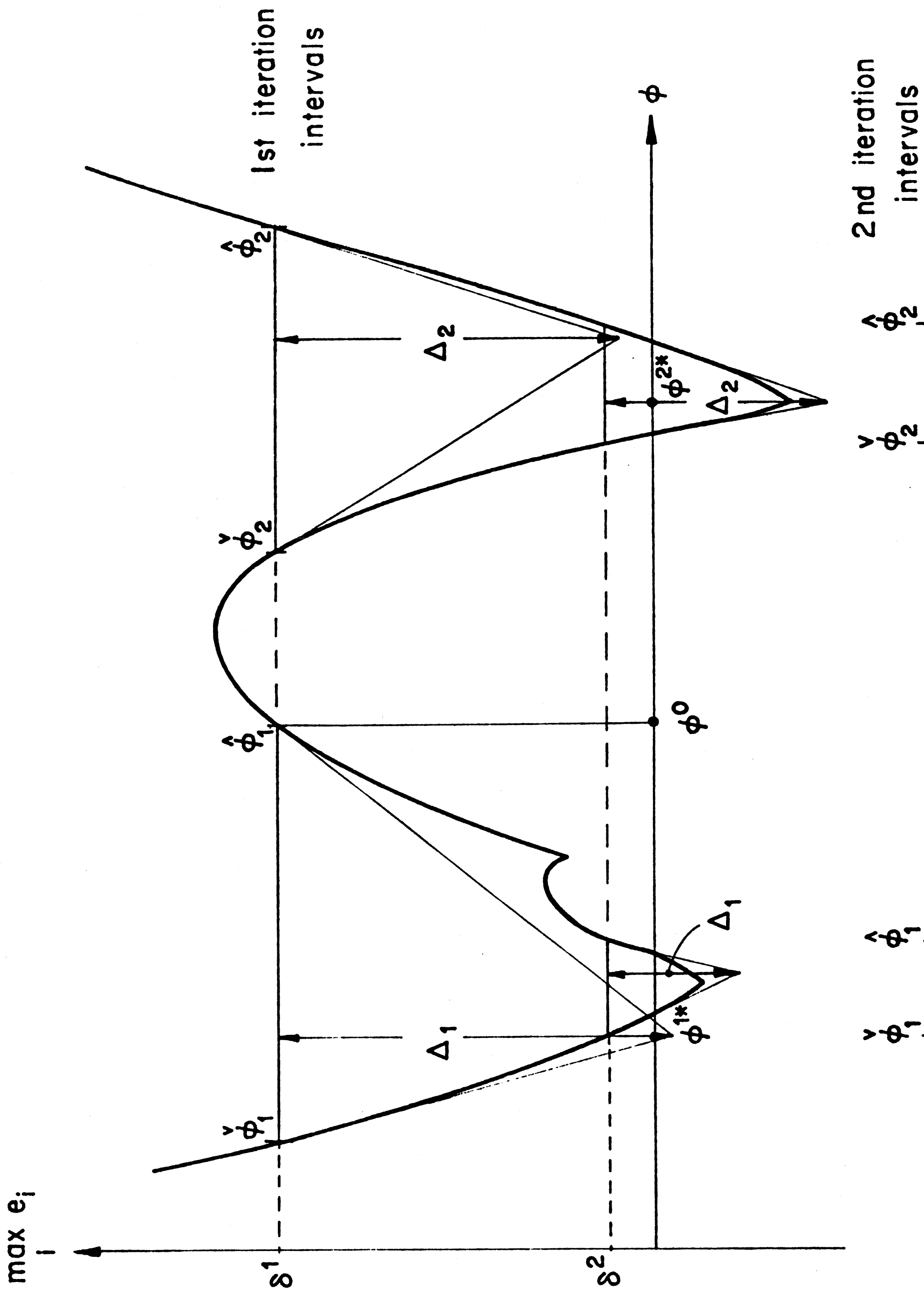

max e_i
δ^1
δ^2
φ
φ^o
φ^1*
φ^2*
Δ_1
Δ_2
1st iteration intervals
2nd iteration intervals

Fig. 1

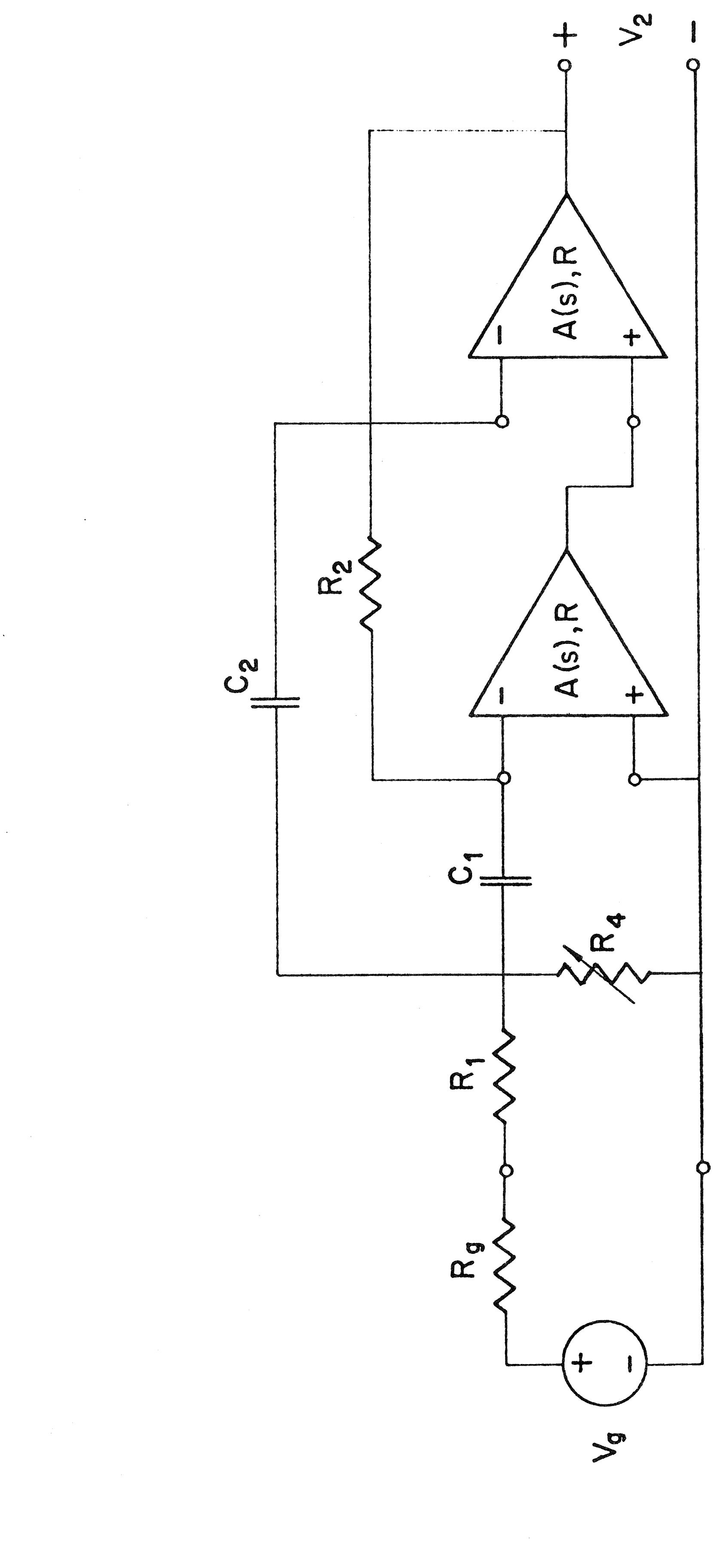


Fig. 2

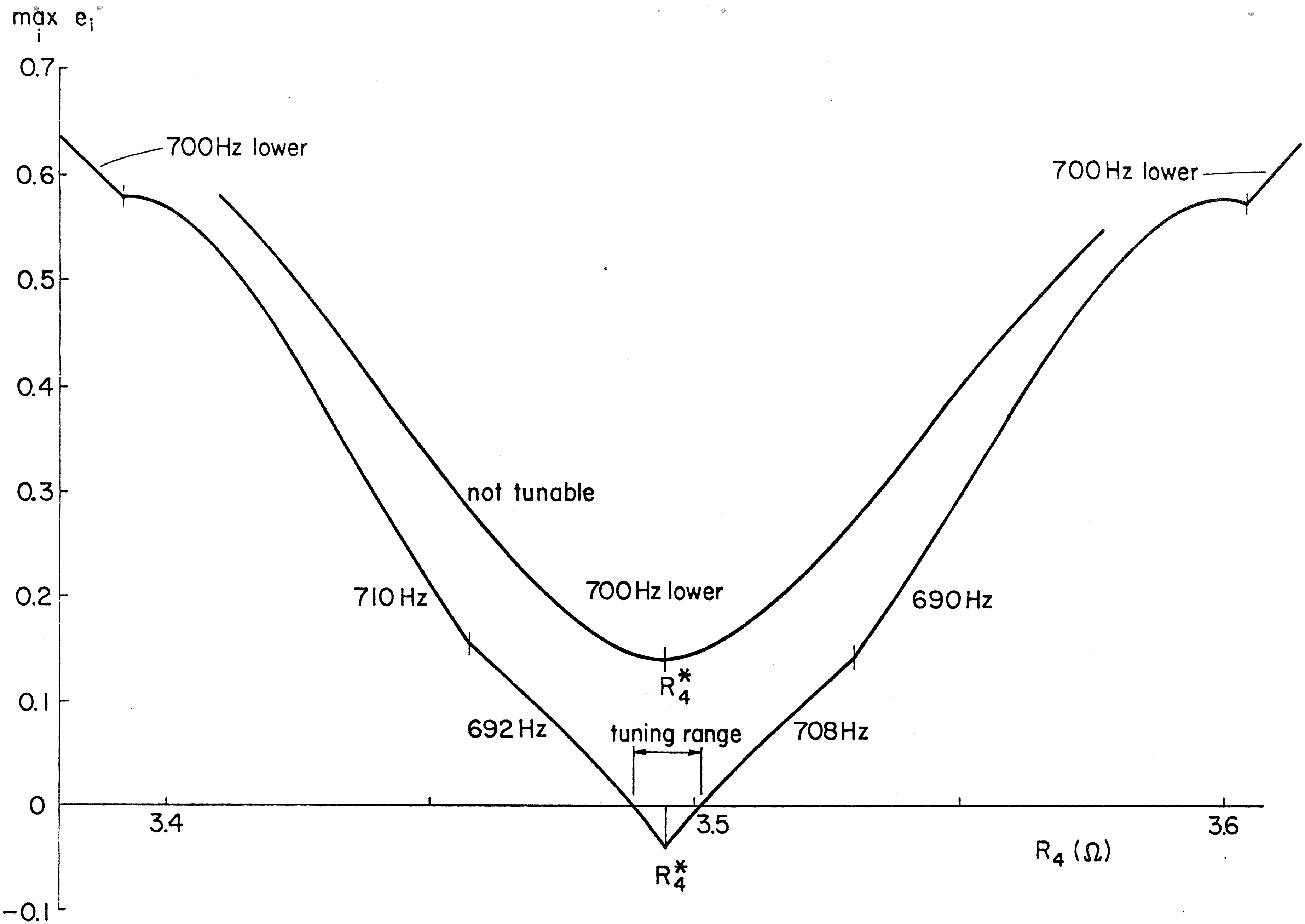
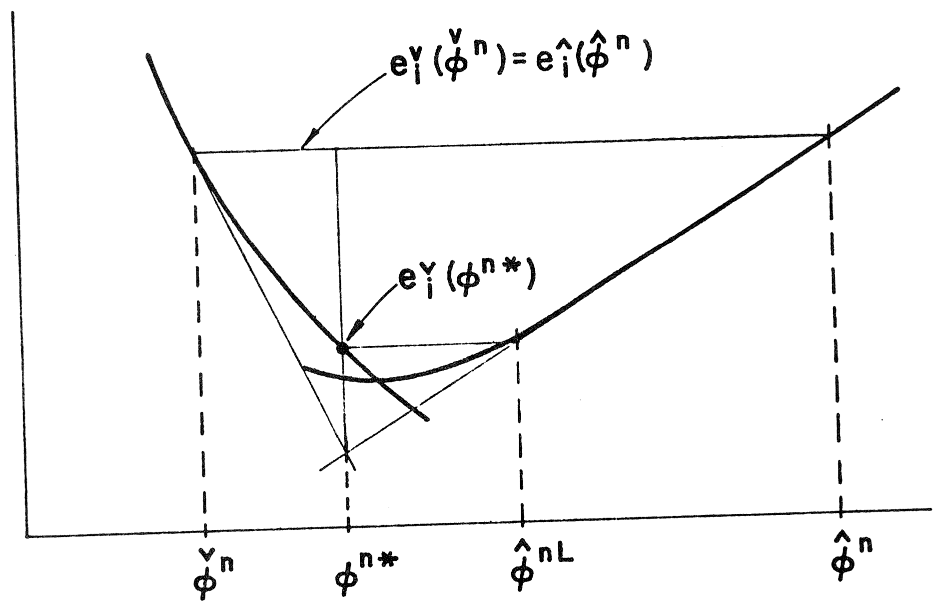

max e_i
i
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
700Hz lower
700Hz lower
not tunable
710Hz
700Hz lower
690Hz
R*4
692Hz
tuning range
708Hz
3.4
3.5
3.6
R*4
R4 (Ω)

Fig. 5



$e_i^{\vee}(\check{\phi}^n) = e_i^{\wedge}(\hat{\phi}^n)$
$e_i^{\vee}(\phi^{n*})$
$\check{\phi}^n$
ϕ^{n*}
$\hat{\phi}^{nL}$
$\hat{\phi}^n$

Fig. 6

SOC-240

A ONE-DIMENSIONAL MINIMAX ALGORITHM BASED ON BIQUADRATIC MODELS AND ITS APPLICATION IN CIRCUIT DESIGN

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Key Words: Design centering, tolerancing, tuning, minimax design, biquadratic functions, one-dimensional optimization

Abstract: This paper exploits the biquadratic behaviour w.r.t. a variable exhibited in the frequency domain by certain lumped, linear circuits. A globally convergent and extremely efficient minimax algorithm is developed and tested to optimize the frequency response w.r.t. any circuit parameter. It is shown that the algorithm converges to the global minimax optimum and that the rate of convergence is at least of second order. The algorithm is based on the linearization of error functions at boundary points of valid intervals. Boundary points of the region of acceptable designs are explicitly calculated and an algorithm to exactly determine the region itself for the general nonconvex case is presented and illustrated.

Description:

Related Work: SOC-37, SOC-87, SOC-211, SOC-229.

Price: $ 6.00.