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Abstract This letter shows that the necessary conditions for an optimum in nonlinear minimax approximation problems do not require a qualification analogous to the Kuhn-Tucker constraint qualification.

Conditions for optimality in the Chebyshev or minimax sense are of considerable interest in circuit and system analysis and design. Furthermore, it is well known that nonlinear minimax approximation problems can be reformulated and solved by nonlinear programming [1]. Assuming differentiability of the functions concerned, that a point ϕ^0 is a local optimum, and that the Kuhn-Tucker constraint qualification holds at ϕ^0, the Kuhn-Tucker conditions can be shown to hold [2]. The reason for requiring the constraint qualification is the uncertainty in the validity of the Kuhn-Tucker conditions at local optima of certain problems. This letter shows that such an uncertainty does not arise in normal minimax approximation problems.

First, a statement of Farkas' Lemma [2] is required. Let $p_0, p_1, \ldots, p_n$, be an arbitrary set of vectors. There exist $\beta_i \geq 0$ such that

$$p_0 = \sum_{i=1}^{n} \beta_i \, p_i$$

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if and only if

$$\underset{\sim}{p}_0^T \underset{\sim}{q} \geq 0$$

for all $\underset{\sim}{q}$ satisfying

$$\underset{\sim}{p}_i^T \underset{\sim}{q} \geq 0, \ i = 1,2,\ldots,n.$$

Consider the problem

$$\text{minimize } U = \phi_{k+1}$$

subject to

$$\phi_{k+1} \geq f_i(\underset{\sim}{\phi}) \ , \ i \in I$$

with $f_i(\underset{\sim}{\phi})$, $i \in I$, differentiable with respect to $\underset{\sim}{\phi} \triangleq [\phi_1 \ \phi_2 \ \ldots \ \phi_k]^T$. This is equivalent to finding $\min_{\underset{\sim}{\phi}} \max_{i \in I} f_i(\underset{\sim}{\phi})$. Let $\underset{\sim}{\nabla} \triangleq [\partial/\partial\phi_1 \ \partial/\partial\phi_2 \ \ldots \ \partial/\partial\phi_k]^T$.

Now let $\underset{\sim}{p}_0$ correspond to

$$\begin{bmatrix} \underset{\sim}{\nabla} U \\ \dfrac{\partial U}{\partial \phi_{k+1}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{0} \\ 1 \end{bmatrix},$$

and $\underset{\sim}{p}_1, \underset{\sim}{p}_2, \ldots, \underset{\sim}{p}_n$, correspond to

$$\begin{bmatrix} \underset{\sim}{\nabla} \\ \dfrac{\partial}{\partial \phi_{k+1}} \end{bmatrix} (\phi_{k+1}^0 - f_i(\underset{\sim}{\phi}^0)) = \begin{bmatrix} -\underset{\sim}{\nabla} f_i(\underset{\sim}{\phi}^0) \\ 1 \end{bmatrix}, \ i \in I^0$$

where

$$I^0 \triangleq \{i \mid \phi_{k+1}^0 = f_i(\underset{\sim}{\phi}^0)\}$$

and

$$\begin{bmatrix} \underset{\sim}{\phi}^0 \\ \phi_{k+1}^0 \end{bmatrix}$$

is the local optimum under consideration, and which is assumed to exist.

Paraphrasing Farkas' Lemma, there exist $u_i \geq 0$ such that

$$\begin{bmatrix} \underset{\sim}{0} \\ 1 \end{bmatrix} = \sum_{i\epsilon I_0} u_i \begin{bmatrix} -\underset{\sim}{\nabla} f_i(\underset{\sim}{\phi}^0) \\ 1 \end{bmatrix} \tag{1}$$

if and only if

$$\begin{bmatrix} \underset{\sim}{0} \\ 1 \end{bmatrix}^T \begin{bmatrix} \Delta\underset{\sim}{\phi} \\ \Delta\phi_{k+1} \end{bmatrix} \geq 0 \tag{2}$$

for all $\{\Delta\underset{\sim}{\phi}, \Delta\phi_{k+1}\}$ satisfying

$$\begin{bmatrix} -\underset{\sim}{\nabla} f_i(\underset{\sim}{\phi}^0) \\ 1 \end{bmatrix}^T \begin{bmatrix} \Delta\underset{\sim}{\phi} \\ \Delta\phi_{k+1} \end{bmatrix} \geq 0, \; i \; \epsilon \; I^0 \tag{3}$$

where

$$\Delta\underset{\sim}{\phi} \triangleq \underset{\sim}{\phi} - \underset{\sim}{\phi}^0, \; \Delta\phi_{k+1} = \phi_{k+1} - \phi^0_{k+1}.$$

Equation (1) involves the necessary conditions, (2) can be written as $\Delta\phi_{k+1} \geq 0$, and (3) as $-\underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^0)\Delta\underset{\sim}{\phi} + \Delta\phi_{k+1} \geq 0$, $i \; \epsilon \; I^0$. By assumption, $\underset{\sim}{\phi}^0$ is a local optimum and the f_i are differentiable. Hence, it is impossible to find $\Delta\underset{\sim}{\phi}$ such that $\underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^0)\Delta\underset{\sim}{\phi} < 0$ for all $i \; \epsilon \; I^0$. In this case $\Delta\phi_{k+1}$ must be nonnegative. (This can also be seen by examining the statement of the problem). Thus (1) holds for $u_i \geq 0$, $i \; \epsilon \; I^0$ without apparent additional qualification.

REFERENCES

[1] J.W. Bandler, "Conditions for a minimax optimum", <u>IEEE Trans. Circuit Theory</u>, vol. CT-18, July 1971, pp. 476-479.

[2] L.S. Lasdon, <u>Optimization Theory for Large Systems.</u> New York: MacMillan, 1970, ch. 1.