

**A GENERALIZED,  
COMPLEX ADJOINT APPROACH  
TO POWER NETWORK SENSITIVITIES**

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A GENERALIZED, COMPLEX ADJOINT APPROACH TO  
POWER NETWORK SENSITIVITIES

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Abstract

A unified study of the class of adjoint network approaches to power system sensitivity analysis which exploits the Jacobian matrix of the load flow solution is presented. Generalized sensitivity expressions which are easily derived, compactly described and effectively used for calculating first-order changes and gradients of functions of interest are obtained. These generalized sensitivity expressions are common to all modes of formulating the power flow equations, e.g., polar and cartesian. The approach exploits a special complex notation and complex matrix manipulations to define the adjoint system and to derive the sensitivity formulas. The approach is applicable to both real and complex function sensitivities.

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## I. INTRODUCTION

Two kinds of analysis can be distinguished in power system operation and planning studies. In the first kind, which implies the load flow solution [1,2] of the power network, the system states are obtained with the control (independent) variables fixed at particular values. The solution obtained describes the power system steady state behaviour associated with these particular values of the control variables. The second kind of analysis deals with variations in control variables and the resulting effect on either system states or, in general, on a particular function of interest [3-5]. This analysis is usually referred to as sensitivity analysis. The importance of sensitivity analysis has been recognized [6-7] in power system operation and planning studies to supply first-order changes of functions of interest and their gradients required for effective optimization techniques.

The class of adjoint network approaches [6,8-10] incorporating the method of Lagrange multipliers provides the advantage of using the transpose of the Jacobian of the load flow problem as an adjoint matrix of coefficients. When describing adjoint network approaches which exploit the Jacobian of the load flow problem, the sensitivity expressions for different elements are derived according to the mode of formulation used, e.g., polar or cartesian. Different forms of sensitivity expressions have been presented for different studies. A unified sensitivity study for this class of adjoint network approaches has not, however, been previously described.

The impact of the conjugate notation [10,11], which describes the first-order changes of general complex functions in terms of formal

derivatives w.r.t. complex system variables provides a useful tool for describing a generalized adjoint network sensitivity approach, as presented in this paper, where generalized sensitivity expressions are easily derived, compactly described and effectively used subject to any mode of formulation. The adjoint matrix of coefficients is always the transpose of the Jacobian of the original load flow problem and, regardless of the formulation, these generalized sensitivity expressions can be used.

In the first few sections, we briefly describe the notation used and illustrate the problem formulation. For the detailed analytical aspects of the conjugate notation, the reader is referred to [10,11]. We then derive the complex transformation matrices relating different modes of formulating the power flow equations to a standard complex form. This standard complex form is employed in the subsequent sections to define and analyze the adjoint system and to derive the generalized sensitivity formulas. In order to illustrate the novel concepts, two examples of the simplest 2-bus sample power system are employed throughout the paper. Numerical results for a 6-bus sample power system are also presented. The formulas derived, however, are general and can be directly programmed for a general power system of practical size.

## II. NOTATION

In the conjugate notation [10,11], a complex variable

$$\zeta_i = \zeta_{i1} + j \zeta_{i2}, \quad (1)$$

and its complex conjugate  $\zeta_i^*$  replace, as independent quantities, the real and imaginary parts of the variable. Hence, we may express the first-order change of a continuous function of a set of complex

variables arranged in a column vector  $\zeta$ ,

$$\zeta = \zeta_1 + j \zeta_2, \quad (2)$$

and their complex conjugate  $\zeta^*$  in the form

$$\delta f = \left( \frac{\partial f}{\partial \zeta} \right)^T \delta \zeta + \left( \frac{\partial f}{\partial \zeta^*} \right)^T \delta \zeta^*, \quad (3)$$

where  $\delta$  denotes first-order change, T denotes transposition and  $\partial f / \partial \zeta$  and  $\partial f / \partial \zeta^*$  are column vectors representing the formal [12] partial derivatives of  $f$  w.r.t.  $\zeta$ , and  $\zeta^*$ , respectively.

It can be shown [10] that, for a real function  $f$ , we may write

$$\frac{\partial f}{\partial \zeta^*} = \left( \frac{\partial f}{\partial \zeta} \right)^*. \quad (4)$$

### III. BASIC FORMULATION

#### Load Flow Equations

The electric power network can be represented by a system of node equations in the form

$$Y_T V_M = I_M, \quad (5)$$

where

$$Y_T = Y_{T1} + j Y_{T2} \quad (6)$$

is the bus admittance matrix of the power network,

$$V_M = V_{M1} + j V_{M2} \quad (7)$$

is a column vector of the bus voltages, and

$$I_M = I_{M1} + j I_{M2} \quad (8)$$

is a vector of bus currents.

We write the bus loading equations in the matrix form

$$E_M^* I_M = S_M^*, \quad (9)$$

where  $E_M$  is a diagonal matrix of components of  $V_M$  in corresponding order, i.e.,

$$\underline{E}_M \underline{v} = \underline{V}_M, \quad (10)$$

where  $\underline{v}$  is given by

$$\underline{v} \triangleq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \quad (11)$$

and  $\underline{S}_M$  is a vector of the injected bus powers given by

$$\underline{S}_M \triangleq \underline{P}_M + j \underline{Q}_M. \quad (12)$$

Substituting (5) into (9), we get

$$\underline{E}_M^* \underline{Y}_T \underline{V}_M = \underline{S}_M^*. \quad (13)$$

The system of nonlinear equations (13) represents the typical load flow problem, whose solution is required.

### Complex Perturbed Form

The system (13) may be written in the perturbed form

$$\underline{K}^S \delta \underline{V}_M + \overline{\underline{K}}^S \delta \underline{V}_M^* = \delta \underline{S}_M^* - \underline{E}_M^* \delta \underline{Y}_T \underline{V}_M, \quad (14)$$

where  $\delta \underline{V}_M$ ,  $\delta \underline{V}_M^*$ ,  $\delta \underline{S}_M^*$  and  $\delta \underline{Y}_T$  represent first-order changes of  $\underline{V}_M$ ,  $\underline{V}_M^*$ ,  $\underline{S}_M^*$  and  $\underline{Y}_T$ , respectively,

$$\underline{K}^S \triangleq \underline{E}_M^* \underline{Y}_T \quad (15)$$

and  $\overline{\underline{K}}^S$  is a diagonal matrix of components of  $\underline{I}_M$ , i.e.,

$$\overline{\underline{K}}^S \underline{v} = \underline{I}_M. \quad (16)$$

We write (14) in the form

$$\underline{K}^S \delta \underline{V}_M + \overline{\underline{K}}^S \delta \underline{V}_M^* = \underline{d}^S, \quad (17)$$

where we have defined

$$\underline{d}^S \triangleq \delta \underline{S}_M^* - \underline{E}_M^* \delta \underline{Y}_T \underline{V}_M. \quad (18)$$

Note that for constant  $\underline{Y}_T$ ,  $\underline{d}^S$  of (18) is simply  $\delta \underline{S}_M^*$  and (17) rigorously represents a set of linear equations to be solved in the well-known Newton-Raphson iterative method.

Slack Bus

The equation of (17) corresponding to the slack bus of specified voltage is replaced by

$$\bar{k}_{\sim n}^T \delta V_{\sim M} + \bar{k}_{\sim n}^T \delta V_{\sim M}^* = \delta V_n^*, \quad (19)$$

where we have assigned the last bus, namely the nth bus, as a slack bus,

$$\bar{k}_{\sim n} = 0 \quad (20)$$

and

$$\bar{k}_{\sim n} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}. \quad (21)$$

Observe that in the special application to the load flow solution, the equation corresponding to the slack bus may be eliminated.

Generator Buses

Consider the equation of (17) corresponding to a voltage-controlled or generator bus g. Let

$$\tilde{S}_g \triangleq P_g + j |V_g|, \quad (22)$$

hence

$$\delta \tilde{S}_g^* = \delta P_g - j \delta |V_g|. \quad (23)$$

Since

$$2P_g = V_g I_g^* + V_g^* I_g, \quad (24)$$

then

$$2\delta P_g = V_g \delta I_g^* + I_g^* \delta V_g + V_g^* \delta I_g + I_g \delta V_g^*. \quad (25)$$

Using (5), we write  $I_g$  as

$$I_g = y_{\sim g}^T V_{\sim M}, \quad (26)$$

where  $\underline{y}_g^T$  represents the corresponding row of the bus admittance matrix  $\underline{Y}_T$ , hence

$$\delta \underline{I}_g = \underline{y}_g^T \delta \underline{V}_{\sim M} + \underline{V}_{\sim M}^T \delta \underline{y}_g. \quad (27)$$

Also,

$$\delta |V_g| = \delta (V_g V_g^*)^{1/2} = (V_g \delta V_g^* + V_g^* \delta V_g) / (2 |V_g|). \quad (28)$$

Using (25)-(28), it is straightforward to show that  $\delta \underline{S}_g^*$  of (23) is given by

$$\delta \underline{S}_g^* = \underline{k}_g^T \delta \underline{V}_{\sim M} + \overline{\underline{k}}_g^T \delta \underline{V}_{\sim M}^* + \underline{V}_g^* \underline{V}_{\sim M}^T \delta \underline{y}_g / 2 + \underline{V}_g \underline{V}_{\sim M}^{*T} \delta \underline{y}_g^* / 2, \quad (29)$$

where

$$\underline{k}_g \triangleq (V_g^* / 2) \underline{y}_g + [\underline{y}_g^{*T} \underline{V}_{\sim M}^* / 2 - j V_g^* / (2 |V_g|)] \underline{\mu}_g \quad (30)$$

and

$$\overline{\underline{k}}_g \triangleq (V_g / 2) \underline{y}_g^* + [\underline{y}_g^T \underline{V}_{\sim M} / 2 - j V_g / (2 |V_g|)] \underline{\mu}_g \quad (31)$$

and where  $\underline{\mu}_g$  is a column vector of unity  $g$ th element and zero other elements. Using (29), the equation of (17) corresponding to the  $g$ th bus is replaced by

$$\underline{k}_g^T \delta \underline{V}_{\sim M} + \overline{\underline{k}}_g^T \delta \underline{V}_{\sim M}^* = \underline{d}_g, \quad (32)$$

where

$$\underline{d}_g = \delta P_g - j \delta |V_g| - \underline{V}_g^* \underline{V}_{\sim M}^T \delta \underline{y}_g / 2 - \underline{V}_g \underline{V}_{\sim M}^{*T} \delta \underline{y}_g^* / 2. \quad (33)$$

### Standard Complex Form

We write (17), including (19) for slack bus and (33) for generator buses, in the form

$$\underline{K} \delta \underline{V}_{\sim M} + \overline{\underline{K}} \delta \underline{V}_{\sim M}^* = \underline{d}. \quad (34)$$

Note that the elements of  $\delta \underline{V}_{\sim M}$  and  $\delta \underline{V}_{\sim M}^*$ , namely,  $\delta V_i$  and  $\delta V_i^*$ ,  $i = 1, \dots, n$  can be replaced by the relative quantities  $\delta V_i / |V_i|$  and  $\delta V_i^* / |V_i|$ , respectively. In this case the elements  $k_{ij}$  and  $\overline{k}_{ij}$  of the  $i$ th row of the coefficient matrices  $\underline{K}$  and  $\overline{\underline{K}}$  are replaced by  $|V_j| k_{ij}$  and  $|V_j| \overline{k}_{ij}$ , respectively. Note also that we could equally well specify  $|V_g|^2$



instead of  $|V_g|$  for a generator bus. In this case  $|V_g|^2$  replaces  $|V_g|$  in (22) as a control variable and the required modifications for subsequent derivation can be performed in a straightforward manner.

#### IV. MODES OF FORMULATION

In the previous section, we have considered the complex formulation of power system equations. We shall exploit this formulation to derive compact forms of sensitivity expressions. In this section, we investigate, via suitable transformations, the relationship between the complex formulation and other formulations. This investigation provides the possibility of formulating the adjoint equations to be solved in the same mode as the original load flow problem. Hence, the available Jacobian of the load flow may be used in solving the adjoint system.

##### Transformation for Rectangular Formulation

We define the transformation matrix

$$\underset{\sim}{L}^q \triangleq \begin{bmatrix} \underset{\sim}{L}_1 & \underset{\sim}{L}_1^* \\ \underset{\sim}{L}_2 & \underset{\sim}{L}_2^* \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{1} \\ -\underset{\sim}{j} & \underset{\sim}{j} \end{bmatrix}, \quad (35)$$

where  $\underset{\sim}{1}$  is the identity matrix of order  $n$  and

$$\underset{\sim}{j} \triangleq j \underset{\sim}{1}, \quad (36)$$

hence

$$(\underset{\sim}{L}^q)^{-1} = \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{j} \\ \underset{\sim}{1} & -\underset{\sim}{j} \end{bmatrix}, \quad (37)$$

n denoting the number of buses in the power network. It follows, using

$$\begin{bmatrix} \zeta_{\sim 1} \\ \zeta_{\sim 2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1^{\sim n} & 1^{\sim n} \\ -j^{\sim n} & j^{\sim n} \end{bmatrix} \begin{bmatrix} \zeta_{\sim} \\ \zeta_{\sim}^* \end{bmatrix} \quad (38)$$

and (7), that

$$\begin{bmatrix} V_{\sim M1} \\ V_{\sim M2} \end{bmatrix} = \begin{bmatrix} L_{\sim 1} & L_{\sim 1}^* \\ L_{\sim 2} & L_{\sim 2}^* \end{bmatrix} \begin{bmatrix} V_{\sim M} \\ V_{\sim M}^* \end{bmatrix}, \quad (39)$$

hence

$$\begin{bmatrix} \delta V_{\sim M1} \\ \delta V_{\sim M2} \end{bmatrix} = \begin{bmatrix} L_{\sim 1} & L_{\sim 1}^* \\ L_{\sim 2} & L_{\sim 2}^* \end{bmatrix} \begin{bmatrix} \delta V_{\sim M} \\ \delta V_{\sim M}^* \end{bmatrix}. \quad (40)$$

Using the perturbed form (40), it is straightforward to show that (34) can be written in the form

$$\begin{bmatrix} (K_{\sim 1} + \bar{K}_{\sim 1}) & (-K_{\sim 2} + \bar{K}_{\sim 2}) \\ -(K_{\sim 2} + \bar{K}_{\sim 2}) & (-K_{\sim 1} + \bar{K}_{\sim 1}) \end{bmatrix} \begin{bmatrix} \delta V_{\sim M1} \\ \delta V_{\sim M2} \end{bmatrix} = \begin{bmatrix} d_{\sim 1} \\ -d_{\sim 2} \end{bmatrix}, \quad (41)$$

where we have set

$$K_{\sim} = K_{\sim 1} + j K_{\sim 2}, \quad (42)$$

$$\bar{K}_{\sim} = \bar{K}_{\sim 1} + j \bar{K}_{\sim 2}, \quad (43)$$

and

$$d_{\sim} = d_{\sim 1} + j d_{\sim 2}. \quad (44)$$

The  $2n \times 2n$  matrix of coefficients in (41), denoted by  $K_{\sim}^{crt}$ , constitutes the well-known Jacobian matrix of the flow problem in rectangular form. Moreover, writing (34) in the form

$$\begin{bmatrix} \underline{K} & \underline{\bar{K}} \end{bmatrix} \begin{bmatrix} \delta \underline{V}_M \\ \delta \underline{V}_M^* \end{bmatrix} = \underline{d}, \quad (45)$$

it follows that

$$\begin{bmatrix} \underline{K} & \underline{\bar{K}} \end{bmatrix} = \begin{bmatrix} \underline{K}^q & \underline{\bar{K}}^q \end{bmatrix} \begin{bmatrix} L_1 & L_1^* \\ L_2 & L_2^* \end{bmatrix}, \quad (46)$$

where  $\underline{K}^q$  and  $\underline{\bar{K}}^q$  are formed directly from the Jacobian of (41) as

$$\underline{K}^q = (\underline{K}_1 + \underline{\bar{K}}_1) + j(\underline{K}_2 + \underline{\bar{K}}_2) \quad (47)$$

and

$$\underline{\bar{K}}^q = (-\underline{K}_2 + \underline{\bar{K}}_2) - j(-\underline{K}_1 + \underline{\bar{K}}_1). \quad (48)$$

Observe that (46) relates the Jacobian of the complex formulation (34) to the Jacobian of the rectangular formulation (41).

### Transformation for Polar Formulation

For polar formulation, we set

$$V_i = |V_i| \angle \delta_i, \quad i = 1, \dots, n, \quad (49)$$

where  $V_i$  are elements of  $\underline{V}_M$ , and we define the vectors

$$|\underline{V}| \triangleq \begin{bmatrix} |V_1| \\ \vdots \\ |V_n| \end{bmatrix} \quad (50)$$

and

$$\tilde{\delta} \triangleq \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}. \quad (51)$$

Then, we define the transformation matrix

$$\tilde{L}^p \triangleq \begin{bmatrix} \tilde{L}_{\delta} & \tilde{L}_{\delta}^* \\ \tilde{L}_{v} & \tilde{L}_{v}^* \end{bmatrix}, \quad (52)$$

where  $\tilde{L}_{\delta}$ ,  $\tilde{L}_{\delta}^*$ ,  $\tilde{L}_{v}$  and  $\tilde{L}_{v}^*$  are diagonal matrices whose elements represent the formal partial derivatives  $\partial \delta_i / \partial V_i$ ,  $\partial \delta_i / \partial V_i^*$ ,  $\partial |V_i| / \partial V_i$  and  $\partial |V_i| / \partial V_i^*$ , respectively, hence

$$\tilde{L}_{\delta} \triangleq \text{diag} \{L_{\delta i}\} \quad (53)$$

and

$$\tilde{L}_{v} \triangleq \text{diag} \{L_{v i}\}, \quad (54)$$

where

$$L_{\delta i} = -j / (2 V_i) \quad (55)$$

and

$$L_{v i} = V_i^* / (2 |V_i|). \quad (56)$$

The inverse of  $\tilde{L}^p$  is given by

$$(\tilde{L}^p)^{-1} = \begin{bmatrix} \tilde{L}_{\delta} & \tilde{L}_{v} \\ \tilde{L}_{\delta}^* & \tilde{L}_{v}^* \end{bmatrix}, \quad (57)$$

where  $\tilde{L}_{\delta}$ ,  $\tilde{L}_{\delta}^*$ ,  $\tilde{L}_{v}$  and  $\tilde{L}_{v}^*$  are diagonal matrices whose elements are the partial derivatives  $\partial V_i / \partial \delta_i$ ,  $\partial V_i^* / \partial \delta_i$ ,  $\partial V_i / \partial |V_i|$  and  $\partial V_i^* / \partial |V_i|$ , respectively, hence

$$\tilde{L}_{\delta} \triangleq \text{diag} \{ \tilde{L}_{\delta i} \} \quad (58)$$

and

$$\tilde{L}_{v} \triangleq \text{diag} \{ \tilde{L}_{vi} \} , \quad (59)$$

where

$$\tilde{L}_{\delta i} = jV_i \quad (60)$$

and

$$\tilde{L}_{vi} = V_i / |V_i| . \quad (61)$$

Similarly to (40), we may write

$$\begin{bmatrix} \delta \tilde{\delta} \\ \delta \tilde{|V|} \end{bmatrix} = \begin{bmatrix} \tilde{L}_{\delta} & \tilde{L}_{\delta}^* \\ \tilde{L}_{v} & \tilde{L}_{v}^* \end{bmatrix} \begin{bmatrix} \delta \tilde{V}_M \\ \delta \tilde{V}_M^* \end{bmatrix} . \quad (62)$$

Using the perturbed form (62), it is straightforward to show that (34) can also be written in the form

$$\begin{bmatrix} \tilde{K}_1^p & \tilde{K}_1^{\bar{p}} \\ -\tilde{K}_2^p & -\tilde{K}_2^{\bar{p}} \end{bmatrix} \begin{bmatrix} \delta \tilde{\delta} \\ \delta \tilde{|V|} \end{bmatrix} = \begin{bmatrix} d_1 \\ -d_2 \end{bmatrix} , \quad (63)$$

where we have set

$$\tilde{K}^p = \tilde{K}_1^p + j \tilde{K}_2^p \quad (64)$$

and

$$\tilde{K}^{\bar{p}} = \tilde{K}_1^{\bar{p}} + j \tilde{K}_2^{\bar{p}} , \quad (65)$$

and where the matrices  $\tilde{K}^p$  and  $\tilde{K}^{\bar{p}}$  are related to  $\tilde{K}$  and  $\tilde{\bar{K}}$  through the relationship

$$[\tilde{K} \quad \tilde{\bar{K}}] = [\tilde{K}^p \quad \tilde{K}^{\bar{p}}] \begin{bmatrix} \tilde{L}_{\delta} & \tilde{L}_{\delta}^* \\ \tilde{L}_{v} & \tilde{L}_{v}^* \end{bmatrix} . \quad (66)$$

The  $2n \times 2n$  matrix of coefficients in (63), denoted by  $\tilde{K}^{plr}$ , constitutes the well-known Jacobian matrix of the load flow problem in polar form. Observe that (66) relates the Jacobian of the complex

formulation (34) to the Jacobian of the polar formulation (63), where  $\tilde{K}^P$  and  $\overline{\tilde{K}}^P$  are formed directly from the Jacobian of (63).

At the end of this section, we illustrate the foregoing concepts by two simple examples.

Example 1

Consider first the 2-bus sample power system of Fig. 1 which consists of a load bus and a slack bus. The solution of the load flow equations (13) is given by

$$V_1 = 0.7352 - j 0.2041$$

and

$$S_2 = 5.6705 + j 1.0706.$$

Note that  $S_2$  is the injected power at bus 2. The matrices  $\tilde{K}$  and  $\overline{\tilde{K}}$  of (45) are given by

$$\tilde{K} = \begin{bmatrix} (8.0852 - j 12.0097) & (-8.4934 + j 13.4802) \\ 0 & 0 \end{bmatrix}$$

and

$$\overline{\tilde{K}} = \begin{bmatrix} (-5.2623 + j 5.5411) & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, using cartesian coordinates, the matrix of coefficients of (41) has, using (42) and (43), the form

$$\tilde{K}^{crt} = \begin{bmatrix} 2.8229 & -8.4934 & 17.5508 & -13.4802 \\ 0 & 1 & 0 & 0 \\ 6.4686 & -13.4802 & -13.3475 & 8.4934 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is the Jacobian of the load flow problem in cartesian coordinates when the slack bus equations are included.

For the polar formulation, the matrices  $\tilde{L}_\delta$  and  $\tilde{L}_V$  of (57) are given by

$$\tilde{L}_\delta = \begin{bmatrix} (0.2041 + j 0.7352) & 0 \\ 0 & j \end{bmatrix}$$

and

$$\tilde{L}_V = \begin{bmatrix} (0.9636 - j 0.2675) & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, using (52), (57) and (66), the matrices  $\tilde{K}^p$  and  $\bar{K}^p$  are given by

$$\tilde{K}^p = \begin{bmatrix} (13.4802 + j 8.4934) & (-13.4802 - j 8.4934) \\ 0 & -j \end{bmatrix}$$

and

$$\bar{K}^p = \begin{bmatrix} (-1.9745 - j 9.8031) & (-8.4934 + j 13.4802) \\ 0 & 1 \end{bmatrix},$$

from which the matrix of coefficients of (63) has the form

$$\tilde{K}^{plr} = \begin{bmatrix} 13.4802 & -13.4802 & -1.9745 & -8.4934 \\ 0 & 0 & 0 & 1 \\ -8.4934 & 8.4934 & 9.8031 & -13.4802 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

which is the Jacobian of the load flow problem in polar coordinates when the slack bus equations are included.

### Example 2

Now, consider the 2-bus sample power system of Fig. 2 which consists of a generator bus and a slack bus. The solution of the load flow equations (13) is given by

$$\delta_1 = -0.1995 \text{ rad,}$$

$$Q_1 = 1.9929$$

and

$$S_2 = 4.2742 - j 1.7131.$$

The matrices  $\tilde{K}$  and  $\overline{\tilde{K}}$  of (45) are given by

$$\tilde{K} = \begin{bmatrix} (2.3920 - j 9.4199) & (-4.4300 + j 8.2864) \\ 0 & 0 \end{bmatrix}$$

and

$$\overline{\tilde{K}} = \begin{bmatrix} (2.1938 + j 8.4398) & (-4.4300 - j 8.2864) \\ 0 & 1 \end{bmatrix}.$$

Hence, using cartesian coordinates, the matrix of coefficients of (41) has, using (42) and (43), the form

$$\tilde{K}^{crt} = \begin{bmatrix} 4.5858 & -8.8600 & 17.8597 & -16.5729 \\ 0 & 1 & 0 & 0 \\ 0.9802 & 0 & -0.1982 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is the Jacobian of the load flow problem in cartesian coordinates when the slack bus equations are included.

For the polar formulation, the matrices  $\tilde{L}_{\delta}$  and  $\tilde{L}_V$  of (57) are given by

$$\tilde{L}_{\delta} = \begin{bmatrix} (0.1784 + j 0.8822) & 0 \\ 0 & j \end{bmatrix}$$

and

$$\tilde{L}_V = \begin{bmatrix} (0.9802 - j 0.1982) & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, using (52), (57) and (66), the matrices  $\tilde{K}^p$  and  $\overline{\tilde{K}}^p$  are given by

$$\tilde{K}^p = \begin{bmatrix} 16.5729 & -16.5729 \\ 0 & -j \end{bmatrix}$$



and

$$\bar{K}_{\sim}^p = \begin{bmatrix} 0.9556 - j 1.0 & -8.8600 \\ 0 & 1 \end{bmatrix},$$

from which the matrix of coefficients of (63) has the form

$$\bar{K}_{\sim}^{plr} = \begin{bmatrix} 16.5729 & -16.5729 & 0.9556 & -8.8600 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is the Jacobian of the load flow problem in polar coordinates when the slack bus equations are included.

#### V. COMPLEX ADJOINT ANALYSIS

In this section, we derive the required sensitivity expressions using the compact complex form (34). We exploit the relationships derived in the previous section to provide flexibility in solving the resulting adjoint system of equations in other modes of formulation. We have shown that, using cartesian coordinates, (34) has the form

$$\bar{K}_{\sim}^{crt} \begin{bmatrix} \delta V_{\sim M1} \\ \delta V_{\sim M2} \end{bmatrix} = \begin{bmatrix} d_{\sim 1} \\ -d_{\sim 2} \end{bmatrix}, \quad (67)$$

where the  $2n \times 2n$  matrix of coefficients  $\bar{K}_{\sim}^{crt}$  which constitutes the Jacobian matrix of the load flow problem in rectangular form is given from (41). Also, using polar coordinates, (34) has the form

$$\bar{K}_{\sim}^{plr} \begin{bmatrix} \delta \delta_{\sim} \\ \delta |V|_{\sim} \end{bmatrix} = \begin{bmatrix} d_{\sim 1} \\ -d_{\sim 2} \end{bmatrix}, \quad (68)$$

where the  $2n \times 2n$  matrix of coefficients  $\tilde{K}^{plr}$  which constitutes the Jacobian matrix of the load flow problem in polar form is given from (63).

### Standard Complex Form

We write (34) in the form

$$\begin{bmatrix} \tilde{K} & \overline{\tilde{K}} \\ \overline{\tilde{K}}^* & \tilde{K}^* \end{bmatrix} \begin{bmatrix} \delta V_{\tilde{M}} \\ \delta V_{\tilde{M}}^* \end{bmatrix} = \begin{bmatrix} d \\ d^* \end{bmatrix}. \quad (69)$$

It can be shown [8] that the matrix of coefficients of (69), denoted by  $\tilde{K}^{cmp}$ , has the same rank as that of (67) and the system of equations (69) is consistent if and only if the system (67) is consistent.

For a real function  $f$ , we may write, using (3) and (4)

$$\delta f = \begin{bmatrix} \hat{\mu}^T & \hat{\mu}^{*T} \end{bmatrix} \begin{bmatrix} \delta V_{\tilde{M}} \\ \delta V_{\tilde{M}}^* \end{bmatrix} + \delta f_{\rho}, \quad (70)$$

where we have defined

$$\hat{\mu} \triangleq \frac{\partial f}{\partial V_{\tilde{M}}} \quad (71)$$

and used

$$\frac{\partial f}{\partial V_{\tilde{M}}} = \left( \frac{\partial f}{\partial V_{\tilde{M}}^*} \right)^*, \quad (72)$$

$\delta f_{\rho}$  denoting the change in  $f$  due to changes in other variables in terms of which  $f$  may be explicitly expressed. Hence, from (69)

$$\delta f = \begin{bmatrix} \hat{\mu}^T & \hat{\mu}^{*T} \\ \sim & \sim \end{bmatrix} \begin{bmatrix} K & \bar{K} \\ \sim & \sim \\ \bar{K}^* & K^* \\ \sim & \sim \end{bmatrix}^{-1} \begin{bmatrix} d \\ \sim \\ d^* \\ \sim \end{bmatrix} + \delta f_\rho \quad (73)$$

or

$$\delta f = \begin{bmatrix} \hat{V}^T & \hat{V}^{*T} \\ \sim & \sim \end{bmatrix} \begin{bmatrix} d \\ \sim \\ d^* \\ \sim \end{bmatrix} + \delta f_\rho, \quad (74)$$

where

$$\begin{bmatrix} K^T & \bar{K}^{*T} \\ \sim & \sim \\ \bar{K}^T & K^{*T} \\ \sim & \sim \end{bmatrix} \begin{bmatrix} \hat{V} \\ \sim \\ \hat{V}^* \\ \sim \end{bmatrix} = \begin{bmatrix} \hat{\mu} \\ \sim \\ \hat{\mu}^* \\ \sim \end{bmatrix} \quad (75)$$

or, simply

$$\begin{bmatrix} K^T & \bar{K}^{*T} \\ \sim & \sim \end{bmatrix} \begin{bmatrix} \hat{V} \\ \sim \\ \hat{V}^* \\ \sim \end{bmatrix} = \hat{\mu}. \quad (76)$$

Hence, the first-order change of the real function  $f$  and corresponding gradients can be evaluated by solving (75) and substituting into (74).

### Cartesian Coordinates

Similarly to (70), we may write, using the rectangular formulation

$$\delta f = \begin{bmatrix} \hat{\mu}_r^T & \hat{\mu}_s^T \\ \sim & \sim \end{bmatrix} \begin{bmatrix} \delta V_{M1} \\ \sim \\ \delta V_{M2} \\ \sim \end{bmatrix} + \delta f_\rho, \quad (77)$$

where we have defined

$$\hat{\mu}_r \triangleq \frac{\partial f}{\partial V_{M1}} \quad (78)$$

and

$$\hat{\mu}_s \triangleq \frac{\partial f}{\partial \hat{V}_{M2}} . \quad (79)$$

Hence, from (41)

$$\delta f = \begin{bmatrix} \hat{V}_r^T & \hat{V}_s^T \end{bmatrix} \begin{bmatrix} d_1 \\ -d_2 \end{bmatrix} + \delta f_\rho , \quad (80)$$

where

$$\begin{bmatrix} (K_1 + \bar{K}_1)^T & -(K_2 + \bar{K}_2)^T \\ (-K_2 + \bar{K}_2)^T & (-K_1 + \bar{K}_1)^T \end{bmatrix} \begin{bmatrix} \hat{V}_r \\ \hat{V}_s \end{bmatrix} = \begin{bmatrix} \hat{\mu}_r \\ \hat{\mu}_s \end{bmatrix} . \quad (81)$$

Observe that the matrix of coefficients of (81) is the transpose of the Jacobian matrix of the load flow problem in rectangular form (67).

Theorem 1

- (a) The solution vectors  $\hat{V}_r$  and  $\hat{V}_s$  of the adjoint system of equations (81) are given by

$$\hat{V}_r = 2 \operatorname{Re}\{\hat{V}\}$$

and

$$\hat{V}_s = 2 \operatorname{Im}\{\hat{V}\} ,$$

where  $\hat{V}$  is given from (75).

- (b) The RHS vectors  $\hat{\mu}_r$  and  $\hat{\mu}_s$  of the adjoint system of equations (81) are given by

$$\hat{\underline{\mu}} = L_1^T \hat{\underline{\mu}}_r + L_2^T \hat{\underline{\mu}}_s,$$

where  $\hat{\underline{\mu}}$  is given by (71) and  $L_1$  and  $L_2$  are given by (35).

Proof

Comparing (74) and (80), and using (66), we get

$$\hat{\underline{V}} = (\hat{\underline{V}}_r + j \hat{\underline{V}}_s)/2. \quad (82)$$

From (82), the first part of the theorem is proved. Now, multiplying (81) from the left by the transpose of  $L^q$  of (35) and using the relation

$$2 \begin{bmatrix} (K_1 + \bar{K}_1)^T & -(K_2 + \bar{K}_2)^T \\ (-K_2 + \bar{K}_2)^T & (-K_1 + \bar{K}_1)^T \end{bmatrix} = \begin{bmatrix} K^{qT} & K^{q*T} \\ \bar{K}^{qT} & \bar{K}^{q*T} \end{bmatrix} \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}, \quad (83)$$

it follows from (46) and (82) that

$$\begin{bmatrix} K^T & \bar{K}^{*T} \\ \bar{K}^T & K^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underline{V}} \\ \hat{\underline{V}}^* \end{bmatrix} = \begin{bmatrix} L_1^T & L_2^T \\ L_1^{*T} & L_2^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underline{\mu}}_r \\ \hat{\underline{\mu}}_s \end{bmatrix}, \quad (84)$$

hence, from (75)

$$\begin{bmatrix} \hat{\underline{\mu}} \\ \hat{\underline{\mu}}^* \end{bmatrix} = \begin{bmatrix} L_1^T & L_2^T \\ L_1^{*T} & L_2^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underline{\mu}}_r \\ \hat{\underline{\mu}}_s \end{bmatrix} \quad (85)$$

or, simply

$$\hat{\underline{\mu}} = [L_1^T \quad L_2^T] \begin{bmatrix} \hat{\underline{\mu}}_r \\ \hat{\underline{\mu}}_s \end{bmatrix} \quad (86)$$

The relationship (86) could also be derived by applying, formally, the chain rule of differentiation using the definitions (71), (78) and (79).

Observe that equation (82) relates the solution of the adjoint system (81) to that of (76), and equation (86) relates the RHS of (81) to that of (76).

### Polar Coordinates

Using the polar formulation, we may write

$$\delta f = \begin{bmatrix} \hat{\mu}_S^T & \hat{\mu}_V^T \end{bmatrix} \begin{bmatrix} \delta \delta \\ \delta |V| \end{bmatrix} + \delta f_\rho, \quad (87)$$

where we have defined

$$\hat{\mu}_\sigma \triangleq \frac{\partial f}{\partial \delta} \quad (88)$$

and

$$\hat{\mu}_V \triangleq \frac{\partial f}{\partial |V|}. \quad (89)$$

Hence, from (68)

$$\delta f = \begin{bmatrix} \hat{V}_\delta^T & \hat{V}_V^T \end{bmatrix} \begin{bmatrix} d_1 \\ -d_2 \end{bmatrix} + \delta f_\rho, \quad (90)$$

where

$$\begin{bmatrix} K_{\sim 1}^{PT} & -K_{\sim 2}^{PT} \\ \bar{K}_{\sim 1}^{PT} & -\bar{K}_{\sim 2}^{PT} \end{bmatrix} \begin{bmatrix} \hat{V}_\delta \\ \hat{V}_V \end{bmatrix} = \begin{bmatrix} \hat{\mu}_\delta \\ \hat{\mu}_V \end{bmatrix}. \quad (91)$$

The matrix of coefficients of (91) is the transpose of the Jacobian matrix of the load flow problem in the polar form.

Theorem 2

- (a) The solution vectors  $\hat{\underline{V}}_{\delta}$  and  $\hat{\underline{V}}_v$  of the adjoint system of equations (91) are given by

$$\hat{\underline{V}}_{\delta} = 2 \operatorname{Re}\{\hat{\underline{V}}\}$$

and

$$\hat{\underline{V}}_v = 2 \operatorname{Im}\{\hat{\underline{V}}\},$$

where  $\hat{\underline{V}}$  is given from (75).

- (b) The RHS vectors  $\hat{\underline{\mu}}_{\delta}$  and  $\hat{\underline{\mu}}_v$  of the adjoint system of equations (91) are given by

$$\hat{\underline{\mu}} = \underline{L}_{\delta}^T \hat{\underline{\mu}}_{\delta} + \underline{L}_v^T \hat{\underline{\mu}}_v,$$

where  $\hat{\underline{\mu}}$  is given by (71) and  $\underline{L}_{\delta}$  and  $\underline{L}_v$  are given by (53) and (54).

Proof

Comparing (74) and (90), and using (44), we get

$$\hat{\underline{V}} = (\hat{\underline{V}}_{\delta} + j \hat{\underline{V}}_v)/2. \quad (92)$$

From (92), the first part of the theorem is proved. Now, multiplying (90) from left by the transpose of  $\underline{L}^P$  of (52) and using the relation

$$2 \begin{bmatrix} \underline{K}_{\sim 1}^{PT} & -\underline{K}_{\sim 2}^{PT} \\ \underline{\bar{K}}_{\sim 1}^{PT} & -\underline{\bar{K}}_{\sim 2}^{PT} \end{bmatrix} = \begin{bmatrix} \underline{K}_{\sim}^{PT} & \underline{K}^{P*T} \\ \underline{\bar{K}}_{\sim}^{PT} & \underline{\bar{K}}^{P*T} \end{bmatrix} \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}, \quad (93)$$

it follows from (64) and (92) that

$$\begin{bmatrix} \tilde{K}^T & \tilde{K}^{*T} \\ \tilde{K}^T & \tilde{K}^{*T} \end{bmatrix} \begin{bmatrix} \hat{V} \\ \hat{V}^* \end{bmatrix} = \begin{bmatrix} L_{\tilde{\delta}}^T & L_{\tilde{v}}^T \\ L_{\tilde{\delta}}^{*T} & L_{\tilde{v}}^{*T} \end{bmatrix} \begin{bmatrix} \hat{\mu}_{\tilde{\delta}} \\ \hat{\mu}_{\tilde{v}} \end{bmatrix}, \quad (94)$$

hence, from (85)

$$\begin{bmatrix} \hat{\mu} \\ \hat{\mu}^* \end{bmatrix} = \begin{bmatrix} L_{\tilde{\delta}}^T & L_{\tilde{v}}^T \\ L_{\tilde{\delta}}^{*T} & L_{\tilde{v}}^{*T} \end{bmatrix} \begin{bmatrix} \hat{\mu}_{\tilde{\delta}} \\ \hat{\mu}_{\tilde{v}} \end{bmatrix} \quad (95)$$

or, simply

$$\hat{\mu} = [L_{\tilde{\delta}}^T \quad L_{\tilde{v}}^T] \begin{bmatrix} \hat{\mu}_{\tilde{\delta}} \\ \hat{\mu}_{\tilde{v}} \end{bmatrix} \quad \blacksquare \quad (96)$$

Again, the relationship (96) could also be derived by applying, formally, the chain rule of differentiation using the definitions (71), (88) and (89).

Equation (92) relates the solution of the adjoint system (91) to that of (76), and equation (96) relates the RHS of (91) to that of (76).

### Remarks

We remark that using (82) or (92), the adjoint system can be formulated and solved in a convenient mode, preferably the same formulation as the original load flow problem, while the first-order change of  $f$  and corresponding gradients may be derived compactly using the adjoint variables  $\hat{V}$ . On the other hand, the relations (86) and (96) allow the use of more elegant formal derivatives which, in many cases, facilitate the formulation. For example, consider the function



$$f = \sigma |V_i - V_j|^2 = \sigma (V_i - V_j)(V_i^* - V_j^*), \quad (97)$$

where  $V_i$  and  $V_j$  are the  $i$ th and  $j$ th components of  $\tilde{V}_M$ , respectively, and  $\sigma$  is a real scalar or variable. Note that  $f$  of (97) may represent, for example, the power loss in line  $ij$ . For the polar formulation,  $\hat{\mu}_V$  and  $\hat{\mu}_\delta$  of (91) are calculated as follows. The  $i$ th and  $j$ th components of  $\hat{\mu}_\delta$  and  $\hat{\mu}_V$  are given by

$$\begin{aligned} \hat{\mu}_{\delta i} = \sigma [ & -2 (|V_i| \cos \delta_i - |V_j| \cos \delta_j) |V_i| \sin \delta_i \\ & + 2 (|V_i| \sin \delta_i - |V_j| \sin \delta_j) |V_i| \cos \delta_i ], \end{aligned}$$

$$\begin{aligned} \hat{\mu}_{\delta j} = \sigma [ & 2 (|V_i| \cos \delta_i - |V_j| \cos \delta_j) |V_j| \sin \delta_j \\ & - 2 (|V_i| \sin \delta_i - |V_j| \sin \delta_j) |V_j| \cos \delta_j ], \end{aligned}$$

$$\begin{aligned} \hat{\mu}_{V i} = \sigma [ & 2 (|V_i| \cos \delta_i - |V_j| \cos \delta_j) \cos \delta_i \\ & + 2 (|V_i| \sin \delta_i - |V_j| \sin \delta_j) \sin \delta_i ] \end{aligned}$$

and

$$\begin{aligned} \hat{\mu}_{V j} = \sigma [ & 2 (|V_i| \cos \delta_i - |V_j| \cos \delta_j) \cos \delta_j \\ & - 2 (|V_i| \sin \delta_i - |V_j| \sin \delta_j) \sin \delta_j ]. \end{aligned}$$

All other components are zero. On the other hand, one may calculate

$$\hat{\mu} = \sigma \begin{bmatrix} 0 \\ \vdots \\ (V_i^* - V_j^*) \\ \vdots \\ -(V_i^* - V_j^*) \\ \vdots \\ 0 \end{bmatrix},$$

and use (95) to calculate  $\hat{\mu}_V$  and  $\hat{\mu}_\delta$ , where  $(L^{pT})^{-1}$  is the transpose of  $(L^p)^{-1}$  of (57). In this example, the derivation of the formal derivatives is clearly easier.

We also remark that other forms of power flow equations can be handled in a similar way. The previous theorems can be easily generalized for other formulations provided that transformations similar to (35) and (52) are defined.

We illustrate the foregoing concepts by the two simple examples considered before.

Example 3

For the first system, as shown in Fig. 1, consider the function

$$f = |V_1|^2 = V_1 V_1^* .$$

From (71),

$$\hat{\mu} = \begin{bmatrix} V_1^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7352 + j0.2041 \\ 0 \end{bmatrix}$$

and (76) has the solution

$$\hat{V} = \begin{bmatrix} 0.0562 + j0.0892 \\ 1.6788 + j0.0 \end{bmatrix} .$$

Also, for the polar formulation, we have from (88) and (89)

$$\hat{\mu}_\delta = 0$$

and

$$\hat{\mu}_V = \begin{bmatrix} 2|V_1| \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5261 \\ 0 \end{bmatrix} ,$$

and (91) has the solution

$$\hat{V}_\delta = \begin{bmatrix} 0.1123 \\ 3.3577 \end{bmatrix}$$

and

$$\hat{V}_V = \begin{bmatrix} 0.1783 \\ 0 \end{bmatrix} .$$

Note that the  $\hat{\tilde{V}}_{\delta}$  and  $\hat{\tilde{V}}_v$  obtained for the polar formulation and  $\hat{\tilde{V}}$  satisfy (92).

Example 4

For the second system, as shown in Fig. 2, consider the function

$$f = \delta_1 = \tan^{-1} \left[ \frac{V_1 + V_1^*}{j(V_1^* - V_1)} \right] .$$

From (71)

$$\hat{\tilde{\mu}} = \begin{bmatrix} -j/(2V_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1101 - j0.5445 \\ 0 \end{bmatrix}$$

and (76) has the solution

$$\hat{\tilde{V}} = \begin{bmatrix} 0.0302 - j0.0288 \\ 0.2673 + j0.5 \end{bmatrix} .$$

Also, for the polar formulation, we have from (88) and (89)

$$\hat{\tilde{\mu}}_{\delta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\hat{\tilde{\mu}}_v = 0$$

and (91) has the solution

$$\hat{\tilde{V}}_{\delta} = \begin{bmatrix} 0.0603 \\ 0.5346 \end{bmatrix}$$

and

$$\hat{\tilde{V}}_v = \begin{bmatrix} -0.0577 \\ 1.0 \end{bmatrix} .$$

Observe that the obtained  $\hat{\tilde{V}}_{\delta}$  and  $\hat{\tilde{V}}_v$  for the polar formulation and  $\hat{\tilde{V}}$  satisfy (92).

## VI. GRADIENT CALCULATIONS

In the previous section, we have derived the adjoint systems in different modes of formulation and investigated the relationships between the corresponding excitation and solution vectors. In power system studies such as contingency analysis, the first-order change of  $f$  is of prime interest. The first-order change  $\delta f$  can be calculated from (74), (80) and (90). On the other hand, the derivatives of  $f$  w.r.t. control variables are required to be calculated, for example, in planning studies.

In the following, we consider the buses to be ordered such that subscripts  $l=1, 2, \dots, n_L$  identify load buses,  $g = n_L+1, \dots, n_L + n_G$  identify generator buses and  $n = n_L + n_G+1$  identifies the slack bus.

The vector  $\tilde{d}$  of (34) is now partitioned into subvectors associated with the sets of load, generator and slack buses of appropriate dimension in the form

$$\tilde{d} = \begin{bmatrix} \tilde{d}_L \\ \tilde{d}_G \\ \tilde{d}_n \end{bmatrix}, \quad (98)$$

where  $\tilde{d}_L$  has elements  $d_l$  given from (18) by

$$d_l = \delta S_l^* - V_l^* V_M^T \delta y_{\tilde{l}}, \quad (99)$$

$y_{\tilde{l}}^T$  representing the corresponding row of the bus admittance matrix  $Y_{\tilde{T}}$ ,  $\tilde{d}_G$  has elements  $d_g$  given by (33) and  $\tilde{d}_n$  is  $\delta V_n^*$  from (19). Also, the vector  $\hat{V}$  of (74) is partitioned correspondingly in the form

$$\hat{V} = \begin{bmatrix} \hat{V}_{\tilde{L}} \\ \hat{V}_{\tilde{G}} \\ \hat{V}_{\tilde{n}} \end{bmatrix}. \quad (100)$$

Note that the above formulation leads to expressing the vector  $\tilde{d}$  solely in terms of variations in control variables, the gradients in terms of which

can be obtained by writing (74) in the form

$$\begin{aligned} \delta f = & \hat{\underline{V}}_L^T \underline{d}_L + \hat{\underline{V}}_G^T \underline{d}_G + \hat{V}_n d_n + \left(\frac{\partial f}{\partial \rho}\right)^T \delta \rho \\ & + \hat{\underline{V}}_L^{*T} \underline{d}_L^* + \hat{\underline{V}}_G^{*T} \underline{d}_G^* + \hat{V}_n^* d_n^* + \left(\frac{\partial f}{\partial \rho}\right)^{*T} \delta \rho^* . \end{aligned} \quad (101)$$

The first term of (101) is given, using (99), by

$$\begin{aligned} \hat{\underline{V}}_L^T \underline{d}_L &= \sum_{\ell=1}^{n_L} \hat{V}_\ell \underline{d}_\ell \\ &= \sum_{\ell=1}^{n_L} (\hat{V}_\ell \delta S_\ell^*) - \sum_{\ell=1}^{n_L} \sum_{m=1}^n (\hat{V}_\ell V_\ell^* V_m \delta Y_{\ell m}) , \end{aligned} \quad (102)$$

where  $Y_{\ell m}$  is an element of  $\underline{Y}_T$ , which is assumed, for simplicity, to be a symmetric admittance matrix (the case of unsymmetric admittance matrix can be analyzed in a similar straightforward way), or

$$\begin{aligned} \hat{\underline{V}}_L^T \underline{d}_L &= \sum_{\ell=1}^{n_L} (\hat{V}_\ell \delta S_\ell^*) + \sum_{\ell=1}^{n_L} \sum_{\substack{m=1 \\ m \neq \ell}}^n \hat{V}_\ell V_\ell^* (V_m - V_\ell) \delta y_{\ell m} \\ &\quad - \sum_{\ell=1}^{n_L} (\hat{V}_\ell V_\ell^* V_\ell \delta y_{\ell 0}) , \end{aligned} \quad (103)$$

where  $y_{\ell m}$  denotes the admittance of line  $\ell m$  connecting load bus  $\ell$  with bus  $m$  ( $=\ell, g$  or  $n$ ), and  $y_{\ell 0}$  is the shunt admittance at bus  $\ell$ . The second term of (101) is given, using (33) by

$$\begin{aligned} \hat{\underline{V}}_G^T \underline{d}_G &= \sum_{g=n_L+1}^{n-1} \hat{V}_g \underline{d}_g \\ &= \sum_{g=n_L+1}^{n-1} \hat{V}_g (\delta P_g - j \delta |V_g|) \end{aligned}$$

$$- \sum_{g=n_L+1}^{n-1} \sum_{m=1}^n \hat{V}_g \operatorname{Re}\{V_g^* V_m \delta Y_{gm}\} \quad (104)$$

or

$$\begin{aligned} \hat{V}_G^T d_G = & \sum_{g=n_L+1}^{n-1} \hat{V}_g (\delta P_g - j\delta |V_g|) + \sum_{g=n_L+1}^{n-1} \sum_{\substack{m=1 \\ m \neq g}}^n \hat{V}_g \operatorname{Re}\{V_g^* (V_m - V_g) \delta y_{gm}\} \\ & - \sum_{g=n_L+1}^{n-1} \hat{V}_g \operatorname{Re}\{V_g^* V_g \delta y_{g0}\} , \end{aligned} \quad (105)$$

where  $y_{gm}$  denotes the admittance of line  $gm$  connecting generator bus  $g$  with bus  $m$  ( $=l, g$  or  $n$ ), and  $y_{g0}$  is the shunt admittance at bus  $g$ . The third term of (101) is given, using (19) by

$$\hat{V}_n d_n = \hat{V}_n \delta V_n^* . \quad (106)$$

The fourth term of (101) is simply the first-order change of  $f$  due to changes in other variables  $\rho$  in terms of which the function  $f$  may be explicitly expressed.

Equations (103)-(106) provide useful information for gradient evaluation since they provide direct expressions w.r.t. the control variables of interest. The derivatives of the function  $f$  w.r.t. the control variables are obtained as follows, where we temporarily assume that  $\rho$  does not contain such control variables.

### Load Bus Control Variables

From (103) and its complex conjugate, the derivatives of  $f$  w.r.t. the demand  $S_\ell$  and  $S_\ell^*$  at load bus  $\ell$  is given by

$$\frac{df}{dS_\ell} = \hat{V}_\ell^* \quad (107)$$

and

$$\frac{df}{dS_l^*} = \hat{V}_l . \quad (108)$$

### Generator Bus Control Variables

From (105) and its complex conjugate, the derivatives of  $f$  w.r.t. the real generated power  $P_g$  and the voltage magnitude  $|V_g|$  at generator bus  $g$  are given by

$$\frac{df}{dS_g} = \hat{V}_g^* \quad (109)$$

and

$$\frac{df}{d\tilde{S}_g^*} = \hat{V}_g , \quad (110)$$

where  $\tilde{S}_g$  is given by (22).

### Slack Bus Control Variables

From (106) and its complex conjugate, the derivatives of  $f$  w.r.t. the slack bus voltage  $V_n$  and  $V_n^*$  are given by

$$\frac{df}{dV_n} = \hat{V}_n^* \quad (111)$$

and

$$\frac{df}{dV_n^*} = \hat{V}_n . \quad (112)$$

In practice, the phase angle of the slack bus voltage is set to zero as a reference angle. Hence, the slack bus has only one effective real control variable.

Line Control Variables

The derivatives of  $f$  w.r.t. line control variables  $y_{ij}$ , can be obtained from (103) and (105) and their complex conjugate as follows. For  $y_{ll}$ , between load buses  $l$  and  $l'$ , we have from (103) and its complex conjugate

$$\frac{df}{dy_{ll}} = (\hat{V}_l V_l^* - \hat{V}_{l'} V_{l'}^*) (V_{l'} - V_l) \quad (113)$$

and

$$\frac{df^*}{dy_{ll}^*} = (\hat{V}_l^* \hat{V}_{l'} - \hat{V}_{l'}^* V_{l'}) (V_{l'}^* - V_l^*) . \quad (114)$$

For  $y_{l0}$  between load bus  $l$  and ground, we have from (103) and its complex conjugate

$$\frac{df}{dy_{l0}} = -\hat{V}_l V_l^* V_l \quad (115)$$

and

$$\frac{df^*}{dy_{l0}^*} = -\hat{V}_l^* V_l V_l^* . \quad (116)$$

For  $y_{gg}$  between generator buses  $g$  and  $g'$ , we have from (105) and its complex conjugate

$$\frac{df}{dy_{gg'}} = (\hat{V}_{g1} V_g^* - \hat{V}_{g',1} V_{g'}^*) (V_{g'} - V_g) \quad (117)$$

and

$$\frac{df^*}{dy_{gg'}^*} = (\hat{V}_{g1} V_g - \hat{V}_{g',1} V_{g'}) (V_{g'}^* - V_g^*) , \quad (118)$$

where

$$\hat{V}_m = \hat{V}_{m1} + j \hat{V}_{m2} \quad (119)$$

and  $m$  is a bus index. For  $y_{g0}$  between generator bus  $g$  and ground, we have from (105)



$$\frac{df}{dy_{g0}} = \frac{df}{dy_{g0}^*} = - \hat{V}_{g1} V_g^* V_g . \quad (120)$$

For  $y_{lg}$  between load bus  $l$  and generator bus  $g$ , we have from (103) and (105) and their complex conjugate

$$\frac{df}{dy_{lg}} = (\hat{V}_{g1} V_g^* - \hat{V}_l V_l^*) (V_l - V_g) , \quad (121)$$

and

$$\frac{df}{dy_{lg}^*} = (\hat{V}_{g1} V_g - \hat{V}_l^* V_l) (V_l^* - V_g^*) . \quad (122)$$

For  $y_{ln}$  between load bus  $l$  and the slack bus  $n$ , we have from (103) and its complex conjugate

$$\frac{df}{dy_{ln}} = \hat{V}_l V_l^* (V_n - V_l) \quad (123)$$

and

$$\frac{df}{dy_{ln}^*} = \hat{V}_l^* V_l (V_n^* - V_l^*) . \quad (124)$$

Finally, for  $y_{gn}$  between generator bus  $g$  and the slack bus  $n$ , we have from (105) and its complex conjugate

$$\frac{df}{dy_{gn}} = \hat{V}_{g1} V_g^* (V_n - V_g) \quad (125)$$

and

$$\frac{df}{dy_{gn}^*} = \hat{V}_{g1} V_g (V_n^* - V_g^*) . \quad (126)$$

### Special Considerations

If  $\rho$  of (101) contains some of the above control variables, the partial derivatives of  $f$  w.r.t. appropriate control variables must be added to the expressions obtained.

When any of the control variables  $u_k$  is a function of some real design

variables we write

$$\delta u_k = \sum_i \frac{\partial u_k}{\partial \zeta_{ki}} \Delta \zeta_{ki}, \quad (127)$$

where  $\zeta_{ki}$  is the  $i$ th design variable associated with  $u_k$  and  $\Delta \zeta_{ki}$  denotes the change in  $\zeta_{ki}$ . Hence,

$$\frac{df}{d\zeta_{ki}} = \frac{df}{du_k} \frac{\partial u_k}{\partial \zeta_{ki}}. \quad (128)$$

The control variables associated with other power system components, e.g., transformers, which are represented in the bus admittance matrix  $\underline{Y}_T$  can be easily considered. The corresponding sensitivity expressions may be derived in a similar straightforward manner.

Equations (107)-(118) and (120)-(126) compactly define the required formal derivatives of the real function  $f$  w.r.t. complex control variables. In practice, gradients w.r.t. real and imaginary parts of the defined control variables are of direct interest. These gradients are simply obtained from

$$\frac{df}{du_{k1}} = 2 \operatorname{Re} \left\{ \frac{df}{du_k} \right\} \quad (129)$$

and

$$\frac{df}{du_{k2}} = -2 \operatorname{Im} \left\{ \frac{df}{du_k} \right\}, \quad (130)$$

where the complex control variable  $u_k$  is given by

$$u_k = u_{k1} + j u_{k2}. \quad (131)$$

Table I summarizes the derived expressions of function gradients w.r.t. real control variables of practical interest.

Example 5

Using the values of  $\hat{V}$  obtained, we have for the first system

$$\frac{df}{dP_1} = 2 \hat{V}_{11} = 0.1123 ,$$

$$\frac{df}{dQ_1} = 2 \hat{V}_{12} = 0.1783 ,$$

$$\frac{df}{dV_{21}} = 2 \hat{V}_{21} = 3.3577 ,$$

$$\frac{df}{B_{10}} = 2 |V_1|^2 \hat{V}_{12} = 0.1038 ,$$

$$\frac{df}{dG_{12}} = 2 \operatorname{Re}\{\hat{V}_1 V_1^* (V_2 - V_1)\} = -0.0192 ,$$

and

$$\frac{df}{dB_{12}} = -2 \operatorname{Im}\{\hat{V}_1 V_1^* (V_2 - V_1)\} = -0.0502 ,$$

where  $G_{mm}$  and  $B_{mm}$  denote, respectively, the conductance and susceptance of line  $mm'$  connecting buses  $m$  and  $m'$ ,  $m'=0$  denotes the ground.

Example 6

For the second system, we have

$$\frac{df}{dP_1} = 2 \hat{V}_{11} = 0.0603 ,$$

$$\frac{df}{d|V_1|} = 2 \hat{V}_{12} = -0.0577 ,$$

$$\frac{df}{dV_{21}} = 2 \hat{V}_{21} = 0.5346 ,$$

$$\frac{df}{dB_{10}} = 0.0 ,$$

$$\frac{df}{dG_{12}} = 2 \hat{V}_{11} \operatorname{Re}\{V_1^* (V_2 - V_1)\} = 0.0044$$

and

$$\frac{df}{dB_{12}} = -2 \hat{V}_{11} \operatorname{Im}\{V_1^* V_2\} = -0.0108.$$

The gradients obtained can be easily checked by small perturbations about the base case values.

## VII. SENSITIVITY OF COMPLEX FUNCTIONS

In the previous sections, we have derived the required sensitivity expressions and gradients for a general real function. The relationships between different modes of formulation have been investigated and expressions relating the RHS and solution vector of corresponding adjoint systems have been derived.

The sensitivities of a general complex function can be obtained using the previous formulas derived simply by considering the real and imaginary parts separately. In this case, only the RHS of the adjoint system of equations has to be changed. In other words, only one forward and one backward substitutions are required for each real function, provided that the LU factors of the formed matrix of coefficients are stored and that the base case point remains unchanged.

In this section, we show how the compact complex formulation can be exploited to formulate the adjoint system corresponding to a general complex function and to derive the required sensitivities. The relationships between different modes of formulation are again investigated for the complex function case.

For a complex function  $f$ , we may write, using (3)

$$\delta f = \begin{bmatrix} \hat{\mu}^T \\ \tilde{\mu} \end{bmatrix} \begin{bmatrix} \delta V_{\tilde{M}} \\ \delta V_{\tilde{M}}^* \end{bmatrix} + \delta f_{\rho}, \quad (132)$$

where we have defined

$$\hat{\mu} \triangleq \frac{\partial f}{\partial V_{\tilde{M}}} \quad (133)$$

and

$$\tilde{\mu} \triangleq \frac{\partial f}{\partial V_{\tilde{M}}^*}, \quad (134)$$

$\delta f_{\rho}$  being the change in  $f$  due to changes in other variables in terms of which  $f$  may be explicitly expressed. Hence, from (69)

$$\delta f = \begin{bmatrix} \hat{\mu}^T \\ \tilde{\mu} \end{bmatrix} \begin{bmatrix} K & \bar{K} \\ \bar{K}^* & K^* \end{bmatrix}^{-1} \begin{bmatrix} d \\ d^* \end{bmatrix} + \delta f_{\rho} \quad (135)$$

or

$$\delta f = \begin{bmatrix} \hat{V}^T \\ \tilde{V} \end{bmatrix} \begin{bmatrix} d \\ d^* \end{bmatrix} + \delta f_{\rho}, \quad (136)$$

where

$$\begin{bmatrix} K^T & \bar{K}^{*T} \\ \bar{K}^T & K^{*T} \end{bmatrix} \begin{bmatrix} \hat{V} \\ \tilde{V} \end{bmatrix} = \begin{bmatrix} \hat{\mu} \\ \tilde{\mu} \end{bmatrix}, \quad (137)$$

which represents the adjoint system of equations to be solved. The first-order change of the complex function  $f$  can be evaluated by solving

(137) and substituting into (136).

The relationships between the adjoint solution of different modes of formulation are derived as follows. Let

$$f = f_1 + j f_2, \quad (138)$$

hence

$$\delta f = \delta f_1 + j \delta f_2, \quad (139)$$

and let  $\hat{\underline{V}}_r^1$  and  $\hat{\underline{V}}_s^1$  be the solution vector of the adjoint system (81) using cartesian coordinates for the real function  $f_1$ . Similarly, let  $\hat{\underline{V}}_r^2$  and  $\hat{\underline{V}}_s^2$  be the solution vector of (81) for the real function  $f_2$ . Hence, using (80) and (136), one may write

$$\hat{\underline{V}}_d^T + \hat{\underline{V}}_d^{T*} = (\hat{\underline{V}}_r^1 d_1 - \hat{\underline{V}}_s^1 d_2) + j(\hat{\underline{V}}_r^2 d_1 - \hat{\underline{V}}_s^2 d_2), \quad (140)$$

hence, from (44),

$$\hat{\underline{V}} = (\hat{\underline{V}}_r^1 - \hat{\underline{V}}_s^2)/2 + j(\hat{\underline{V}}_s^1 + \hat{\underline{V}}_r^2)/2 \quad (141)$$

and

$$\hat{\underline{V}} = (\hat{\underline{V}}_r^1 + \hat{\underline{V}}_s^2)/2 + j(-\hat{\underline{V}}_s^1 + \hat{\underline{V}}_r^2)/2. \quad (142)$$

Equations (141) and (142) relate the solutions of the adjoint system (81) for both  $f_1$  and  $f_2$  to the solution of (137) for the complex function  $f$ .

Similarly, let  $\hat{\underline{V}}_\delta^1$  and  $\hat{\underline{V}}_v^1$  be the solution vector of the adjoint system (91) using polar coordinates for the real function  $f_1$ . Also, let  $\hat{\underline{V}}_\delta^2$  and  $\hat{\underline{V}}_v^2$  be the solution vector of (91) for the real function  $f_2$ . Hence, using (90) and (136), one may write

$$\hat{\underline{V}}_d^T + \hat{\underline{V}}_d^{T*} = (\hat{\underline{V}}_\delta^1 d_1 - \hat{\underline{V}}_v^1 d_2) + j(\hat{\underline{V}}_\delta^2 d_1 - \hat{\underline{V}}_v^2 d_2), \quad (143)$$

hence, from (44)

$$\hat{\underline{V}} = (\hat{\underline{V}}_\delta^1 - \hat{\underline{V}}_v^2)/2 + j(\hat{\underline{V}}_v^1 + \hat{\underline{V}}_\delta^2)/2 \quad (144)$$

and

$$\hat{\underline{V}} = (\hat{\underline{V}}_\delta^1 + \hat{\underline{V}}_v^2)/2 + j(-\hat{\underline{V}}_v^1 + \hat{\underline{V}}_\delta^2)/2. \quad (145)$$

Equations (144) and (145) relate the solutions of the adjoint system (91) for both  $f_1$  and  $f_2$  to the solution of (137) for the complex function  $f$ .

For gradient calculations, we proceed as before and use the partitioned forms (98), (100) and

$$\hat{\tilde{V}} = \begin{bmatrix} \hat{\tilde{V}}_L \\ \hat{\tilde{V}}_G \\ \hat{\tilde{V}}_n \end{bmatrix}, \quad (146)$$

and we write (74) in the form

$$\begin{aligned} \delta f = & \hat{\tilde{V}}_L^T d_{\tilde{L}} + \hat{\tilde{V}}_G^T d_{\tilde{G}} + \hat{\tilde{V}}_n d_n + \left(\frac{\partial f}{\partial \rho}\right)^T \delta \rho \\ & + \hat{\tilde{V}}_L^T d_{\tilde{L}}^* + \hat{\tilde{V}}_G^T d_{\tilde{G}}^* + \hat{\tilde{V}}_n d_n^* + \left(\frac{\partial f}{\partial \rho^*}\right)^T \delta \rho^*. \end{aligned} \quad (147)$$

The first, second and third terms of (147) are given by (103), (105) and (106) respectively. The fifth term of (147) is given, using (99), by

$$\begin{aligned} \hat{\tilde{V}}_L^T d_{\tilde{L}}^* = & \sum_{\ell=1}^{n_L} (\hat{\tilde{V}}_\ell \delta S_\ell) + \sum_{\ell=1}^{n_L} \sum_{\substack{m=1 \\ m \neq \ell}}^n \hat{\tilde{V}}_\ell V_\ell (V_m^* - V_\ell^*) \delta y_{\ell m}^* \\ & - \sum_{\ell=1}^n \hat{\tilde{V}}_\ell V_\ell V_\ell^* \delta y_{\ell 0}^*. \end{aligned} \quad (148)$$

Also, the sixth term of (147) is given, using (33) by

$$\begin{aligned} \hat{\tilde{V}}_G^T d_{\tilde{G}}^* = & \sum_{g=n_L+1}^{n-1} \hat{\tilde{V}}_g (\delta P_g + j \delta |V_g|) + \sum_{g=n_L+1}^{n-1} \sum_{\substack{m=1 \\ m \neq g}}^n \hat{\tilde{V}}_g \operatorname{Re}\{V_g^* (V_m - V_g)\} \delta y_{gm} \\ & - \sum_{g=n_L+1}^{n-1} \hat{\tilde{V}}_g \operatorname{Re}\{V_g^* V_g\} \delta y_{g0} \end{aligned} \quad (149)$$

and the seventh term of (147) is given, using (19) by

$$\hat{\bar{V}}_n d_n^* = \hat{\bar{V}}_n \delta V_n . \quad (150)$$

Equations (103), (105), (106), (148), (149) and (150) provide useful information for gradient evaluation of the complex function  $f$  w.r.t. the control variables of interest. Under the assumption that  $\rho$  does not contain such control variables, the derivatives of the complex function  $f$  are obtained as follows.

#### Load Bus Control Variables

From (103) and (148), the derivatives of  $f$  w.r.t. the demand  $S_\ell$  and  $S_\ell^*$  at load bus  $\ell$  is given by

$$\frac{df}{dS_\ell} = \hat{\bar{V}}_\ell \quad (151)$$

and

$$\frac{df}{dS_\ell^*} = \hat{\bar{V}}_\ell . \quad (152)$$

#### Generator Bus Control Variables

From (105) and (149), the derivatives of  $f$  w.r.t. the generator control variables are given by

$$\frac{df}{\tilde{dS}_g} = \hat{\bar{V}}_g \quad (153)$$

and

$$\frac{df}{\tilde{dS}_g^*} = \hat{\bar{V}}_g , \quad (154)$$

where  $\tilde{S}_g$  is given by (22).



Slack Bus Control Variables

From (106) and (150), the derivatives of  $f$  w.r.t. the slack bus voltage  $V_n$  and  $V_n^*$  are given by

$$\frac{df}{dV_n} = \hat{V}_n \quad (155)$$

and

$$\frac{df}{dV_n^*} = \hat{V}_n^* \quad (156)$$

Line Control Variables

The derivatives of  $f$  w.r.t. line control variables  $y_{ij}$  can be obtained from (103), (105), (148) and (149) as follows. For  $y_{ll'}$  between load buses  $l$  and  $l'$ , we have from (103) and (148)

$$\frac{df}{dy_{ll'}} = (\hat{V}_l V_l^* - \hat{V}_{l'} V_{l'}^*) (V_{l'} - V_l) \quad (157)$$

and

$$\frac{df}{dy_{ll'}^*} = (\hat{V}_l V_l - \hat{V}_{l'} V_{l'}) (V_{l'}^* - V_l^*) \quad (158)$$

For  $y_{l0}$  between load bus  $l$  and ground, we have from (103) and (148)

$$\frac{df}{dy_{l0}} = -\hat{V}_l V_l^* V_l \quad (159)$$

and

$$\frac{df}{dy_{l0}^*} = -\hat{V}_l V_l V_l^* \quad (160)$$

For  $y_{gg}$  between generator buses  $g$  and  $g'$ , we have from (105) and (149)

$$\frac{df}{dy_{gg'}} = \frac{1}{2} [(\hat{V}_g + \hat{V}_{g'})V_g^* - (\hat{V}_{g'} + \hat{V}_g)V_{g'}^*] (V_{g'} - V_g) \quad (161)$$

and

$$\frac{df}{dy_{gg}^*} = \frac{1}{2} [(\hat{V}_g + \hat{\bar{V}}_g) V_g - (\hat{V}_g + \hat{\bar{V}}_g) V_g] (V_g^* - V_g^*) . \quad (162)$$

For  $y_{g0}$  between generator bus  $g$  and ground, we have from (105) and (149)

$$\frac{df}{dy_{g0}} = \frac{df}{dy_{g0}^*} = -\frac{1}{2} (\hat{V}_g + \hat{\bar{V}}_g) V_g^* V_g . \quad (163)$$

For  $y_{lg}$  between load bus  $l$  and generator bus  $g$ , we have from (103), (105), (148) and (149)

$$\frac{df}{dy_{lg}} = \left[ \frac{1}{2} (\hat{V}_g + \hat{\bar{V}}_g) V_g^* - \hat{V}_l V_l^* \right] (V_l - V_g) \quad (164)$$

and

$$\frac{df}{dy_{lg}^*} = \left[ \frac{1}{2} (\hat{V}_g + \hat{\bar{V}}_g) V_g - \hat{\bar{V}}_l V_l \right] (V_l^* - V_g^*) . \quad (165)$$

For  $y_{ln}$  between load bus  $l$  and the slack bus  $n$ , we have from (103) and (148)

$$\frac{df}{dy_{ln}} = \hat{V}_l V_l^* (V_n - V_l) \quad (166)$$

and

$$\frac{df}{dy_{ln}^*} = \hat{\bar{V}}_l V_l (V_n^* - V_l^*) . \quad (167)$$

Finally, for  $y_{gn}$  between generator bus  $g$  and the slack bus  $n$ , we have from (105) and (149)

$$\frac{df}{dy_{gn}} = \frac{1}{2} (\hat{V}_g + \hat{\bar{V}}_g) V_g^* (V_n - V_g) \quad (168)$$

and

$$\frac{df}{dy_{gn}^*} = \frac{1}{2} (\hat{V}_g + \hat{\bar{V}}_g) V_g (V_n^* - V_g^*) . \quad (169)$$

Remarks

If  $\rho$  of (147) contains any of the above control variables, the partial derivatives of  $f$  w.r.t. appropriate control variables must be added to the expressions (151)-(169).

Equations (151)-(169) compactly define the required formal derivatives of the complex function  $f$  w.r.t. complex control variables. The gradients of  $f$  w.r.t. real and imaginary parts of the control variables are obtained using

$$\frac{df}{du_{k1}} = \frac{df}{du_k} + \frac{df}{du_k^*} \quad (170)$$

and

$$\frac{df}{du_{k2}} = j \left( \frac{df}{du_k} - \frac{df}{du_k^*} \right), \quad (171)$$

where  $u_k$  is given by (131).

Expressions of forms (170) and (171) can be directly obtained from (151)-(169).

Example 7

Now, we consider the first 2-bus system and the complex function

$$f = V_1 = V_{11} + j V_{12}.$$

Using cartesian coordinates, the adjoint system solutions for  $V_{11}$  and  $V_{12}$  are given, respectively, by

$$\hat{V}_1^r = \begin{bmatrix} 0.0883 \\ 2.3144 \end{bmatrix},$$

$$\hat{V}_1^s = \begin{bmatrix} 0.1161 \\ 0.2041 \end{bmatrix},$$

$$\underset{\sim}{V}^2_r = \begin{bmatrix} 0.0428 \\ 0.1117 \end{bmatrix}$$

and

$$\underset{\sim}{V}^2_s = \begin{bmatrix} -0.0187 \\ 0.7352 \end{bmatrix} ,$$

hence, from (141) and (142)

$$\underset{\sim}{\hat{V}} = \begin{bmatrix} 0.0535 + j 0.0794 \\ 0.7896 + j 0.1579 \end{bmatrix}$$

and

$$\underset{\sim}{\hat{V}} = \begin{bmatrix} 0.0348 - j 0.0366 \\ 1.5248 - j 0.0462 \end{bmatrix} .$$

The derivatives of  $f$  w.r.t. control variables are calculated, using the derived expressions, as follows. For  $S_1$ ,

$$\frac{df}{dS_1} = \underset{\sim}{\hat{V}}_1 = 0.0348 - j 0.0366$$

and

$$\frac{df}{dS_1^*} = \underset{\sim}{\hat{V}}_1 = 0.0535 + j 0.0794 ,$$

hence, from (170) and (171)

$$\frac{df}{dP_1} = 0.0883 - j 0.0428$$

and

$$\frac{df}{dQ_1} = 0.1161 - j 0.0187 .$$

For  $V_2$ ,

$$\frac{df}{dV_2} = \underset{\sim}{\hat{V}}_2 = 1.5248 - j 0.0462$$

and

$$\frac{df}{dV_2^*} = \hat{V}_2 = 0.7896 + j 0.1579,$$

hence, from (170)

$$\frac{df}{dV_{21}} = 2.3144 + j 0.1117.$$

For  $y_{10}$ ,

$$\frac{df}{dy_{10}} = -|V_1|^2 \hat{V}_1 = -0.0311 - j 0.0462$$

and

$$\frac{df}{dy_{10}^*} = -|V_1|^2 \hat{V}_1^* = -0.0203 + j 0.0213,$$

hence, from (170) and (171)

$$\frac{df}{dG_{10}} = -0.0514 - j 0.0249$$

and

$$\frac{df}{dB_{10}} = 0.0676 - j 0.0109.$$

For  $y_{12}$ ,

$$\frac{df}{dy_{12}} = \hat{V}_1 V_1^* (V_2 - V_1) = -0.0080 + j 0.0231$$

and

$$\frac{df}{dy_{12}^*} = \hat{V}_1^* V_1 (V_2^* - V_1^*) = -0.0022 - j 0.0127,$$

hence, from (170) and (171)

$$\frac{df}{dG_{12}} = -0.0102 + j 0.0104$$

and

$$\frac{df}{dB_{12}} = - 0.0358 - j 0.0059 .$$

#### VIII. APPLICATIONS TO A 6-BUS SAMPLE POWER SYSTEM

In this section, we present some of the numerical results obtained for a 6-bus power system [13] using the sensitivity formulas derived in the paper.

The system consists of three specified load buses ( $l = 1,2,3$ ), two generator buses ( $g = 4,5$ ), the slack bus ( $n=6$ ) and eight transmission lines ( $t = 7, \dots, 14$ ). The single line diagram for this system is shown in Fig. 3. The line and bus data are shown, respectively, in Tables II and III. All values shown are in per unit. The application of the adjoint network approach results in the load flow solution shown in Table IV.

Examples of sensitivities of bus states, namely  $|V_1|$ ,  $Q_4$ ,  $\delta_1$ , and  $\delta_4$  w.r.t. system bus and line control variables are shown in Tables V to VIII. The estimated effects of the line and circuit outages on the different states, based on first-order changes, are also shown.

Observe that the sensitivities w.r.t. non-existing elements, e.g., the shunt parameters in Tables V - VIII can be evaluated as well.

Although the sensitivities of a general function can be evaluated using the same adjoint matrix of coefficients at the load flow solution and by defining the RHS of the adjoint equations corresponding to the function considered, these sensitivities can also be obtained, directly, using the results of Tables V to VIII. For example, consider the function

$$f = |I_{14}|^2 = |V_1 - V_4|^2 \cdot |Y_{14}|^2, \quad (172)$$

which may denote the loading of line 1,4. The sensitivities of this function w.r.t. a control variable  $u_k$  is given by

$$\frac{df}{du_k} = \frac{\partial f}{\partial u_k} + 2|V_1 - V_4||Y_{14}|^2 \frac{\partial |V_1 - V_4|}{\partial u_k}, \quad (173)$$

which, when substituting values at the load flow solution and noting that  $|V_4|$  is constant, reduces to

$$\frac{df}{du_k} = \frac{\partial f}{\partial u_k} - 1.6871 \frac{\partial |V_1|}{\partial u_k} - 4.8588 \frac{\partial \delta_1}{\partial u_k} + 4.8588 \frac{\partial \delta_4}{\partial u_k}.$$

Now, let  $u_k$  denote the conductance of line 2,4. Hence, from Tables V, VII and VIII, we get

$$\frac{df}{dG_{24}} = - 0.0324.$$

Similarly, if  $u_k$  denotes the susceptance of line 2,4, we get

$$\frac{df}{dB_{24}} = - 0.0932.$$

The effect of line 2,4 outage on the function considered can be estimated using the relation

$$\delta f = - \frac{df}{dG_{24}} G_{24} - \frac{df}{dB_{24}} B_{24}, \quad (174)$$

where we have set the changes in line conductance and susceptance, respectively, to  $-G_{24}$  and  $-B_{24}$ . Substituting the values of  $G_{24}$  (=0.5882) and  $B_{24}$  (= -2.3529) in (174), we get

$$\delta f = 0.019 - 0.219 = - 0.200,$$

which is identical to the result presented in the Tellegen's theorem approach of [13] where the function  $f = |I_{14}|^2$  was considered, directly, in the adjoint simulation without state transformations.

## IX. CONCLUSIONS

A unified study for the class of adjoint network approaches to power system sensitivity analysis which exploits the Jacobian matrix of the load flow solution has been presented. Generalized sensitivity expressions which are easily derived, compactly described and effectively used for calculating first-order changes and gradients of functions of interest have been obtained. These generalized sensitivity expressions are common to all modes of formulation, e.g., polar and cartesian.

A first step towards deriving these generalized sensitivity expressions has been performed where we have utilized a special complex notation to compactly describe the transformations relating different ways of formulating power network equations to a standard complex form. This special notation and the derived transformations have been used to effectively derive the required sensitivity expressions only by matrix manipulations.

The use of these generalized sensitivity expressions requires only the solution of an adjoint system of linear equations, the matrix of coefficients of which is simply the transpose of the Jacobian matrix of the load flow solution in any mode of formulation. These generalized sensitivity expressions are applicable to both real and complex modes of performance functions as well as the control variables defined in a particular study.



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TABLE I  
DERIVATIVES OF A REAL FUNCTION f W.R.T. CONTROL VARIABLES

Control Variable	Description	Derivative
$P_l$	demand real power	$2 \hat{V}_{l1}$
$Q_l$	demand reactive power	$2 \hat{V}_{l2}$
$P_g$	generator real power	$2 \hat{V}_{g1}$
$ V_g $	generator bus voltage magnitude	$2 \hat{V}_{g2}$
$V_{n1}$	real component of slack bus voltage	$2 \hat{V}_{n1}$
$G_{ll}$	conductance between two load buses	$2 \operatorname{Re}\{(\hat{V}_l V_l^* - \hat{V}_l V_l^*)(V_l - V_l)\}$
$B_{ll}$	susceptance between two load buses	$-2 \operatorname{Im}\{(\hat{V}_l V_l^* - \hat{V}_l V_l^*)(V_l - V_l)\}$
$G_{l0}$	shunt conductance of a load bus	$-2  V_l ^2 \hat{V}_{l1}$
$B_{l0}$	shunt susceptance of a load bus	$2  V_l ^2 \hat{V}_{l2}$
$G_{gg}$	conductance between two generator buses	$2 \operatorname{Re}\{(\hat{V}_{g1} V_{g1}^* - \hat{V}_{g1} V_{g1}^*)(V_{g1} - V_{g1})\}$
$B_{gg}$	susceptance between two generator buses	$-2 \operatorname{Im}\{(\hat{V}_{g1} V_{g1}^* - \hat{V}_{g1} V_{g1}^*)(V_{g1} - V_{g1})\}$
$G_{g0}$	shunt conductance of a generator bus	$-2  V_g ^2 \hat{V}_{g1}$
$B_{g0}$	shunt susceptance of a generator bus	0
$G_{lg}$	conductance between load and generator buses	$2 \operatorname{Re}\{(\hat{V}_{g1} V_{g1}^* - \hat{V}_l V_l^*)(V_l - V_{g1})\}$
$B_{lg}$	susceptance between load and generator buses	$-2 \operatorname{Im}\{(\hat{V}_{g1} V_{g1}^* - \hat{V}_l V_l^*)(V_l - V_{g1})\}$

Control Variable	Description	Derivative
$G_{ln}$	conductance between load and slack buses	$2 \operatorname{Re}\{\hat{V}_l V_l^* (V_n - V_l)\}$
$B_{ln}$	susceptance between load and slack buses	$-2 \operatorname{Im}\{\hat{V}_l V_l^* (V_n - V_l)\}$
$G_{gn}$	conductance between generator and slack buses	$2\hat{V}_{g1} \operatorname{Re}\{V_g^* (V_n - V_g)\}$
$B_{gn}$	susceptance between generator and slack buses	$-2 \hat{V}_{g1} \operatorname{Im}\{V_g^* V_n\}$

TABLE II  
LINE DATA FOR 6-BUS POWER SYSTEM

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Line No.	Terminal Buses	Resistance $R_t$ (pu)	Reactance $X_t$ (pu)	Number of Lines
1	1,4	0.05	0.20	1
2	1,5	0.025	0.10	2
3	2,3	0.10	0.40	1
4	2,4	0.10	0.40	1
5	2,5	0.05	0.20	1
6	2,6	0.01875	0.075	4
7	3,4	0.15	0.60	1
8	3,6	0.0375	0.15	2

---

TABLE III  
BUS DATA FOR 6-BUS POWER SYSTEM

Bus Index, m	Bus Type	$P_m$ (pu)	$Q_m$ (pu)	$ V_m  \angle \delta_m$ (pu)
1	load	-2.40	0	- $\angle -$
2	load	-2.40	0	- $\angle -$
3	load	-1.60	-0.40	- $\angle -$
4	generator	-0.30	-	1.02 $\angle -$
5	generator	1.25	-	1.04 $\angle -$
6	slack	-	-	1.04 $\angle 0$

TABLE IV  
LOAD FLOW SOLUTION OF 6-BUS POWER SYSTEM

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Load Buses

$$V_1 = 0.9787 \angle -0.6602$$

$$V_2 = 0.9633 \angle -0.2978$$

$$V_3 = 0.9032 \angle -0.3036$$

Generator Buses

$$Q_4 = 0.7866, \quad \delta_4 = -0.5566$$

$$Q_5 = 0.9780, \quad \delta_5 = -0.4740$$

Slack Bus

$$P_6 = 6.1298, \quad Q_6 = 1.3546$$

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TABLE V  
6-BUS SYSTEM: SENSITIVITIES OF  $|V_1|$

Line Quantities				
Line	Total Derivatives		Contingency Effect	
	Conductance	Susceptance	Outage of One Line	Outage of Circuit
1,4	-0.006326	-0.005283	0.017421	0.017421
1,5	-0.011838	-0.008884	0.027880	0.055760
2,3	0.000027	-0.000012	0.000044	0.000044
2,4	-0.000207	-0.000597	-0.001282	0.001282
2,5	0.000163	0.000294	-0.001192	-0.001192
2,6	-0.000002	0.000039	-0.000123	-0.000494
3,4	-0.000265	-0.000443	0.000591	0.000591
3,6	-0.000017	-0.000120	0.000362	0.000724

Load Bus Quantities - Total Derivatives				
Bus	Real Power	Reactive Power	Shunt Conductance	Shunt Susceptance
1	0.029522	0.070273	-0.028275	-0.067306
2	-0.000131	-0.000005	0.000122	0.000005
3	0.000378	0.000169	-0.000308	-0.000138

Generator Bus Quantities - Total Derivatives				
Bus	Voltage Magnitude	Real Power	Shunt Conductance	Shunt Susceptance
4	0.357365	0.002243	-0.002334	0.0
5	0.732004	-0.001804	0.001951	0.0

TABLE VI  
6-BUS SYSTEM: SENSITIVITIES OF  $Q_4$

Line Quantities				
Line	Total Derivatives		Contingency Effect	
	Conductance	Susceptance	Outage of One Line	Outage of Circuit
1,4	-0.056140	-0.044515	0.143437	0.143437
1,5	0.065943	0.060168	-0.205565	-0.411130
2,3	0.000236	0.004289	-0.009954	-0.009954
2,4	0.256340	0.022413	0.098051	0.098051
2,5	-0.015503	0.028010	-0.150048	-0.150048
2,6	0.046139	0.039093	-0.086459	-0.345835
3,4	0.243148	-0.031249	0.144371	0.144371
3,6	0.062174	0.056610	-0.128837	-0.257674

Load Bus Quantities - Total Derivatives				
Bus	Real Power	Reactive Power	Shunt Conductance	Shunt Susceptance
1	-0.457852	-0.358531	0.438519	0.343391
2	-0.115872	-0.168723	0.107512	0.156551
3	-0.127525	-0.258052	0.104029	0.210506

Generator Bus Quantities - Total Derivatives				
Bus	Voltage Magnitude	Real Power	Shunt Conductance	Shunt Susceptance
4	7.51274	-0.550625	0.572870	0.0
5	-4.66462	-0.219233	0.237122	0.0



TABLE VII  
6-BUS SYSTEM: SENSITIVITIES OF  $\delta_1$

Line Quantities				
Line	Total Derivatives		Contingency Effect	
	Conductance	Susceptance	Outage of One Line	Outage of Circuit
1,4	0.001197	-0.010358	0.050152	0.050152
1,5	-0.004594	-0.016180	0.070737	0.141473
2,3	-0.001609	0.000178	-0.001366	-0.001366
2,4	-0.010354	-0.031650	0.068981	0.068981
2,5	-0.011653	-0.025839	0.107885	0.107885
2,6	-0.005283	-0.025867	0.077008	0.308030
3,4	-0.020029	-0.036084	0.054530	0.054530
3,6	-0.002723	-0.019449	0.058881	0.117762

Load Bus Quantities - Total Derivatives				
Bus	Real Power	Reactive Power	Shunt Conductance	Shunt Susceptance
1	0.309969	-0.002339	-0.296880	0.002240
2	0.085296	0.026631	-0.079143	-0.024709
3	0.061420	0.027332	-0.050104	-0.022297

Generator Bus Quantities - Total Derivatives				
Bus	Voltage Magnitude	Real Power	Shunt Conductance	Shunt Susceptance
4	0.192793	0.208858	-0.217296	0.0
5	0.271949	0.223549	-0.241790	0.0

TABLE VIII  
6-BUS SYSTEM: SENSITIVITIES OF  $\delta_4$

Line Quantities				
Line	Total Derivatives		Contingency Effect	
	Conductance	Susceptance	Outage of One Line	Outage of Circuit
1,4	-0.006119	0.005953	-0.035213	-0.035213
1,5	-0.004959	-0.011033	0.046087	0.092174
2,3	-0.000725	-0.000212	0.000073	0.000073
2,4	-0.017094	-0.051045	0.110050	0.110050
2,5	-0.006343	-0.016282	0.069157	0.069157
2,6	-0.005360	-0.024608	0.072997	0.291989
3,4	-0.028650	-0.050482	0.067952	0.067952
3,6	-0.003276	-0.023336	0.017661	0.035321

Load Bus Quantities - Total Derivatives				
Bus	Real Power	Reactive Power	Shunt Conductance	Shunt Susceptance
1	0.222333	0.005176	-0.212945	-0.004957
2	0.081031	0.026460	-0.075185	-0.024551
3	0.073688	0.032826	-0.060111	-0.026778

Generator Bus Quantities - Total Derivatives				
Bus	Voltage Magnitude	Real Power	Shunt Conductance	Shunt Susceptance
4	-0.047087	0.281747	-0.293130	0.0
5	0.272518	0.164929	-0.178387	0.0

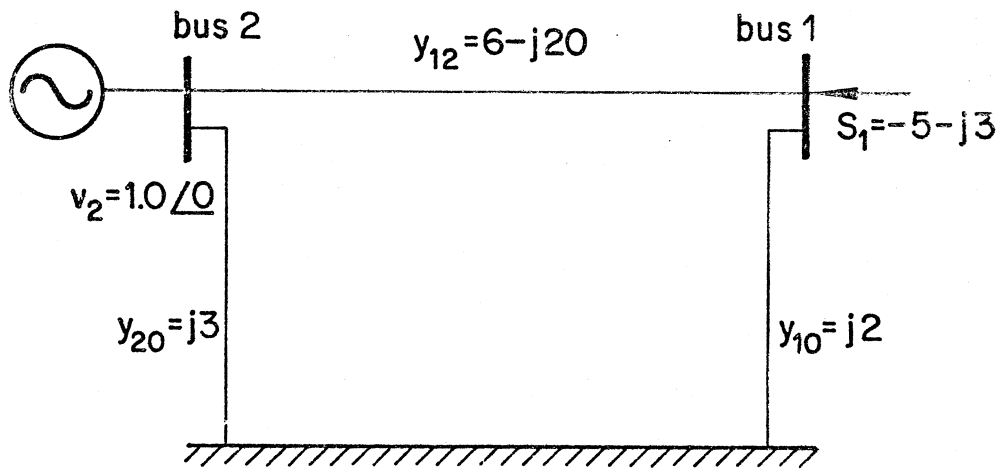


Fig. 1 2-bus load-slack sample power system

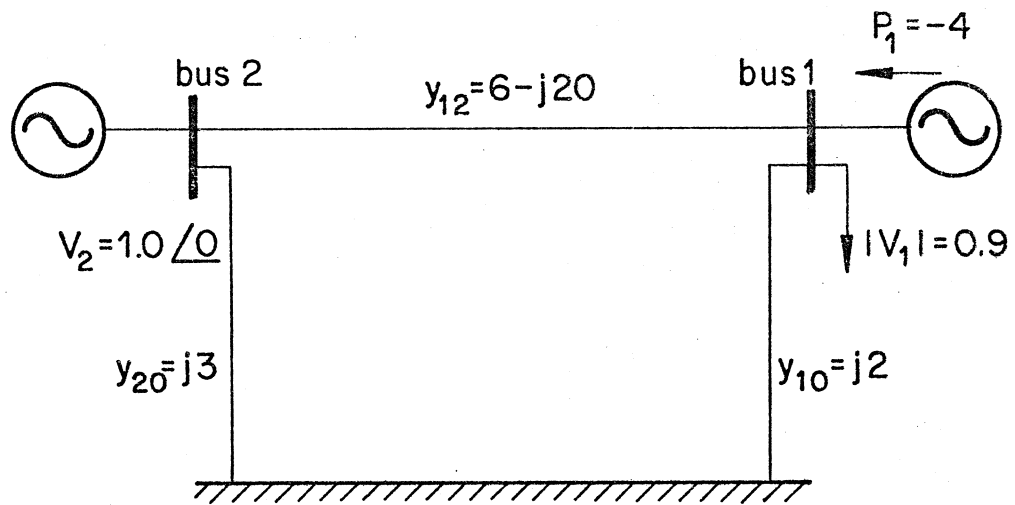


Fig. 2 2-bus generator-slack sample power system

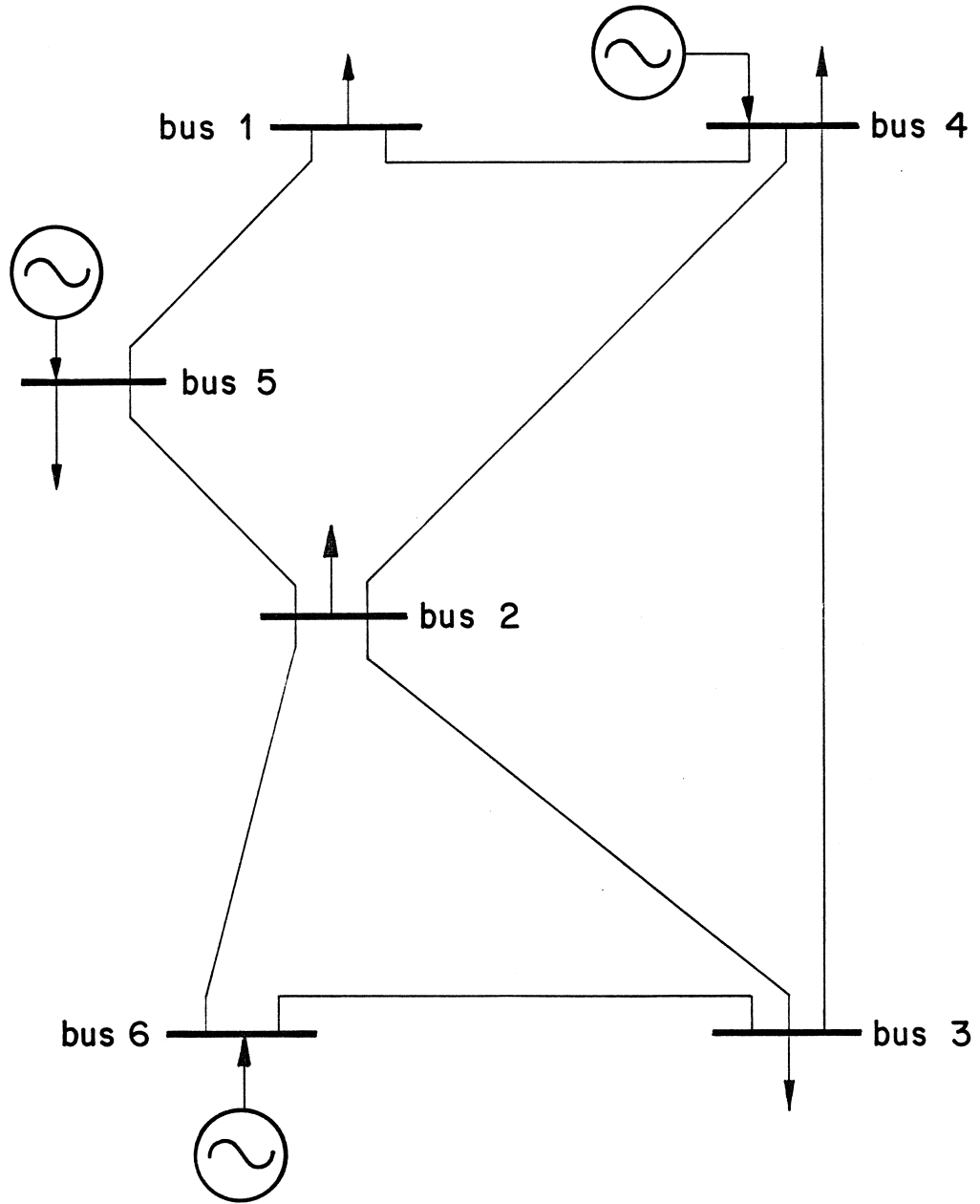


Fig. 3 6-bus sample power system