THEORY OF COMPUTER-AIDED DESIGN OF MICROWAVE MULTIPLEXERS

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Abstract

A completely general and attractive theory is presented for computer-oriented simulation, sensitivity analysis and design of multiplexing networks represented by branches connected at arbitrarily spaced and defined junctions along a main cascade. This theory permits an efficient and fast analytical and numerical investigation of responses and sensitivities of all functions of interest w.r.t. any variable parameter, including frequency. Thevenin equivalent circuits at any reference plane and their sensitivities are also expressed analytically and calculated systematically. Thus, responses such as common port return loss, channel output return loss, insertion or transducer loss, gain slope and group delay can be handled exactly and efficiently. Any junction model can be accomodated. A comprehensive set of explicit analytical formulas is also provided in the text and by tables for various response and sensitivity calculations. The basic analysis approach, namely, the forward and reverse analysis approach, is fully demonstrated and can be further developed for more general applications in network simulation, sensitivity analysis and design procedures.

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I. INTRODUCTION

This paper presents a general theory for computer-aided design of multiplexing networks, which can be represented by a cascaded circuit having channels connected at various arbitrarily spaced and defined junctions to the cascaded circuit (Fig. 1). An exact analysis approach for cascaded structures originated by Bandler et al. [1-3] is used in our multiplexer network to yield an efficient and fast analytical and numerical investigation of responses and sensitivities of all functions of interest w.r.t. any variable parameter, including frequency. Thevenin equivalent circuits at any reference plane and their sensitivities are also evaluated exactly and systematically using this approach. Thus, responses such as common port return loss, channel output return loss, insertion or transducer loss, gain slope and group delay can be handled exactly and efficiently. Channel models can be fully imbedded into multiplexer structures, e.g., of the waveguide manifold type and any desired junction model can be accommodated, e.g., models from Marcuvitz [4].

This paper is organized such that the new theory is introduced from first principles and then we proceed to various advanced multiplexer response and sensitivity expressions. The basic analysis method used in this theory, namely the forward and reverse analysis method [1], is made possible in our multiplexer structure through systematic reductions of 3-port junction models to 2-port representations, which are demonstrated in Section II. In Section III, we fully illustrate the forward and reverse analysis method and simultaneously introduce basic definitions of systematic reference planes, subnetworks and general multiplexer configurations. Then, we start the multiplexer simulation and sensitivity analysis, first with channel output voltage responses in Section IV, second with Thevenin equivalent circuits in Section V. These two sections, i.e., IV and V, constitute the basic and key simulation and sensitivity analysis procedures. In Sections VI, VII and VIII, various formulas are derived for the calculation of responses and sensitivities of common port and channel output port reflection coefficient, return loss, insertion loss, gain slope and group delay. The simulation and sensitivity analysis at subnetwork levels are discussed in Section

IX. A complete set of tables is provided to show the flexibility of our general theory and to permit efficient computations of responses and sensitivities for various situations where, for example, the variable can exist in a junction, spacing, channel or everywhere, the junction can be a series or parallel connection, the main cascade termination can be S.C. or O.C., etc. An appendix is provided for conversions between different forms of subnetwork matrix representations which will be helpful in the reduction of various 3-port junctions to 2-port representations.

While our theory is general, we have in mind a typical multiplexer network consisting of a waveguide manifold with series connected filters and short-circuited main cascade termination. First- and second-order sensitivity expressions are derived with respect to every possible design variable, such as filter coupling coefficients, input-output transformer ratios, waveguide spacings or section lengths, non-ideal junction susceptances and frequency.

II. REDUCTION OF A 3-PORT JUNCTION TO A 2-PORT REPRESENTATION

The forward and reverse analysis approach [1-3] provides a powerful tool for cascaded network analysis. By reducing certain 3-port junctions to 2-port representations, various response and sensitivity formulas are derived and efficient computational schemes are constructed for multiplexing networks.

Consider the 3-port network shown in Fig. 2. The 2-port representation can be obtained by terminating one port in the original network and studying the external behaviour of the remaining ports. For different terminated ports, we have the following two cases.

<u>Case 1</u> <u>Port 3 Terminated</u>

Suppose the 3-port network of Fig. 2 is characterized by a hybrid matrix **H** such that

$$\begin{bmatrix} V_1 \\ I_1 \\ I_3 \end{bmatrix} = \mathbf{H} \begin{bmatrix} V_2 \\ I_2 \\ V_3 \end{bmatrix} , \tag{1}$$

where

$$\mathbf{H} \triangleq \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} . \tag{2}$$

Then we have the transmission representation between ports 1 and 2 given by

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \mathbf{A} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} , \tag{3}$$

where

$$\mathbf{A} \triangleq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} , \tag{4}$$

$$a_{ij} = (-1)^{j-1} [h_{ij} - h_{i3} h_{3j} / (Y_3 + h_{33})] \tag{5}$$

and port 3 is terminated as

$$Y_3 = (-I_3)/V_3 \ . \tag{6}$$

Also, we can express the linear combination

$$\boldsymbol{\alpha}^T \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \boldsymbol{\beta}^T \begin{bmatrix} V_3 \\ -I_3 \end{bmatrix} , \tag{7}$$

where

$$\boldsymbol{\alpha}^T = [-h_{31} \quad h_{32}] \tag{8}$$

$$\boldsymbol{\beta}^T = [\quad h_{33} \quad 1 \ ]. \tag{9}$$

Case 2 Port 2 Terminated

Consider the hybrid matrix $\bar{\mathbf{H}}$ given by

$$\begin{bmatrix} V_1 \\ I_1 \\ I_2 \end{bmatrix} = \bar{\mathbf{H}} \begin{bmatrix} V_3 \\ I_3 \\ V_2 \end{bmatrix} , \tag{10}$$

where

$$\bar{\mathbf{H}} \triangleq \begin{bmatrix} \bar{h}_{11} & \bar{h}_{12} & \bar{h}_{13} \\ \bar{h}_{21} & \bar{h}_{22} & \bar{h}_{23} \\ \bar{h}_{31} & \bar{h}_{32} & \bar{h}_{33} \end{bmatrix} . \tag{11}$$

We have the transmission representation between ports 1 and 3 given by

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \mathbf{D} \begin{bmatrix} V_3 \\ -I_3 \end{bmatrix} , \tag{12}$$

where

$$\mathbf{D} \triangleq \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} , \tag{13}$$

$$d_{ij} = (-1)^{j-1} [\bar{h}_{ij} - \bar{h}_{i3} \bar{h}_{3j} / (Y_2 + \bar{h}_{33})] , \tag{14}$$

and the equivalent termination of port 2 is given by

$$Y_2 = (-I_2)/V_2 \ . \tag{15}$$

Discussion

In some cases, e.g., for a special ideal 3-port network consisting of a pure parallel connection, neither $\mathbf{H}$ nor $\bar{\mathbf{H}}$ exists. The corresponding hyrid matrix expressions then take the form

$$\begin{bmatrix} V_1 \\ I_1 \\ V_3 \end{bmatrix} = \mathbf{H} \begin{bmatrix} V_2 \\ I_2 \\ I_3 \end{bmatrix} \tag{16}$$

and

$$\begin{bmatrix} V_1 \\ I_1 \\ V_2 \end{bmatrix} = \bar{\mathbf{H}} \begin{bmatrix} V_3 \\ I_3 \\ I_2 \end{bmatrix} . \tag{17}$$

Simultaneously, Y_3 and Y_2 in equations (5) and (14) are replaced by $Z_3 = 1/Y_3$ and $Z_2 = 1/Y_2$, respectively, and equations (8) and (9) become

$$\boldsymbol{\alpha}^T = [\, h_{31} \quad -h_{32}] \;, \tag{18}$$

and

$$\boldsymbol{\beta}^T = [\, 1 \qquad h_{33}] \;. \tag{19}$$

Formulas are provided in the Appendix for various conversions between different hybrid matrices, e.g., $\mathbf{H}$ and $\bar{\mathbf{H}}$, the admittance matrix and the impedance matrix.

III. THE FORWARD AND REVERSE ANALYSIS APPROACH IN THE MULTIPLEXER STRUCTURE

Introduction

The forward and reverse analysis method is an exact and efficient approach to network analysis for cascaded structures [1 - 3, 5]. In this approach, all calculations are applied directly to the given network, no auxiliary or adjoint networks being required. Response functions, sensitivities, or large change effects are expressed analytically in terms of the variable parameters in the network. All parts of the network to be kept constant are reduced to a few 2-element vectors appearing as constants in the analysis.

In the previous section, we have reduced general 3-port junctions to 2-port representations so that forward and reverse analysis can be readily carried through these junctions in different desired directions, either along the main cascade or into any desired channel. As a consequence, all functions of interest and their exact sensitivities w.r.t. any variable in the multiplexer, including frequency, are calculated systematically and efficiently by cascaded analysis only.

Reference Planes

Consider the multiplexer consisting of N sections, as shown in Fig. 3. A typical section, the kth section depicted in Fig. 4(a), has a junction, a spacing and n(k) elements of branch k considered in cascade as the kth channel. All reference planes in the entire multiplexer are defined uniformly and numbered consecutively beginning from the main cascade termination, which is designated reference plane 1. The source port is designated reference plane 2N+2. The termination of the kth channel is called reference plane $\tau(k)$ and the channel main cascade connection is reference plane $\sigma(k)$, k = 1, 2, ..., N, where

$$\begin{aligned} &\tau(1) = 2N + 3, && \\ &\sigma(k) = \tau(k) + n(k), && k = 1, 2, \ldots, N\,, \\ &\tau(k) = \sigma(k-1) + 1, && k = 2, 3, \ldots, N\,. \end{aligned} \tag{20}$$

Two-port matrix and vector representations $\mathbf{A}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\mathbf{D}$ (in equations (4), (8), (9) and (13)) are calculated for each branch/junction combination and are denoted as $\mathbf{A}_{2k}$, $\boldsymbol{\alpha}_{2k}$, $\boldsymbol{\beta}_{2k}$ and $\mathbf{D}_{2k}$ for the kth junction. Elements in every channel and spacing in every section are represented by chain matrices $\mathbf{A}_i$, where i is the index of the reference plane at the output of the corresponding element or spacing. See Figs. 3 and 4(a).

Let

$$I_r = \{1, 2, 3, \ldots, \sigma(N)\} \tag{21}$$

be the index set containing indexes of all reference planes and

$$I = \{i \mid i \in I_r, i \neq 2N+2, i \neq \sigma(k), k = 1, 2, \ldots, N\} \tag{22}$$

be the index set containing subscripts of all $\mathbf{A}$ matrices which can logically be defined using subscript of the associated output reference plane.

Cascade Analysis [1]

Let

$$\mathbf{y}^i \triangleq \begin{bmatrix} y_1^i \\ y_2^i \end{bmatrix}, \tag{23}$$

where $y_1^{\,i}$ and $y_2^{\,i}$ are the voltage and the current, respectively, at reference plane i, $i \in I_r$, as shown in Fig. 5.

The principal concepts using forward and reverse analysis in multiplexer simulation are summarized in Table I. Figures 3 and 4 can be referred to for associated reference planes and chain matrices. The basic iteration is $\mathbf{y}^{i+1} = \mathbf{A}_i\, \mathbf{y}^i$ for $i \in I$ and $\mathbf{y}^{2k+1} = \mathbf{D}_{2k}\, \mathbf{y}^{\sigma(k)}$ for $k = 1, 2, \ldots, N$, where $\mathbf{A}_i$ or $\mathbf{D}_{2k}$ is a chain matrix and where $\mathbf{y}^{i+1}$ and $\mathbf{y}^i$, or $\mathbf{y}^{2k+1}$ and $\mathbf{y}^{\sigma(k)}$ are the voltage-current vectors at the input and output ports of the corresponding chain matrices respectively.

The forward analysis $(\mathbf{u}^{xi})^T$ is the result of a row vector initialized at reference plane x as either $[1 \;\; 0]$, $[0 \;\; 1]$ or a suitable linear combination and successively premultiplying each corresponding chain matrix by the resulting row vector until reference plane i is reached. The reverse analysis, on the other hand, is similar to the conventional cascaded network analysis. The column vector $\mathbf{v}^{ix}$ is obtained by initializing a column vector as either $[1 \;\; 0]^T$, $[0 \;\; 1]^T$ or a suitable linear combination at reference plane x and successively postmultiplying each chain matrix by the resulting column vector until reference plane i is reached. Also, the unit vectors necessary for the analysis are given by

$$\mathbf{e}_1 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{24}$$

and

$$\mathbf{e}_2 \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{25}$$

The result of the analysis between reference planes i and j is defined as

$$\mathbf{Q}_{ij} \triangleq [\mathbf{p}_{ij} \;\; \mathbf{q}_{ij}] \triangleq \begin{bmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{bmatrix}, \tag{26}$$

where

$$\mathbf{p}_{ij} \triangleq \begin{bmatrix} A_{ij} \\ C_{ij} \end{bmatrix}, \mathbf{q}_{ij} \triangleq \begin{bmatrix} B_{ij} \\ D_{ij} \end{bmatrix} \tag{27}$$

and where A_{ij}, B_{ij}, C_{ij} and D_{ij} are the equivalent chain matrix elements between reference planes i and j and are expressed in the form $\mathbf{u}^T \mathbf{A}\ \mathbf{v}$ to facilitate sensitivity, first-order change, and large change analysis [1]. Table II can be referred to for various definitions associated with A_{ij}, B_{ij}, C_{ij} and D_{ij}.

Sensitivity Analysis

From the definitions of A, $\mathbf{u}^{i,x+1}$ and $\mathbf{v}^{xj}$, we can write

$$\begin{aligned} A_{ij} &= (\mathbf{u}_1^{i,x+1})^T \mathbf{A}_x \mathbf{v}_1^{xj} \\ &= \mathbf{e}_1^T \mathbf{A}_{i-1} \mathbf{A}_{i-2} \cdots \mathbf{A}_x \cdots \mathbf{A}_{j+1} \mathbf{A}_j \mathbf{e}_1 \end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned} \frac{\partial A_{ij}}{\partial \phi} &= \sum_{x=j}^{i-1} \left[\mathbf{e}_1^T \mathbf{A}_{i-1} \cdots \mathbf{A}_{x+1} \frac{\partial \mathbf{A}_x}{\partial \phi} \mathbf{A}_{x-1} \cdots \mathbf{A}_j \mathbf{e}_1 \right] \\ &= \sum_{x=j}^{i-1} \left[\left(\mathbf{u}_1^{i,x+1} \right)^T \frac{\partial \mathbf{A}_x}{\partial \phi} \mathbf{v}_1^{xj} \right] \\ &= \sum_{\ell \in I_\phi} \left[\left(\mathbf{u}_1^{i,\ell+1} \right)^T \frac{\partial \mathbf{A}_\ell}{\partial \phi} \mathbf{v}_1^{\ell j} \right] \\ &= \sum_{\ell \in I_\phi} \frac{\partial A_{ij}^{\ell}}{\partial \phi} \quad , \end{aligned} \tag{29}$$

where

$$\frac{\partial A_{ij}^{\ell}}{\partial \phi}$$

has been defined in Table II and where I_ϕ is an index set whose elements identify the chain matrices between reference planes i and j containing the variable parameter ϕ. Generally, we have

$$\frac{\partial Q_{ij}}{\partial \phi} = \sum_{\ell \in I_\phi} \frac{\partial Q_{ij}^{\ell}}{\partial \phi} , \tag{30}$$

where Q represents *A*, *B*, *C* or *D*. Table II can be referred to for the definition of

$$\frac{\partial Q_{ij}^{\ell}}{\partial \phi} .$$

Second-order sensitivities can be derived in a similar manner as

$$\frac{\partial^2 Q_{ij}}{\partial \phi \, \partial \psi} = \sum_{\ell \in I_\phi} \sum_{t \in I_\psi} \frac{\partial^2 Q_{ij}^{\ell t}}{\partial \phi \, \partial \psi} , \tag{31}$$

where I_ϕ and I_ψ are index sets, not necessarily disjoint, which identify the chain matrices between reference planes i and j containing variables ϕ and ψ, respectively. Also, we define

$$\frac{\partial^2 Q_{ij}^{\ell t}}{\partial \phi \, \partial \psi}$$

as the second-order sensitivity of Q_{ij} as if variables ϕ and ψ exist only in $\mathbf{A}_\ell$ and $\mathbf{A}_t$ respectively, i.e.,

$$\frac{\partial^2 Q_{ij}^{\ell t}}{\partial \phi \, \partial \psi} \triangleq \begin{cases} \left(\mathbf{u}^{i,\ell+1}\right)^T \dfrac{\partial \mathbf{A}_\ell}{\partial \phi} \begin{bmatrix} \mathbf{v}_1^{\ell,t+1} & \mathbf{v}_2^{\ell,t+1} \end{bmatrix} \dfrac{\partial \mathbf{A}_t}{\partial \psi} \mathbf{v}^{tj} & , \text{ if } \ell > t \\[2ex] \left(\mathbf{u}^{i,\ell+1}\right)^T \dfrac{\partial^2 \mathbf{A}_\ell}{\partial \phi \partial \psi} \mathbf{v}^{\ell j} & , \text{ if } \ell = t \\[2ex] \left(\mathbf{u}^{i,t+1}\right)^T \dfrac{\partial \mathbf{A}_t}{\partial \psi} \begin{bmatrix} \mathbf{v}_1^{t,\ell+1} & \mathbf{v}_2^{t,\ell+1} \end{bmatrix} \dfrac{\partial \mathbf{A}_\ell}{\partial \phi} \mathbf{v}^{\ell j} & , \text{ if } \ell < t \end{cases} \tag{32}$$

where the unsubscripted **u** and **v** indicate the form of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$ or $\mathbf{v}_2$. Specific subscripts, as indicated in Table II, must be provided to distinguish whether Q is *A*, *B*, *C* or *D*.

From equations (29) to (32), we can see that any sensitivity analysis can be reduced to a series of sensitivity analyses as if any variable exists only in a single branch element, junction or spacing subnetwork. The series of individual sensitivity analyses in the forward-reverse analysis method are the basic forms. Therefore, using this approach, possible common factors are saved and can be used for sensitivity analysis w.r.t. other variable parameters. The overall sensitivity can be calculated systematically as a summation of basic sensitivity factors $\partial Q_{ij}^{\ell}/\partial\phi$ over those indexes ℓ, $\ell \in I_{\phi}$, and $\mathbf{A}_{\ell}$ contains the variable parameter ϕ.

Properties of the Analysis

From the definitions of A, B, C and D, we have, from (26)

$$\begin{aligned} \mathbf{y}^{i} &= \mathbf{Q}_{ij}\,\mathbf{y}^{j} \\ &= [\mathbf{p}_{ij}\ \ \mathbf{q}_{ij}]\,\mathbf{y}^{j} \\ &= \begin{bmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{bmatrix} \mathbf{y}^{j}. \end{aligned} \tag{33}$$

The following are interesting properties enjoyed by the $\mathbf{Q}$ matrix and $\mathbf{u}$, $\mathbf{v}$ vectors and can be exploited in our analysis and computations.

The $\mathbf{Q}_{ij}$, $\mathbf{Q}_{i\ell}$ and $\mathbf{Q}_{\ell j}$ matrices are related by

$$\mathbf{Q}_{ij} = \mathbf{Q}_{i\ell}\,\mathbf{Q}_{\ell j}. \tag{34}$$

For the $\mathbf{u}$ and $\mathbf{v}$ vectors, we have

$$[\mathbf{u}_1^{xi}\ \ \mathbf{u}_2^{xi}]^T\,[\mathbf{u}_1^{ij}\ \ \mathbf{u}_2^{ij}]^T = [\mathbf{u}_1^{xj}\ \ \mathbf{u}_2^{xj}]^T \tag{35}$$

and

$$[\mathbf{u}_1^{xi}\ \ \mathbf{u}_2^{xi}]^T\,[\mathbf{v}_1^{ij}\ \ \mathbf{v}_2^{ij}] = [\mathbf{u}_1^{xj}\ \ \mathbf{u}_2^{xj}]^T. \tag{36}$$

If the chain matrices between reference planes x and i are all nonsingular, then from (35) and (36) we find that

$$(\mathbf{u}^{ij})^T = \mathbf{e}^T([\mathbf{u}_1^{xi}\ \ \mathbf{u}_2^{xi}]^T)^{-1}\,[\mathbf{u}_1^{xj}\ \ \mathbf{u}_2^{xj}]^T \tag{37}$$

and

$$\mathbf{v}^{ij} = ([\mathbf{u}_1^{xi}\ \ \mathbf{u}_2^{xi}]^T)^{-1}\,[\mathbf{u}_1^{xj}\ \ \mathbf{u}_2^{xj}]^T\,\mathbf{e}\ , \tag{38}$$

where $\mathbf{e}$ can be $\mathbf{e}_1$, resulting in $\mathbf{u}_1^{ij}$ and $\mathbf{v}_1^{ij}$ or $\mathbf{e}_2$, resulting in $\mathbf{u}_2^{ij}$ and $\mathbf{v}_2^{ij}$.

Notice that, in practical networks, the condition of nonsingularity will usually be satisfied. Thus, under certain circumstances, equations (37) and (38) can be expedited to calculate the equivalent forward and reverse analysis between any two reference planes by performing only a full forward analysis. In certain cases, the matrix inversions can be avoided and computational effort can be saved.

IV. RESPONSE AND SENSITIVITY FORMULAS FOR CHANNEL OUTPUT VOLTAGE

Consider our multiplexer problem and perform forward and reverse analyses between reference planes $\sigma \equiv \sigma(k)$ and $\tau \equiv \tau(k)$, i.e., in channel k. Let $V \equiv V^k$ be the output voltage at the kth channel, as shown in Fig. 4(a). Note that, for an open circuit at the channel termination reference plane τ, we have

$$\mathbf{y}^{\tau} = \begin{bmatrix} V \\ 0 \end{bmatrix} = \mathbf{e}_1 V . \tag{39}$$

Thus, from (33), we find that the voltage and current vector at reference plane σ, namely $\mathbf{y}^{\sigma}$, can be expressed as

$$\begin{aligned} \mathbf{y}^{\sigma} &= [\mathbf{p}_{\sigma\tau} \quad \mathbf{q}_{\sigma\tau}]\, \mathbf{y}^{\tau} = [\mathbf{p}_{\sigma\tau} \quad \mathbf{q}_{\sigma\tau}]\, \mathbf{e}_1 V \\ &= \mathbf{p}_{\sigma\tau} V . \end{aligned} \tag{40}$$

Similarly, from the analysis between reference planes 2k and 1 along the main cascade (see Fig. 3), we have

$$\mathbf{y}^{2k} = \mathbf{q}_{2k,1} I_L \tag{41}$$

for a short circuit main cascade termination, where I_L is the short circuit current at reference plane 1, i.e., $\mathbf{y}^1 = \mathbf{e}_2 I_L$.

For an analysis between reference planes $2N+2$ and 1, we have, as a special case of (41),

$$\begin{aligned} V_S &= \mathbf{e}_1{}^T \mathbf{y}^{2N+2} \\ &= B_{2N+2,1} I_L, \end{aligned} \tag{42}$$

where V_S is the source voltage associated with the common port location (reference plane $2N+2$). Equation (7) indicates, for the 3-port junction at the kth section (Figs. 2 and 4),

$$\boldsymbol{\alpha}^T \mathbf{y}^{2k} = \boldsymbol{\beta}^T \mathbf{y}^{\sigma}, \tag{43}$$

where $\mathbf{y}^{2k}$ corresponds to port 2 and $\mathbf{y}^{\sigma}$ to port 3. Eliminating I_L, $\mathbf{y}^{2k}$ and $\mathbf{y}^{\sigma}$ from (40) - (43), we obtain the following formula to calculate channel output voltage responses:

$$V = \frac{\boldsymbol{\alpha}^T \mathbf{q}_{2k,1} V_S}{\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B_{2N+2,1}} \tag{44}$$

for a short-circuit main cascade termination (at reference plane 1).

Various response formulas can be derived in the same manner for different excitations and terminations. Table III gives a complete set of these various formulas. Table IV summarizes results for special 3-port junctions yielding special values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

For the short-circuit main cascade termination, using $'$ to denote $\partial/\partial\phi$, we have, differentiating (44), the following formula to calculate sensitivities of channel output voltage w.r.t. variable parameters in the multiplexer:

$$V' = \frac{(\boldsymbol{\alpha}^T \mathbf{q}_{2k,1} V_S)' - (\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B_{2N+2,1})' V}{\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B_{2N+2,1}} . \tag{45}$$

For different ϕ, appearing in different parts of the multiplexer, this formula can be simplified as shown in Table V.

Also note that appropriate sensitivity expressions, analogously to (45), exist for various cases considered in Table III.

V. THEVENIN EQUIVALENT CIRCUIT OF A MULTIPLEXER

Formulas have been derived by Bandler et al. to calculate Thevenin equivalent circuits in cascaded structures [1]. These formulas can also be applied to multiplexer analysis, i.e.,

$$V_S^j = \frac{V_S^i}{A_{ij} + Z_S^i C_{ij}} , \tag{46}$$

and

$$Z_S^j = \frac{B_{ij} + Z_S^i D_{ij}}{A_{ij} + Z_S^i C_{ij}} , \tag{47}$$

where V_S^i, Z_S^i, V_S^j and Z_S^j represent Thevenin equivalent voltages and impedances at reference planes i and j, respectively. All forward and reverse analyses in equations (46) and (47) are performed between reference planes i and j as shown in Fig. 6.

Sensitivities of V_S^j and Z_S^j w.r.t. variable parameters in the multiplexer can be evaluated by

$$(V_S^j)' = \frac{(V_S^i)' - [A'_{ij} + Z_S^i C'_{ij} + (Z_S^i)' C_{ij}] V_S^j}{A_{ij} + Z_S^i C_{ij}} \tag{48}$$

and

$$(Z_S^j)' = \frac{[1 \quad Z_S^i] \mathbf{Q}'_{ij} \begin{bmatrix} -Z_S^j \\ 1 \end{bmatrix} + (Z_S^i)' (D_{ij} - Z_S^j C_{ij})}{A_{ij} + Z_S^i C_{ij}} \,. \tag{49}$$

Formulas (46)-(49) can be used to calculate the Thevenin equivalent circuit at any reference plane in the multiplexer looking towards the common port (Fig. 6). Norton equivalent circuits at those reference planes looking away from the common port can also be found as [1]

$$Y_L^i = \frac{C_{ij} + Y_L^j D_{ij}}{A_{ij} + Y_L^j B_{ij}} \tag{50}$$

and

$$I_L^i = 0 \,, \tag{51}$$

where Y_L^i and I_L^i are the Norton equivalent admittance and current, respectively, at reference plane i and Y_L^j is the Norton equivalent admittance at reference plane j. Note that equation (51) is valid for a multiplexer having excitations only at the common port. Also, sensitivity formulas can be derived as

$$(Y_L^i)' = \frac{[-Y_L^i \quad 1] \mathbf{Q}'_{ij} \begin{bmatrix} 1 \\ Y_L^j \end{bmatrix} + (Y_L^j)' (D_{ij} - Y_L^i B_{ij})}{A_{ij} + Y_L^j B_{ij}} \tag{52}$$

and

$$(I_L^i)' = 0 \,. \tag{53}$$

Consider the Thevenin equivalent circuit at the output port of the kth channel (reference plane $j = \tau(k) + 1$), looking towards the source at the source port (reference plane $i = 2N+2$ as shown in Fig. 4(b). This equivalent circuit plays an important role in calculating the reflection coefficient at the channel output ports. Let $\tau \equiv \tau(k)$, $i = 2N+2$, $j = \tau+1$, $\mathbf{Q} \equiv \mathbf{Q}_{2N+2,\tau+1}$, $A \equiv A_{2N+2,\tau+1}$ and $B \equiv B_{2N+2,\tau+1}$. Then we have the following special cases for equations (46) to (49) of the form

$$V_S^{\tau+1} = \frac{V_S}{A}, \tag{54}$$

$$Z_S^{\tau+1} = \frac{B}{A}, \tag{55}$$

$$(V_S^{\tau+1})' = \frac{V_S' - A' V_S^{\tau+1}}{A} \tag{56}$$

and

$$(Z_S^{\tau+1})' = \frac{B' - A' Z_S^{\tau+1}}{A}. \tag{57}$$

For different variables ϕ, appearing in different parts of the multiplexer, equations (56) and (57) can be simplified as shown in Table VI.

As special cases of (50), the equivalent admittance Y_3, defined by equation (6), can be calculated as follows for the kth 3-port junction:

$$Y_3 = Y_L^{\sigma(k)} = \frac{C_{\sigma(k),\tau(k)}}{A_{\sigma(k),\tau(k)}}, \qquad k = 1, 2, \ldots, N. \tag{58}$$

Notice that $Y_L^{\tau(k)} = 0$, provided that the kth channel load is not a short circuit. $Y_L^{\sigma(k)}$ is the Norton equivalent admittance of the kth channel looking into the kth channel-junction from connection reference plane $\sigma(k)$.

Similarly, the equivalent admittance Y_2 defined by (15) can be calculated, for the kth 3-port junction, as

$$Y_2 = Y_L^{2k} = \frac{D_{2k,1}}{B_{2k,1}}, \qquad k = 1,2,\ldots N. \tag{59}$$

Notice that $Y_L^1 \to \infty$ for our short-circuited main cascade termination. Y_L^{2k} is the Norton equivalent admittance at kth junction output port (at main cascade) looking towards the short-circuited main cascade termination.

VI. REFLECTION COEFFICIENT, RETURN LOSS AND THEIR SENSITIVITIES

Consider the circuit of Fig. 7(a). The reflection coefficient can be defined as

$$\rho \triangleq \frac{Z_{in} - R_g}{Z_{in} + R_g} . \tag{60}$$

The return loss for the same circuit is defined as

$$L_R \triangleq -20 \log_{10} |\rho| . \tag{61}$$

At the common port of our multiplexer network of Fig. 1 the term $Z_{in} + R_g$ can be evaluated by $1/Y_L^{2N+2}$, where Y_L^{2N+2} is the Norton equivalent admittance at reference plane 2N+2 and is calculated by (50). Notice that for a short-circuited termination of the main cascade at reference plane 1, we have, similar to (59),

$$Y_L^{2N+2} = \frac{D_{2N+2,1}}{B_{2N+2,1}} . \tag{62}$$

Therefore, the common port reflection coefficient and return loss can be calculated as

$$\rho^0 = \frac{\dfrac{B_{2N+2,1}}{D_{2N+2,1}} - 2R_S}{\dfrac{B_{2N+2,1}}{D_{2N+2,1}}}$$

$$= 1 - \frac{2R_S D_{2N+2,1}}{B_{2N+2,1}} \tag{63}$$

and

$$L_R^0 = -20 \log_{10} \left| 1 - \frac{2R_S D_{2N+2,1}}{B_{2N+2,1}} \right| . \tag{64}$$

The sensitivities of ρ^0 and L_R^0 w.r.t. variable parameters, including frequency, can be calculated by

$$(\rho^0)' = 2\,\frac{R_S\, D_{2N+2,1}\, B'_{2N+2,1} - (R'_S\, D_{2N+2,1} + R_S\, D'_{2N+2,1})B_{2N+2,1}}{(B_{2N+2,1})^2} \tag{65}$$

and

$$(L_R^0)' = \frac{-20}{(\ell n\, 10)|\rho^0|^2}\ \mathrm{Re}\,\{(\rho^0)^*\,(\rho^0)'\}\,, \tag{66}$$

where * signifies the complex conjugate.

Using equation (55), the reflection coefficient and the return loss at the kth channel output port (reference plane $\tau(k)+1$) can be computed by

$$\rho^k = \frac{Z_S^{\tau+1} - R_L^k}{Z_S^{\tau+1} + R_L^k}$$

$$= \frac{B - AR_L^k}{B + AR_L^k} \tag{67}$$

and

$$L_R^k = -\,20\,\log_{10}\left|\frac{B - AR_L^k}{B + AR_L^k}\right|\quad , \tag{68}$$

where $\tau \equiv \tau(k)$, $A \equiv A_{2N+2,\,\tau+1}$ and $B \equiv B_{2N+2,\,\tau+1}$.

The sensitivities of ρ^k and $L_R{}^k$ w.r.t. variable parameters, including frequency, can be evaluated by

$$(\rho^k)' = \frac{(1-\rho^k)\,B' - (1+\rho^k)[\,A' R_L^k + A(R_L^k)']}{B + AR_L^k} \tag{69}$$

and

$$(L_R^k)' = \frac{-20}{(\ell n\, 10)|\,\rho^k|^2}\ \mathrm{Re}\,\{(\rho^k)^*\,(\rho^k)'\}\,. \tag{70}$$

VII. INSERTION LOSS AND GAIN SLOPE COMPUTATIONS

Consider the circuit of Fig. 7(b), for which the insertion loss is given by [6]

$$L_I = -\ 20 \log_{10} |V| - 20 \log_{10}\left(\frac{R_g + R_L}{R_L}\right), \tag{71}$$

where V is the output voltage for a unit voltage excitation, R_g and R_L are the source and load resistances, respectively.

Using equation (44), we can obtain the insertion loss for the kth channel of our multiplexer from

$$L_I^k = -\ 20 \log_{10} \left| \frac{\boldsymbol{\alpha}^T \mathbf{q}_{2k,1}}{\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B_{2N+2,1}} \right|$$

$$-\ 20 \log_{10}\left(\frac{R_S + R_L}{R_L}\right), \tag{72}$$

where $\sigma \equiv \sigma(k)$, $\tau \equiv \tau(k)$, $\boldsymbol{\alpha} \equiv \boldsymbol{\alpha}_{2k}$, $\boldsymbol{\beta} \equiv \boldsymbol{\beta}_{2k}$ and $R_L = R_L{}^k$.

The gain slope of a network is defined as the first-order derivative of the insertion loss w.r.t. frequency. In our multiplexer network, the gain slope for the kth channel can be calculated, using $d \equiv \boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B_{2N+2,1}$, by

$$S_G^k = \frac{\partial}{\partial\omega} L_I^k$$

$$= \frac{-\ 20}{\ell n\, 10} \left| \frac{d}{\boldsymbol{\alpha}^T \mathbf{q}_{2k,1}} \right|^2 \mathrm{Re}\left\{ \left(\frac{\boldsymbol{\alpha}^T \mathbf{q}_{2k,1}}{d} \right)^* \right.$$

$$\left[\left(\boldsymbol{\alpha}^T \mathbf{q}'_{2k,1} + \mathbf{q}^T_{2k,1} \boldsymbol{\alpha}' \right) d - \boldsymbol{\alpha}^T \mathbf{q}_{2k,1} \left(\mathbf{p}^T_{\sigma\tau} \boldsymbol{\beta}' B_{2N+2,1} \right. \right.$$

$$\left. \left. \left. + \ \boldsymbol{\beta}^T \mathbf{p}'_{\sigma\tau} B_{2N+2,1} + \boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B'_{2N+2,1} \right) \right] / d^2 \right\}. \tag{73}$$

The gain slope can also be calculated directly from sensitivities of channel output voltages w.r.t. frequency as

$$S_G^k = \frac{\partial}{\partial\omega}\left(-20\log_{10}|V|\right) =$$

$$= -\frac{20}{\ell n\, 10\, |V|^2}\,\mathrm{Re}\left\{ V^* \frac{\partial V}{\partial\omega}\right\}\,, \tag{74}$$

where $V \equiv V^k$ is the kth channel output voltage.

Using Table V to substitute $\partial V/\partial\omega$ in equation (74), we also arrive at equation (73).

VIII. GROUP DELAY AND ITS SENSITIVITIES

The exact group delay can be defined as [6,7]

$$T_G \triangleq -\mathrm{Im}\left\{\frac{1}{V}\frac{\partial V}{\partial\omega}\right\}\,, \tag{75}$$

where V is the output voltage. In multiplexer networks the exact group delay from the source port to the kth channel output port, namely $T_G{}^k$ can be calculated, using Table V as

$$T_G = -\mathrm{Im}\left\{\frac{(\boldsymbol{\alpha}^T\mathbf{q}'_{2k,1} + \mathbf{q}^T_{2k,1}\boldsymbol{\alpha}')}{\boldsymbol{\alpha}^T\mathbf{q}_{2k,1}} - \frac{(\boldsymbol{\beta}^T\mathbf{p}'_{\sigma\tau} + \mathbf{p}^T_{\sigma\tau}\boldsymbol{\beta}')}{\boldsymbol{\beta}^T\mathbf{p}_{\sigma\tau}} - \frac{B'_{2N+2,1}}{B_{2N+2,1}}\right\}\,, \tag{76}$$

where $T_G \equiv T_G{}^k$, $\boldsymbol{\beta} \equiv \boldsymbol{\beta}_{2k}$, $\boldsymbol{\alpha} \equiv \boldsymbol{\alpha}_{2k}$, $\sigma \equiv \sigma(k)$, $\tau \equiv \tau(k)$ and $\partial/\partial\omega$ is denoted as $'$.

The exact group delay sensitivity for the multiplexer network can be calculated using the formula

$$\frac{\partial T_G}{\partial\phi} = -\mathrm{Im}\Bigg\{\Bigg(\boldsymbol{\alpha}^T\frac{\partial^2\mathbf{q}_{2k,1}}{\partial\phi\partial\omega} + \frac{\partial\boldsymbol{\alpha}^T}{\partial\phi}\frac{\partial\mathbf{q}_{2k,1}}{\partial\omega} + \frac{\partial\boldsymbol{\alpha}^T}{\partial\omega}\frac{\partial\mathbf{q}_{2k,1}}{\partial\phi}$$

$$+ \frac{\partial^2\boldsymbol{\alpha}^T}{\partial\phi\partial\omega}\mathbf{q}_{2k,1}\Bigg)\Big/\left(\boldsymbol{\alpha}^T\mathbf{q}_{2k,1}\right)$$

$$-\left(\boldsymbol{\alpha}^T\frac{\partial\mathbf{q}_{2k,1}}{\partial\omega} + \frac{\partial\boldsymbol{\alpha}^T}{\partial\omega}\mathbf{q}_{2k,1}\right)\left(\boldsymbol{\alpha}^T\frac{\partial\mathbf{q}_{2k,1}}{\partial\phi} + \frac{\partial\boldsymbol{\alpha}^T}{\partial\phi}\mathbf{q}_{2k,1}\right)$$

$$\Big/\left(\boldsymbol{\alpha}^T\mathbf{q}_{2k,1}\right)^2$$

$$- \left(\boldsymbol{\beta}^T \frac{\partial^2 \mathbf{p}_{\sigma\tau}}{\partial\phi\partial\omega} + \frac{\partial \boldsymbol{\beta}^T}{\partial\phi} \frac{\partial \mathbf{p}_{\sigma\tau}}{\partial\omega} + \frac{\partial \boldsymbol{\beta}^T}{\partial\omega} \frac{\partial \mathbf{p}_{\sigma\tau}}{\partial\phi} \right.$$

$$\left. + \frac{\partial^2 \boldsymbol{\beta}^T}{\partial\phi\partial\omega} \mathbf{p}_{\sigma\tau} \right) / \left(\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} \right)$$

$$+ \left(\boldsymbol{\beta}^T \frac{\partial \mathbf{p}_{\sigma\tau}}{\partial\omega} + \frac{\partial \boldsymbol{\beta}^T}{\partial\omega} \mathbf{p}_{\sigma\tau} \right) \left(\boldsymbol{\beta}^T \frac{\partial \mathbf{p}_{\sigma\tau}}{\partial\phi} + \frac{\partial \boldsymbol{\beta}^T}{\partial\phi} \mathbf{p}_{\sigma\tau} \right)$$

$$/ \left(\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} \right)^2 - \frac{\partial^2}{\partial\phi\partial\omega} \left(B_{2N+2,1} \right) / B_{2N+2,1}$$

$$\left. + \frac{\partial B_{2N+2,1}}{\partial\phi} \frac{\partial B_{2N+2,1}}{\partial\omega} / B^2_{2N+2,1} \right\} . \tag{77}$$

For different ϕ, appearing in different parts of the multiplexer, this formula can be simplified as shown in Table VII.

IX. TYPICAL EXAMPLES OF SUBNETWORKS AND THEIR SENSITIVITIES

As discussed in the previous sections, the multiplexer simulation requires, as a fundamental step, all subnetworks such as junctions, spacings and channel elements to be represented by 2-port chain matrices $\mathbf{A}_i$, $i \in I$, where I has been defined in (22), and $\mathbf{D}_{2k}$, $k = 1, 2, \ldots, N$. The basic sensitivity analyses are expressed in the form of $\mathbf{u}^T(\partial\mathbf{A}/\partial\phi)\mathbf{v}$ for first-order sensitivities or the form of (32) for second-order sensitivities. These basic sensitivity analyses are formulated as if any variable exists only in a single subnetwork. This section discusses typical examples of subnetworks, namely, junctions, spacings and channels, their chain matrix representations, first-order and second-order sensitivities. Tables VIII, IX and X present complete summaries of these results.

Series and shunt inductors and capacitors as well as ladder section subnetworks are considered first. See Fig. 8.

Second, we deal with the series and parallel 3-port junctions shown in Fig. 9, where Y_a, Y_b and Y_c or Z_a, Z_b and Z_c are non-ideal reactive junction admittances or impedances. Formulas introduced in Section II are applied to appropriate 2-port equivalent chain matrix representations of these junctions. Consider the series 3-port junction of Fig. 9(a) as an example. We can obtain the chain matrix between port 1 and port 2 by terminating port 3, where the termination is represented by Y_3. In this example, the **H** matrix defined by equation (2) is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 \\ Y_a + Y_b & -1 & Y_a \\ -Y_b & 1 & Y_c \end{bmatrix} . \tag{78}$$

Applying (5), we have the 2-port representation as

$$\mathbf{A} = \begin{bmatrix} 1+\dfrac{Y_b}{Y_3+Y_c} & \dfrac{1}{Y_3+Y_c} \\ Y_a+Y_b+\dfrac{Y_a Y_b}{Y_3+Y_c} & 1+\dfrac{Y_a}{Y_3+Y_c} \end{bmatrix} . \tag{79}$$

The waveguide subnetwork (Fig. 10) is considered next.

Finally, the unterminated lossless filter shown in Fig. 11 is handled by solving an nxn system of linear equations with the coefficient matrix **Z** defined by

$$\mathbf{Z} = s\mathbf{1} + j\mathbf{M} , \tag{80}$$

where **M** is an nxn real, symmetrical matrix whose (a,b)th element, namely M_{ab} represents the coupling coefficient between ,e.g., the ath and the bth cavities for a waveguide cavity filter [8]. Also

$$s \triangleq j\frac{\omega_0}{\Delta\omega}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right), \tag{81}$$

where ω_0 and $\Delta\omega$ are the center frequency and bandwidth parameter of the filter. The variables n_1 and n_2 are the input and output transformer ratios, respectively. The sets of linear equations are solved for the simulation and sensitivity analysis with right-hand-sides as unit n-vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_a$ and $\mathbf{e}_b$, respectively, where the a and b indices identify the (a,b)th element of **M**, namely M_{ab}, as the variable parameter. Correspondingly, p_1, q_1, p_n, q_n, p_a, q_a, p_b and q_b are the 1st, nth, ath and bth components of the vectors **p** and **q**, respectively. Note that vectors **p** and **q** in Tables VIII, IX and X are defined only for the unterminated filter simulation and sensitivity analysis as solution vectors of the linear equations indicated in these tables. They should not be confused with similar symbols defined in (27).

Suppose we construct a multiplexer using subnetwork models shown in Figs. 8, 9, 10 and 11. Then all chain matrix representations required, namely, $\mathbf{A}_i$, $i \in I$, and $\mathbf{D}_{2k}$, $k = 1, 2, ..., N$, as indicated in Figs. 3 and 4(a), can be obtained from Table VIII. Terms of $\partial\mathbf{A}_i/\partial\phi$ and $\partial\mathbf{D}_{2k}/\partial\phi$ required in the basic sensitivity analysis can be directly taken from Table IX, while $\partial^2\mathbf{D}_{2k}/(\partial\phi\partial\psi)$ or $\partial^2\mathbf{A}_i/(\partial\phi\partial\psi)$, required in (32) are provided in Table X. Performing the appropriate forward and reverse analyses to obtain the necessary **u** and **v** vectors and formulating Q and Q′ according to Table II, all responses and their sensitivities are readily computed by implementing the relevant formulas given in previous sections.

X. CONCLUSIONS

A general theory has been presented for the simulation, sensitivity analysis and design of microwave multiplexers. Thevenin equivalents at any reference plane and various frequency responses as well as their sensitivities are expressed analytically and calculated systematically. A computer algorithm can be developed to implement this theory and to compute all responses and sensitivities efficiently using the results of forward and reverse analysis. Various properties of the forward and reverse analysis approach can be exploited to save computational effort.

Further design procedures, such as centering, tolerancing and tuning of multiplexers can be developed from this theory. It is also possible to develop the forward and reverse analysis approach for a more general electrical network structure.

APPENDIX

CONVERSIONS BETWEEN DIFFERENT MATRIX REPRESENTATIONS OF A N-PORT NETWORK

Under certain circumstances, it is desirable to convert matrix representations of a network from one form to other forms (e.g., the hybrid matrices between case 1 and case 2 in Section II.) This can be achieved in the following way.

Consider the set of linear equations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} . \tag{A1}$$

Exchanging x_ℓ and y_k, we have

$$\mathbf{B} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{\ell-1} \\ y_k \\ x_{\ell+1} \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{k-1} \\ x_\ell \\ y_{k+1} \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} , \tag{A2}$$

where

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \tag{A3}$$

and

$$b_{ij} = \begin{cases} 1/a_{k\ell}, & i=k, \quad j=\ell \\ a_{ij}/a_{k\ell}, & i\neq k, \quad j=\ell \\ -a_{ij}/a_{k\ell}, & i=k, \quad j\neq\ell \\ a_{ij} - a_{i\ell}\,a_{kj}/a_{k\ell}, & i\neq k, \quad j\neq\ell \end{cases} \tag{A4}$$

Matrix **B** in (A2) does not exist when $a_{k\ell} = 0$. This corresponds to situations in which conversion between hybrid matrices is not physically possible, e.g., the hybrid matrix for a pure series connection of the form given by equation (1) cannot be converted to the hybrid matrix of the form given by equation (16).

The conversions among various hybrid matrices, admittance matrix and impedance matrix of an n-port network can be readily carried out by using equation (A4) with necessary column permutations, as the case may be.

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TABLE I

PRINCIPAL CONCEPTS INVOLVED IN THE MULTIPLEXER SIMULATION

Concept	Definition	Initial Reference Plane	Terminal Reference Plane
Basic iteration	$\mathbf{y}^{i+1} = \mathbf{A}_i \mathbf{y}^i$	–	–
	$\mathbf{y}^{2k+1} = \mathbf{D}_{2k} \mathbf{y}^{\sigma(k)}$	–	–
Forward operation	$(\mathbf{u}^{x,i+1})^T \mathbf{A}_i = (\mathbf{u}^{xi})^T$	x	i
	$(\mathbf{u}^{x,2k+1})^T \mathbf{D}_{2k} = (\mathbf{u}^{x\sigma(k)})^T$	x	σ(k)
Reverse operation	$\mathbf{v}^{i+1,x} = \mathbf{A}_i \mathbf{v}^{ix}$	x	i+1
	$\mathbf{v}^{2k+1,x} = \mathbf{D}_{2k} \mathbf{v}^{\sigma(k)x}$	x	2k+1
Voltage selector	$\mathbf{u}_1{}^{ii} = \mathbf{e}_1$	i	i
Current selector	$\mathbf{u}_2{}^{ii} = \mathbf{e}_2$	i	i
O.C. termination	$\mathbf{v}_1{}^{\tau(k)\tau(k)} = \mathbf{e}_1$	τ(k)	τ(k)
	$\mathbf{v}_1{}^{11} = \mathbf{e}_1$	1	1
S.C. termination	$\mathbf{v}_2{}^{\tau(k)\tau(k)} = \mathbf{e}_2$	τ(k)	τ(k)
	$\mathbf{v}_2{}^{11} = \mathbf{e}_2$	1	1

TABLE II

NOTATION AND DEFINITION OF THE **Q** FORMS

Factor	Identification	Initialization Plane		
		Forward	Reverse	
$(\mathbf{u}^{i,\ell+1})^T(*)\,\mathbf{v}^{\ell j}$	$(+)_{ij}$	i	j	†
$(\mathbf{u}^{i,2k+1})^T(\square)\,\mathbf{v}^{\sigma(k)j}$	$(+)_{ij}$	i	j	††

$*$ denotes $\mathbf{A}_\ell$, $\delta\mathbf{A}_\ell$, $\partial\mathbf{A}_\ell/\partial\phi$ or $\Delta\mathbf{A}_\ell$

$\square$ denotes $\mathbf{D}_{2k}$, $\delta\mathbf{D}_{2k}$, $\partial\mathbf{D}_{2k}/\partial\phi$ or $\Delta\mathbf{D}_{2k}$

$+$ denotes Q, δQ^ℓ, $\partial Q^\ell/\partial\phi$ or ΔQ^ℓ

where $Q = A$ if $\mathbf{u} = \mathbf{u}_1$ and $\mathbf{v} = \mathbf{v}_1$

$Q = B$ if $\mathbf{u} = \mathbf{u}_1$ and $\mathbf{v} = \mathbf{v}_2$

$Q = C$ if $\mathbf{u} = \mathbf{u}_2$ and $\mathbf{v} = \mathbf{v}_1$

$Q = D$ if $\mathbf{u} = \mathbf{u}_2$ and $\mathbf{v} = \mathbf{v}_2$

† either $j \notin [\tau(k), \sigma(k)]$ or $\ell \neq 2k$, $\ell \neq \sigma(k)$

†† $j \in [\tau(k), \sigma(k)]$, $\ell = 2k$

δ denotes first-order change

$\partial/\partial\phi$ denotes partial derivative w.r.t. ϕ

Δ denotes large change

TABLE III

BASIC CHANNEL RESPONSE FORMULAS

$V \equiv V^k, I \equiv I^k, \sigma \equiv \sigma(k), \tau \equiv \tau(k)$

Excitation at Common Port	Channel Output Response	S.C. Main Cascade Termination	O.C. Main Cascade Termination
Voltage source (V_S)	V	$\dfrac{\boldsymbol{\alpha}^T \mathbf{q}_{2k,1} V_S}{\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B_{2N+2,1}}$	$\dfrac{\boldsymbol{\alpha}^T \mathbf{p}_{2k,1} V_S}{\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} A_{2N+2,1}}$
Voltage source (V_S)	I	$\dfrac{\boldsymbol{\alpha}^T \mathbf{q}_{2k,1} V_S}{\boldsymbol{\beta}^T \mathbf{q}_{\sigma\tau} B_{2N+2,1}}$	$\dfrac{\boldsymbol{\alpha}^T \mathbf{p}_{2k,1} V_S}{\boldsymbol{\beta}^T \mathbf{q}_{\sigma\tau} A_{2N+2,1}}$
Current source (I_S)	V	$\dfrac{\boldsymbol{\alpha}^T \mathbf{q}_{2k,1} I_S}{\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} D_{2N+2,1}}$	$\dfrac{\boldsymbol{\alpha}^T \mathbf{p}_{2k,1} I_S}{\boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} C_{2N+2,1}}$
Current source (I_S)	I	$\dfrac{\boldsymbol{\alpha}^T \mathbf{q}_{2k,1} I_S}{\boldsymbol{\beta}^T \mathbf{q}_{\sigma\tau} D_{2N+2,1}}$	$\dfrac{\boldsymbol{\alpha}^T \mathbf{p}_{2k,1} I_S}{\boldsymbol{\beta}^T \mathbf{q}_{\sigma\tau} C_{2N+2,1}}$

TABLE IV

CHANNEL OUTPUT VOLTAGE RESPONSE FORMULAS FOR SPECIAL

3-PORT JUNCTIONS

$V \equiv V^k$, $\sigma \equiv \sigma(k)$ and $\tau \equiv \tau(k)$

Main Cascade Connection	α and β	Cascade Termination	Response Formula (V)
pure series	$\alpha = \beta = e_2$	O.C.	$\dfrac{C_{2k,1}\, V_S}{C_{\sigma\tau}\, A_{2N+2,1}}$
		S.C.	$\dfrac{D_{2k,1}\, V_S}{C_{\sigma\tau}\, B_{2N+2,1}}$
pure parallel	$\alpha = \beta = e_1$	O.C.	$\dfrac{A_{2k,1}\, V_S}{A_{\sigma\tau}\, A_{2N+2,1}}$
		S.C.	$\dfrac{B_{2k,1}\, V_S}{A_{\sigma\tau}\, B_{2N+2,1}}$

TABLE V

PARTIAL DERIVATIVE FORMULAS FOR INDIVIDUAL CHANNEL OUTPUT VOLTAGES FOR DIFFERENT VARIABLES AND SHORT CIRCUIT MAIN CASCADE TERMINATION

$$V \equiv V^k, \sigma \equiv \sigma(k), \tau \equiv \tau(k), d \equiv \boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B_{2N+2,1}, \phi \in \mathbf{A}_\ell$$

Location of Variable ϕ	Position of Variable	Partial Derivative of kth Channel Output Voltage
rth section	$r<k$	$\dfrac{\boldsymbol{\alpha}^T \mathbf{q}'_{2k,1} V_S - \boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B'_{2N+2,1} V}{d}$
rth spacing	$r=k$	$\dfrac{\boldsymbol{\alpha}^T \mathbf{q}'_{2k,1} V_S - \boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B'_{2N+2,1} V}{d}$
rth channel	$r=k$	$\dfrac{-\boldsymbol{\beta}^T(\mathbf{p}'_{\sigma\tau} B_{2N+2,1} + \mathbf{p}_{\sigma\tau} B'_{2N+2,1})\ V}{d}$
rth junction	$r=k$	$\dfrac{\mathbf{q}^T_{2k,1} \boldsymbol{\alpha}' V_S - \mathbf{p}^T_{\sigma\tau}(\boldsymbol{\beta}'\ B_{2N+2,1} + \boldsymbol{\beta} B'_{2N+2,1}) V}{d}$
rth section	$r>k$	$\dfrac{-\ \boldsymbol{\beta}^T \mathbf{p}_{\sigma\tau} B'_{2N+2,1} V}{d}$
source voltage	common port	$\dfrac{\boldsymbol{\alpha}^T \mathbf{q}_{2k,1}}{d}$

$(*)'_{i\,1} \equiv [(*)^{\ell}_{i\,1}]'$, $i = 2N+2, 2k$, $(*)'_{\sigma\tau} \equiv [(*)^{\ell}_{\sigma\tau}]'$ and $*$ denotes $\mathbf{p}$, $\mathbf{q}$ or B

TABLE V (continued)

Location of Variable ϕ	Position of Variable	Partial Derivative of kth Channel Output Voltage
Frequency ω	everywhere	$\{\boldsymbol{\alpha}^T \mathbf{q}'_{2k,1} V_S + \mathbf{q}^T_{2k,1}(\boldsymbol{\alpha}' V_S + \boldsymbol{\alpha} V'_S)$ $- \boldsymbol{\beta}^T \mathbf{p}'_{\sigma\tau} B_{2N+2,1} V$ $- \mathbf{p}^T_{\sigma\tau}(\boldsymbol{\beta}' B_{2N+2,1} + \boldsymbol{\beta} B'_{2N+2,1}) V\}/d$

$$(*)'_{i1} \equiv \sum_{\ell=1}^{i-1} [(*)^{\ell}_{i1}]', \quad i = 2N+2, 2k, \; (*)'_{\sigma\tau} \equiv \sum_{\ell=\tau}^{\sigma-1} [(*)^{\ell}_{\sigma\tau}]' \text{ and } * \text{ denotes } \mathbf{p}, \mathbf{q} \text{ or } B$$

TABLE VI

PARTIAL DERIVATIVE FORMULAS OF THEVENIN EQUIVALENT CIRCUITS FOR DIFFERENT VARIABLES AND SHORT CIRCUIT MAIN CASCADE TERMINATION

$$A \equiv A_{2N+2,\,\tau(k)+1},\ B \equiv B_{2N+2,\,\tau(k)+1},\ V \equiv V_S^{\tau(k)+1},\ Z = Z_S^{\tau(k)+1}$$

Variable ϕ	Thevenin Voltage (a) Thevenin Impedance (b)	Partial Derivative Formula
$\phi \in \mathbf{A}_\ell, \mathbf{D}_\ell$	(a)	$\dfrac{-(A^\ell)' V}{A}$
	(b)	$\dfrac{(B^\ell)' - Z(A^\ell)'}{A}$
$\phi = \omega$ Frequency	(a)	$\dfrac{V_S' - \left[\sum_{\ell=2k}^{2N+1} (A^\ell)' + \sum_{\ell=\tau(k)+1}^{\sigma(k)-1} (A^\ell)'\right] V}{A}$
	(b)	$\dfrac{\sum_{\ell=2k}^{2N+1} [(B^\ell)' - Z(A^\ell)'] + \sum_{\ell=\tau(k)+1}^{\sigma(k)-1} [(B^\ell)' - Z(A^\ell)']}{A}$
$\phi = V_S$ Source voltage	(a)	$\dfrac{1}{A}$
	(b)	0

TABLE VII

PARTIAL DERIVATIVE FORMULAS FOR INDIVIDUAL CHANNEL GROUP DELAYS FOR DIFFERENT VARIABLES AND SHORT CIRCUIT MAIN CASCADE TERMINATION

$$\mathbf{q} \equiv \mathbf{q}_{2k,1}\,,\ \mathbf{p} \equiv \mathbf{p}_{\sigma(k),\tau(k)}\,,\ \boldsymbol{\alpha} \equiv \boldsymbol{\alpha}_{2k}\,,\ \boldsymbol{\beta} \equiv \boldsymbol{\beta}_{2k}$$

$$B \equiv B_{2N+2,1}\,,\ (\dagger) \text{ denotes } \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{p}, \mathbf{q} \text{ or } B,$$

$$(\dagger)_\phi \equiv \frac{\partial(\dagger)}{\partial\phi}\,,\ (\dagger)_\omega \equiv \frac{\partial(\dagger)}{\partial\omega}\,,\ (\dagger)_{\phi\omega} \equiv \frac{\partial^2(\dagger)}{\partial\phi\partial\omega}\,,\ \phi \in \mathbf{A}_\ell$$

Location of Variable ϕ	Position of Variable	Partial Derivative of kth Channel Group Delay
rth section	$r<k$	$-\,\mathrm{Im}\left\{ \dfrac{[\boldsymbol{\alpha}^T(\mathbf{q}_{\phi\omega}\mathbf{q}^T - \mathbf{q}_\omega \mathbf{q}_\phi^T) + \boldsymbol{\alpha}_\omega^T(\mathbf{q}_\phi \mathbf{q}^T - \mathbf{q}\,\mathbf{q}_\phi^T)]\,\boldsymbol{\alpha}}{(\boldsymbol{\alpha}^T\mathbf{q})^2} - \dfrac{B_{\phi\omega}B - B_\omega B_\phi}{(B)^2} \right\}$
rth spacing	$r=k$	$-\,\mathrm{Im}\left\{ \dfrac{[\boldsymbol{\alpha}^T(\mathbf{q}_{\phi\omega}\mathbf{q}^T - \mathbf{q}_\omega \mathbf{q}_\phi^T) + \boldsymbol{\alpha}_\omega^T(\mathbf{q}_\phi \mathbf{q}^T - \mathbf{q}\,\mathbf{q}_\phi^T)]\,\boldsymbol{\alpha}}{(\boldsymbol{\alpha}^T\mathbf{q})^2} - \dfrac{B_{\phi\omega}B - B_\omega B_\phi}{(B)^2} \right\}$
rth channel	$r=k$	$\mathrm{Im}\left\{ \dfrac{[\boldsymbol{\beta}^T(\mathbf{p}_{\phi\omega}\mathbf{p}^T - \mathbf{p}_\omega \mathbf{p}_\phi^T) + \boldsymbol{\beta}_\omega^T(\mathbf{p}_\phi \mathbf{p}^T - \mathbf{p}\,\mathbf{p}_\phi^T)]\,\boldsymbol{\beta}}{(\boldsymbol{\beta}^T\mathbf{p})^2} + \dfrac{B_{\phi\omega}B - B_\omega B_\phi}{(B)^2} \right\}$

TABLE VII (continued)

Location of Variable ϕ	Position of Variable	Partial Derivative of kth Channel Group Delay
rth junction	$r=k$	$-\mathrm{Im}\left\{ \frac{[\mathbf{q}^T(\boldsymbol{\alpha}_{\phi\omega}\boldsymbol{\alpha}^T - \boldsymbol{\alpha}_{\omega}\boldsymbol{\alpha}_{\phi}^T) + \mathbf{q}_{\omega}^T(\boldsymbol{\alpha}_{\phi}\boldsymbol{\alpha}^T - \boldsymbol{\alpha}\boldsymbol{\alpha}_{\phi}^T)]\mathbf{q}}{(\boldsymbol{\alpha}^T\mathbf{q})^2} - \frac{[\mathbf{p}^T(\boldsymbol{\beta}_{\phi\omega}\boldsymbol{\beta}^T - \boldsymbol{\beta}_{\omega}\boldsymbol{\beta}_{\phi}^T) + \mathbf{p}_{\omega}^T(\boldsymbol{\beta}_{\phi}\boldsymbol{\beta}^T - \boldsymbol{\beta}\boldsymbol{\beta}_{\phi}^T)]\mathbf{p}}{(\boldsymbol{\beta}^T\mathbf{p})^2} - \frac{B_{\phi\omega}B - B_{\omega}B_{\phi}}{(B)^2} \right\}$
rth section	$r>k$	$\mathrm{Im}\left\{ \frac{B_{\omega\phi}B - B_{\omega}B_{\phi}}{(B)^2} \right\}$

$$(\dagger)_{\phi} = \frac{\partial(\dagger)^{\ell}}{\partial\phi}$$

$$(\dagger)_{\omega} = \sum_{i=i_2}^{i_1-1} \frac{\partial(\dagger)^{i}}{\partial\omega}$$

$$(\dagger)_{\omega\phi} = \sum_{i=i_2}^{i_1-1} \frac{\partial^2(\dagger)^{i\ell}}{\partial\omega\partial\phi}$$

i_1 and i_2 are initial reference planes for forward and reverse analysis of $(\dagger)$, respectively, and $(\dagger)$ denotes B, $\mathbf{p}$ or $\mathbf{q}$

TABLE VIII

EXAMPLES OF SUBNETWORKS AND CHAIN MATRIX DESCRIPTIONS

Subnetwork	Configuration	Chain Matrix Expression	Chain Matrix Notation
series inductor	Fig. 8(a)	$\begin{bmatrix} 1 & j\omega L \\ 0 & 1 \end{bmatrix}$	**A**
shunt inductor	Fig. 8(b)	$\begin{bmatrix} 1 & 0 \\ 1/(j\omega L) & 1 \end{bmatrix}$	**A**
series capacitor	Fig. 8(c)	$\begin{bmatrix} 1 & 1/(j\omega C) \\ 0 & 1 \end{bmatrix}$	**A**
shunt capacitor	Fig. 8(d)	$\begin{bmatrix} 1 & 0 \\ j\omega C & 1 \end{bmatrix}$	**A**
one section ladder	Fig. 8(e)	$\begin{bmatrix} 1+YZ & Z \\ Y & 1 \end{bmatrix}$	**A**
series 3-port junction (terminatng port 3)	Fig. 9(a)	$\begin{bmatrix} 1+\dfrac{Y_b}{Y_c+Y_3} & \dfrac{1}{Y_c+Y_3} \\ Y_a+Y_b+\dfrac{Y_aY_b}{Y_c+Y_3} & 1+\dfrac{Y_a}{Y_c+Y_3} \end{bmatrix}$	**A**

TABLE VIII (continued)

Subnetwork	Configuration	Chain Matrix Expression	Chain Matrix Notation
series 3-port junction (terminating port 2)	Fig. 9(b)	similar to terminating port 3, but interchange $Y_b \leftrightarrow Y_c$ change $Y_3 \rightarrow Y_2$	**D**
parallel 3-port junction (terminatng port 3)	Fig. 9(c)	$\begin{bmatrix} 1+\dfrac{Z_a}{Z_c+Z_3} & Z_a+Z_b+\dfrac{Z_aZ_b}{Z_c+Z_3} \\ \dfrac{1}{Z_c+Z_3} & 1+\dfrac{Z_b}{Z_c+Z_3} \end{bmatrix}$	**A**
parallel 3-port junction (terminating port 2)	Fig. 9(d)	similar to terminating port 3, but interchange $Z_b \leftrightarrow Z_c$ change $Z_3 \rightarrow Z_2$	**D**
series 3-port junction (terminating port 3) ($Y_a = Y_b$)	Fig. 9(e)	similar to the case $Y_a \neq Y_b$, but change $Y_b \rightarrow Y_a$	**A**
waveguide ($\theta = \beta\ell$)	Fig. 10	$\begin{bmatrix} \cos\theta & jZ\sin\theta \\ j\sin\theta/Z & \cos\theta \end{bmatrix}$	**A**
unterminated filter ($\mathbf{Z} = s\mathbf{1} + j\,\mathbf{M}$)	Fig. 11	$\begin{bmatrix} -\dfrac{n_2}{n_1}\dfrac{q_n}{q_1} & -\dfrac{1}{n_1n_2q_1} \\ n_1n_2\dfrac{q_1^2-p_1q_n}{q_1} & -\dfrac{n_1}{n_2}\dfrac{p_1}{q_1} \end{bmatrix}$ where $\mathbf{Z}\,\mathbf{p} = \mathbf{e}_1$, $\mathbf{Z}\,\mathbf{q} = \mathbf{e}_n$	**A**

TABLE IX

FIRST-ORDER SENSITIVITIES OF THE SUBNETWORKS IN TABLE VIII

Subnetwork	Identification	Partial Derivative of the Chain Matrix
series inductor	$\partial\mathbf{A}/\partial L$	$\begin{bmatrix} 0 & j\omega \\ 0 & 0 \end{bmatrix}$
	$\partial\mathbf{A}/\partial\omega$	$\begin{bmatrix} 0 & jL \\ 0 & 0 \end{bmatrix}$
shunt inductor	$\partial\mathbf{A}/\partial L$	$\begin{bmatrix} 0 & 0 \\ j/(\omega L^2) & 0 \end{bmatrix}$
	$\partial\mathbf{A}/\partial\omega$	$\begin{bmatrix} 0 & 0 \\ j/(\omega^2 L) & 0 \end{bmatrix}$
series capacitor	$\partial\mathbf{A}/\partial C$	$\begin{bmatrix} 0 & j/(\omega C^2) \\ 0 & 0 \end{bmatrix}$
	$\partial\mathbf{A}/\partial\omega$	$\begin{bmatrix} 0 & j/(\omega^2 C) \\ 0 & 0 \end{bmatrix}$
shunt capacitor	$\partial\mathbf{A}/\partial C$	$\begin{bmatrix} 0 & 0 \\ j\omega & 0 \end{bmatrix}$
	$\partial\mathbf{A}/\partial\omega$	$\begin{bmatrix} 0 & 0 \\ jC & 0 \end{bmatrix}$
one section ladder	$\partial\mathbf{A}/\partial\phi$	$\begin{bmatrix} Y'Z+YZ' & Z' \\ Y' & 0 \end{bmatrix}$

TABLE IX (continued)

Subnetwork	Identification	Partial Derivative of the Chain Matrix
series 3-port junction (terminating port 3)	$\partial \mathbf{A}/\partial \phi$ $\phi \in Y_3$	$\mathbf{K}_1$
$(Y \triangleq Y_c + Y_3)$	$\partial \mathbf{A}/\partial \phi$ $\phi \in J$	$\mathbf{K}_2 + \mathbf{K}_3$
	$\partial \mathbf{A}/\partial \omega$	$\mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3$

where

$$\mathbf{K}_1 = \frac{-1}{Y^2} Y_3' \begin{bmatrix} Y_b & 1 \\ Y_a Y_b & Y_a \end{bmatrix}, \quad \mathbf{K}_2 = \frac{-1}{Y^2} Y_c' \begin{bmatrix} Y_b & 1 \\ Y_a Y_b & Y_a \end{bmatrix},$$

$$\mathbf{K}_3 = \frac{1}{Y} \begin{bmatrix} Y_b' & 0 \\ Y_a'(Y_b + Y) + Y_b'(Y_a + Y) & Y_a' \end{bmatrix}$$

Subnetwork		
series 3-port junction (terminating port 2)	similar to terminating port 3, but interchange $Y_b \leftrightarrow Y_c$ change $Y_3 \rightarrow Y_2$ change $\mathbf{A} \rightarrow \mathbf{D}$	

TABLE IX (continued)

Subnetwork	Identification	Partial Derivative of the Chain Matrix
parallel 3-port junction (terminating port 3)	$\partial \mathbf{A}/\partial \phi$ $\phi \in Z_3$	$\mathbf{L}_1$
$Z \triangleq Z_c + Z_3$	$\partial \mathbf{A}/\partial \phi$ $\phi \in \mathbf{J}$	$\mathbf{L}_2 + \mathbf{L}_3$
	$\partial \mathbf{A}/\partial \omega$	$\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3$

where

$$\mathbf{L}_1 = \frac{-1}{Z^2} Z_3' \begin{bmatrix} Z_a & Z_a Z_b \\ 1 & Z_b \end{bmatrix}, \quad \mathbf{L}_2 = \frac{-1}{Z^2} Z_c' \begin{bmatrix} Z_a & Z_a Z_b \\ 1 & Z_b \end{bmatrix},$$

$$\mathbf{L}_3 = \frac{1}{Z} \begin{bmatrix} Z_a' & Z_a'(Z_b + Z) + Z_b'(Z_a + Z) \\ 0 & Z_b' \end{bmatrix}$$

Subnetwork	Partial Derivative of the Chain Matrix
parallel 3-port junction (terminating port 2)	similar to terminating port 3, but interchange $Z_b \leftrightarrow Z_c$ change $Z_3 \rightarrow Z_2$ change $\mathbf{A} \rightarrow \mathbf{D}$
series 3-port junction (terminating port 3) ($Y_a = Y_b$)	similar to the case $Y_a \neq Y_b$, but change $Y_b \rightarrow Y_a$

TABLE IX (continued)

Subnetwork	Identification	Partial Derivative of the Chain Matrix
waveguide ($\theta = \beta\ell$)	$\partial\mathbf{A}/\partial\ell$	$\begin{bmatrix} -\beta\sin\theta & jZ\beta\cos\theta \\ j\beta\cos\theta/Z & -\beta\sin\theta \end{bmatrix}$
	$\partial\mathbf{A}/\partial\omega$	$\begin{bmatrix} -\ell\beta'\sin\theta & jZ\ell\beta'\cos\theta \\ j\ell\beta'\cos\theta/Z & -\ell\beta'\sin\theta \end{bmatrix}$
unterminated filter ($\mathbf{Z} = s\mathbf{1} + j\mathbf{M}$)	$\partial\mathbf{A}/\partial M_{ab}$	$\frac{jw}{2q_1}(p_a q_b + q_a p_b)\,\mathbf{A} + \frac{jw}{q_1}\begin{bmatrix} \frac{n_2}{n_1} q_a q_b & 0 \\ -n_1 n_2\{q_1(p_a q_b + q_a p_b) + p_1 q_a q_b + q_n p_a p_b\} & \frac{n_1}{n_2} p_a p_b \end{bmatrix}$ where $w = \begin{cases} 2 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases}$
	$\partial\mathbf{A}/\partial\omega$	$\frac{s'}{q_1}\mathbf{p}^T\mathbf{q}\,\mathbf{A} - \frac{s'}{q_1}\begin{bmatrix} -\frac{n_2}{n_1}\mathbf{q}^T\mathbf{q} & 0 \\ n_1 n_2\{2q_1\,\mathbf{p}^T\mathbf{q} - p_1\mathbf{q}^T\mathbf{q} - q_n\mathbf{p}^T\mathbf{p}\} & -\frac{n_1}{n_2}\mathbf{p}^T\mathbf{p} \end{bmatrix}$
	$\partial\mathbf{A}/\partial n_1$	$\frac{1}{n_1}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{A}$
	$\partial\mathbf{A}/\partial n_2$	$\frac{1}{n_2}\mathbf{A}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

TABLE X

SECOND-ORDER SENSITIVITIES OF THE SUBNETWORKS IN TABLE VIII

$$(\dagger)_\phi = \frac{\partial(\dagger)}{\partial\phi},\ (\dagger)_\psi = \frac{\partial(\dagger)}{\partial\psi},\ (\dagger)_{\phi\psi} = \frac{\partial^2(\dagger)}{\partial\phi\partial\psi}$$

$(\dagger)$ denotes Y, Z, Y_a, Y_b, Z_a, Z_b, **H**

Subnetwork	Identification	Second-order Derivative of the Chain Matrix
series inductor	$\partial^2\mathbf{A}/(\partial\omega\partial L)$	$\begin{bmatrix} 0 & j \\ 0 & 0 \end{bmatrix}$
shunt inductor	$\partial^2\mathbf{A}/(\partial\omega\partial L)$	$\begin{bmatrix} 0 & 0 \\ 1/(j\omega^2L^2) & 0 \end{bmatrix}$
series capacitor	$\partial^2\mathbf{A}/(\partial\omega\partial C)$	$\begin{bmatrix} 0 & 1/(j\omega^2C^2) \\ 0 & 0 \end{bmatrix}$
shunt capacitor	$\partial^2\mathbf{A}/(\partial\omega\partial C)$	$\begin{bmatrix} 0 & 0 \\ j & 0 \end{bmatrix}$
one section ladder	$\partial^2\mathbf{A}/(\partial\phi\partial\psi)$	$\begin{bmatrix} Y_{\phi\psi}Z+Y_\phi Z_\psi+Y_\psi Z_\phi+YZ_{\phi\psi} & Z_{\phi\psi} \\ Y_{\phi\psi} & 0 \end{bmatrix}$

TABLE X (continued)

Subnetwork	Identification	Second-order Derivative of the Chain Matrix
series 3-port junction (terminating port 3)	$\partial^2\mathbf{A}/(\partial\phi\partial\psi)$ $(Y \triangleq Y_c + Y_3)$	$\left(\frac{-1}{Y^2}Y_{\phi\psi} + \frac{2}{Y^3}Y_\phi Y_\psi\right)\begin{bmatrix} Y_b & 1 \\ Y_aY_b & Y_a \end{bmatrix} - \frac{1}{Y^2}Y_\phi\begin{bmatrix} (Y_b)_\psi & 0 \\ (Y_a)_\psi Y_b + (Y_b)_\psi Y_a & (Y_a)_\psi \end{bmatrix} - \frac{1}{Y^2}Y_\psi\begin{bmatrix} (Y_b)_\phi & 0 \\ (Y_a)_\phi Y_b + (Y_b)_\phi Y_a & (Y_a)_\phi \end{bmatrix} + \frac{1}{Y}\begin{bmatrix} (Y_b)_{\phi\psi} & 0 \\ (Y_a)_{\phi\psi}(Y_b + Y) + (Y_a)_\phi(Y_b)_\psi + (Y_a)_\psi(Y_b)_\phi + (Y_b)_{\phi\psi}(Y_a + Y) & (Y_a)_{\phi\psi} \end{bmatrix}$
series 3-port junction (terminating port 2)		similar to terminating port 3, but interchange $Y_b \leftrightarrow Y_c$ change $Y_3 \rightarrow Y_2$ change $\mathbf{A} \rightarrow \mathbf{D}$
parallel 3-port junction (terminating port 3)	$\partial^2\mathbf{A}/(\partial\phi\partial\psi)$ $(Z \triangleq Z_c + Z_3)$	$\left(\frac{-1}{Z^2}Z_{\phi\psi} + \frac{2}{Z^3}Z_\phi Z_\psi\right)\begin{bmatrix} Z_a & Z_aZ_b \\ 1 & Z_b \end{bmatrix} - \frac{1}{Z^2}Z_\phi\begin{bmatrix} (Z_a)_\psi & (Z_a)_\psi Z_b + (Z_b)_\psi Z_a \\ 0 & (Z_b)_\psi \end{bmatrix} - \frac{1}{Z^2}Z_\psi\begin{bmatrix} (Z_a)_\phi & (Z_a)_\phi Z_b + (Z_b)_\phi Z_a \\ 0 & (Z_b)_\phi \end{bmatrix} + \frac{1}{Z}\begin{bmatrix} (Z_a)_{\phi\psi} & (Z_a)_{\phi\psi}(Z_b + Z) + (Z_a)_\phi(Z_b)_\psi + (Z_a)_\psi(Z_b)_\phi + (Z_b)_{\phi\psi}(Z_a + Z) \\ 0 & (Z_b)_{\phi\psi} \end{bmatrix}$

TABLE X (continued)

Subnetwork	Identification	Second-order Derivative of the Chain Matrix
parallel 3-port junction (terminating port 2)		similar to terminating port 3, but interchange $Z_b \leftrightarrow Z_c$ change $Z_3 \rightarrow Z_2$ change $\mathbf{A} \rightarrow \mathbf{D}$
series 3-port junction (terminating port 3) ($Y_a = Y_b$)		similar to the case $Y_a \neq Y_b$, but change $Y_b \rightarrow Y_a$
any 3-port junction in Table VIII	$\partial^2\mathbf{A}/(\partial\phi\partial\psi)$	$[\mathbf{e}_1\ \mathbf{e}_2]^T[\mathbf{p}\ \mathbf{q}\ \mathbf{r}](\mathbf{H}_\psi[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]\mathbf{H}_\phi - \mathbf{H}_{\phi\psi} + \mathbf{H}_\phi[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]\mathbf{H}_\psi)[\mathbf{p}\ \mathbf{q}]$

where $\mathbf{H}\,\mathbf{p} = \mathbf{e}_1,\quad \mathbf{H}\,\mathbf{q} = \mathbf{e}_2,\quad \mathbf{H}\,\mathbf{r} = \mathbf{e}_3,$

$\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are unit 3− vectors,

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -1 \\ -(Y_a + Y_b) & 1 & Y_b \\ Y_a & -1 & Y_c + Y_3 \end{bmatrix}$$ for series 3-port junction terminated at port 3

$$\mathbf{H} = \begin{bmatrix} 1 & -(Z_a + Z_b) & -Z_b \\ 0 & 1 & 1 \\ 1 & -Z_a & Z_c + Z_3 \end{bmatrix}$$ for parallel 3-port junction terminated at port 3

For the cases of terminating port 2 and $Y_a = Y_b$, interchange and change appropriately

TABLE X (continued)

Subnetwork	Identification	Second-order Derivative of the Chain Matrix
waveguide ($\theta = \beta\ell$)	$\partial^2\mathbf{A}/(\partial\omega\partial\ell)$	$\begin{bmatrix} -\dfrac{\partial\beta}{\partial\omega}(\sin\theta + \theta\cos\theta) & jZ\dfrac{\partial\beta}{\partial\omega}(\cos\theta - \theta\sin\theta) \\ \dfrac{j}{Z}\dfrac{\partial\beta}{\partial\omega}(\cos\theta - \theta\sin\theta) & -\dfrac{\partial\beta}{\partial\omega}(\sin\theta + \theta\cos\theta) \end{bmatrix}$
unterminated filter ($\mathbf{Z} = s\mathbf{1} + j\mathbf{M}$)	$\partial^2\mathbf{A}/(\partial\omega\partial\phi)$ $\phi = M_{ab}$	$-\dfrac{1}{y_{12}}\begin{bmatrix} \dfrac{\partial^2 y_{22}}{\partial\omega\partial\phi} & 0 \\ * & \dfrac{\partial^2 y_{11}}{\partial\omega\partial\phi} \end{bmatrix} - \dfrac{\dfrac{\partial^2 y_{12}}{\partial\omega\partial\phi}}{y_{12}}\mathbf{A} - \dfrac{\dfrac{\partial y_{12}}{\partial\phi}}{y_{12}}\dfrac{\partial\mathbf{A}}{\partial\omega} - \dfrac{\dfrac{\partial y_{12}}{\partial\omega}}{y_{12}}\dfrac{\partial\mathbf{A}}{\partial\phi}$

where

$$* = \frac{\partial^2 y_{11}}{\partial\omega\partial\phi}y_{22} + \frac{\partial y_{11}}{\partial\omega}\frac{\partial y_{22}}{\partial\phi} + \frac{\partial y_{11}}{\partial\phi}\frac{\partial y_{22}}{\partial\omega} + y_{11}\frac{\partial^2 y_{22}}{\partial\omega\partial\phi} - 2\frac{\partial y_{12}}{\partial\phi}\frac{\partial y_{12}}{\partial\omega} - 2y_{12}\frac{\partial^2 y_{12}}{\partial\omega\partial\phi},$$

$$\mathbf{y} \triangleq \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} = \begin{bmatrix} n_1^2 p_1 & n_1 n_2 q_1 \\ n_1 n_2 q_1 & n_2^2 q_n \end{bmatrix},$$

$$\frac{\partial\mathbf{y}}{\partial\phi} = -jw\begin{bmatrix} n_1^2 p_a p_b & \dfrac{n_1 n_2}{2}(p_a q_b + q_a p_b) \\ \dfrac{n_1 n_2}{2}(p_a q_b + q_a p_b) & n_2^2 q_a q_b \end{bmatrix},$$

and

$$w = \begin{cases} 2 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases},$$

$$\frac{\partial\mathbf{y}}{\partial\omega} = \frac{\partial s}{\partial\omega}\begin{bmatrix} n_1^2\mathbf{p}^T\mathbf{p} & n_1 n_2\mathbf{p}^T\mathbf{q} \\ n_1 n_2\mathbf{p}^T\mathbf{q} & n_2^2\mathbf{q}^T\mathbf{q} \end{bmatrix},$$

TABLE X (continued)

Subnetwork	Identification	Second-order Derivative of the Chain Matrix

$$\frac{\partial^2 \mathbf{y}}{\partial\omega\partial\phi} = -\,jw\,\frac{\partial s}{\partial\omega}\begin{bmatrix} n_1^2(p_a \mathbf{g}^T\mathbf{p} + p_b \mathbf{p}^T\mathbf{f}) & \frac{n_1 n_2}{2}(p_a \mathbf{g}^T\mathbf{q} + q_b \mathbf{p}^T\mathbf{f} + p_b \mathbf{f}^T\mathbf{q} + q_a \mathbf{g}^T\mathbf{p}) \\ \frac{n_1 n_2}{2}(p_a \mathbf{g}^T\mathbf{q} + q_b \mathbf{p}^T\mathbf{f} + p_b \mathbf{f}^T\mathbf{q} + q_a \mathbf{g}^T\mathbf{p}) & n_2^2(q_a \mathbf{g}^T\mathbf{q} + q_b \mathbf{f}^T\mathbf{q}) \end{bmatrix}$$

$$\mathbf{Z}\,\mathbf{p} = \mathbf{e}_1, \quad \mathbf{Z}\,\mathbf{q} = \mathbf{e}_n, \quad \mathbf{Z}\,\mathbf{f} = \mathbf{e}_a, \quad \mathbf{Z}\,\mathbf{g} = \mathbf{e}_b$$

A is the chain matrix of the unterminated filter as shown in Table VIII.

$\partial\mathbf{A}/\partial\omega$ and $\partial\mathbf{A}/\partial\phi$ are the first-order sensitivities of the unterminated filter chain matrix as shown in Table IX.

unterminated filter ($\mathbf{Z} = s\mathbf{1} + j\mathbf{M}$)	$\partial^2\mathbf{A}/(\partial\omega\partial n_1)$	$\frac{1}{n_1}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial\mathbf{A}}{\partial\omega}$
	$\partial^2\mathbf{A}/(\partial\omega\partial n_2)$	$\frac{1}{n_2}\frac{\partial\mathbf{A}}{\partial\omega}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
		where $\partial\mathbf{A}/\partial\omega$ can be found in Table IX.

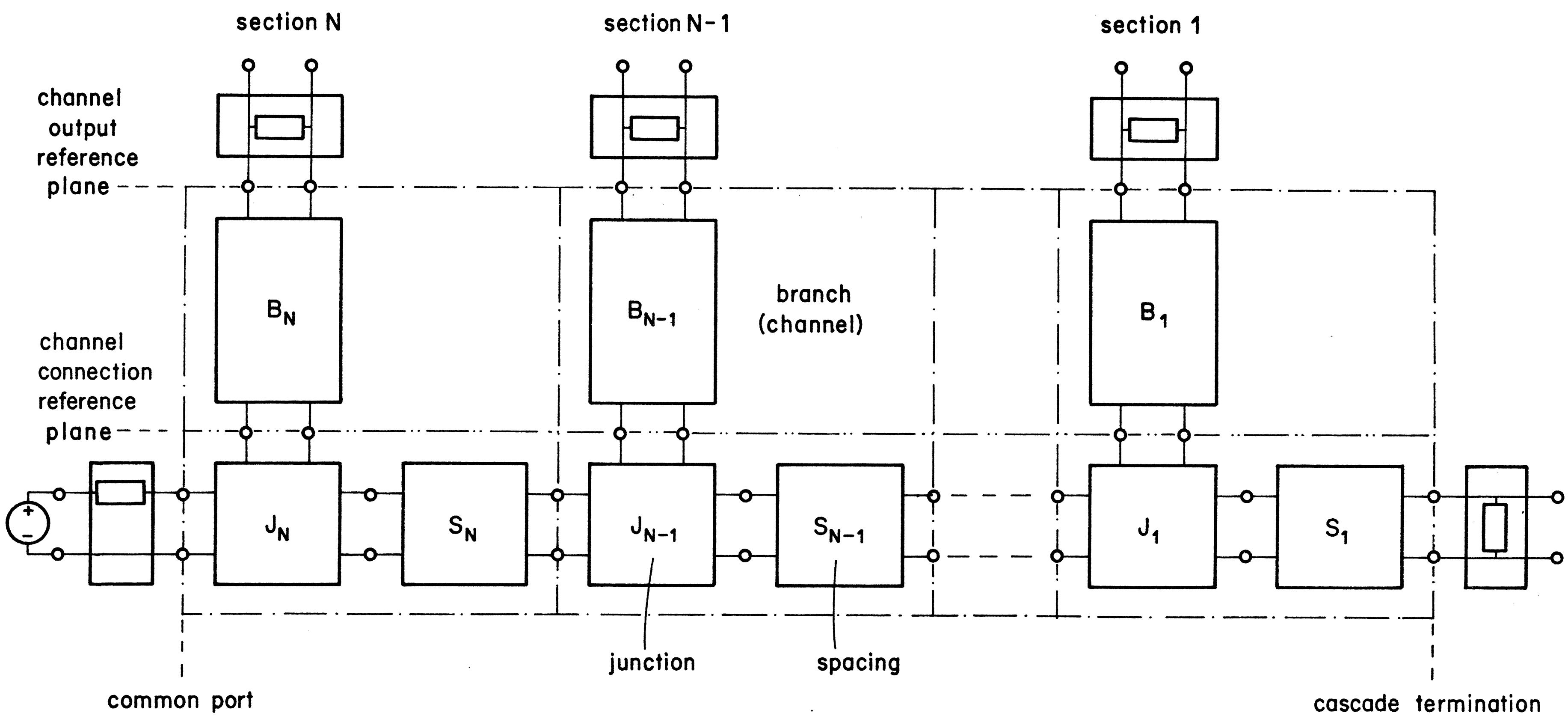


Fig. 1 The multiplexer configuration under consideration. $J_1, J_2, \ldots, J_N$ are arbitrarily defined 3-port junctions, $B_1, B_2, \ldots, B_N$ are terminated branches or channels which may each be represented in reduced cascade form and $S_1, S_2, \ldots, S_N$ are usually waveguide spacing elements.

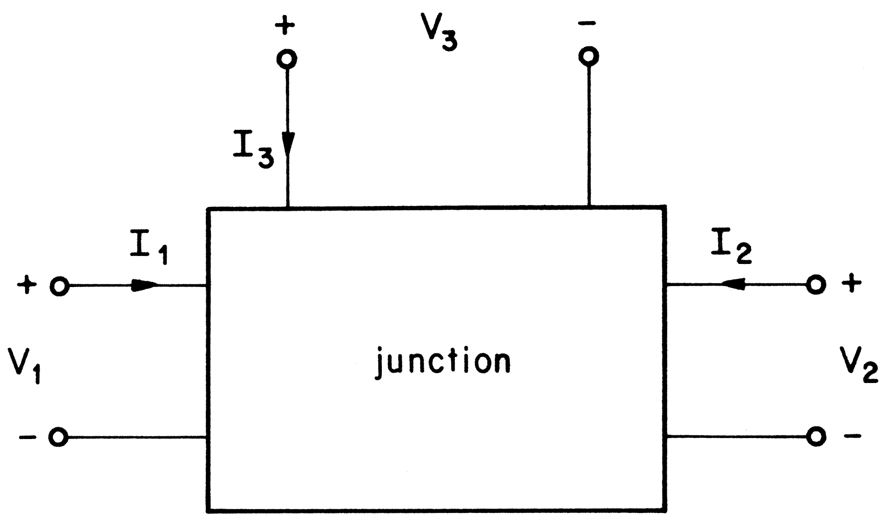


Fig. 2 A 3-port network in which ports 1 and 2 are considered along a main cascade and port 3 represents a channel or branch of the main cascade.

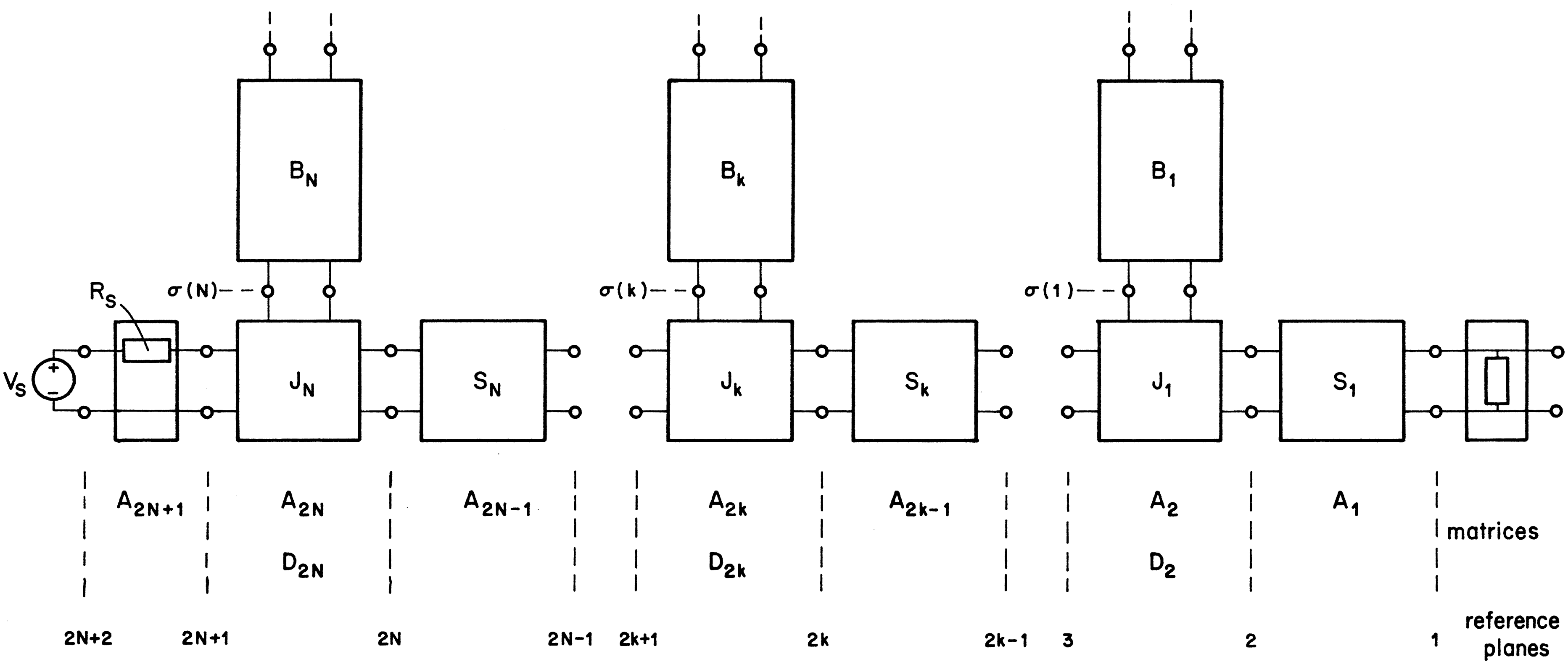


Fig. 3 Illustration of principal concepts involved in multiplexer simulation showing reference planes and transmission matrices.

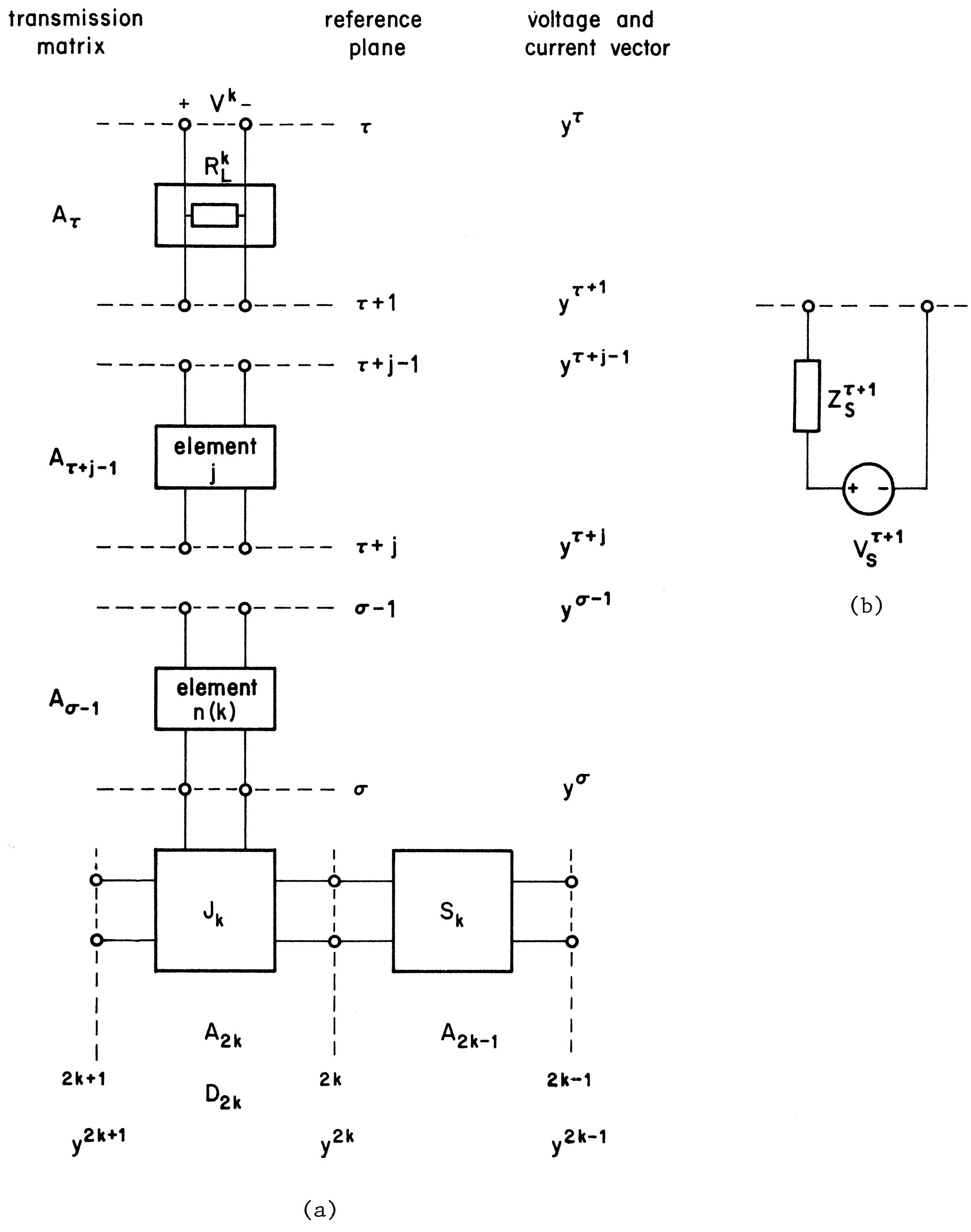
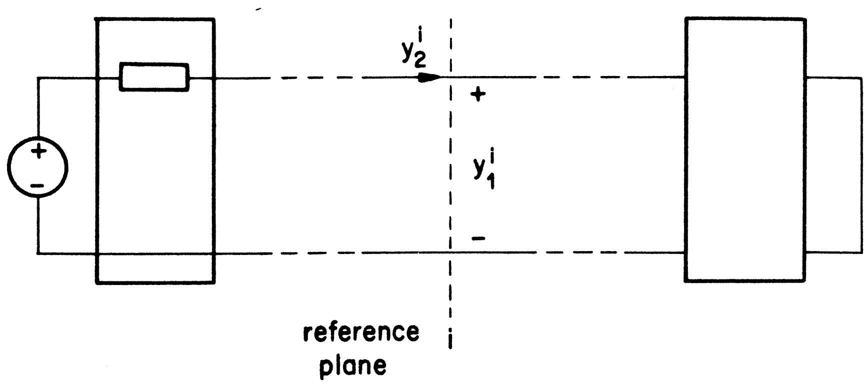
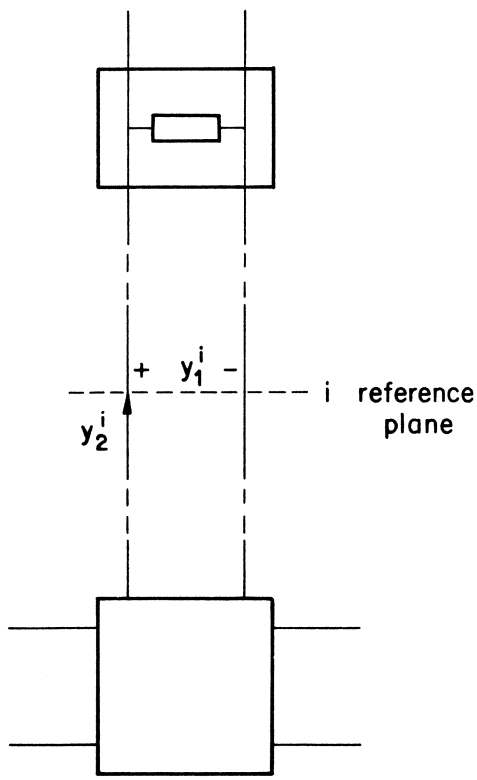


Fig. 4 (a) Detail of the kth multiplexer section showing the reference planes along the branch. (b) Thevenin equivalent circuit at channel output port ($\tau \equiv \tau(k)$, $\sigma \equiv \sigma(k)$).



(a)



(b)

Fig. 5 Illustration of vector $\mathbf{y}^i$ containing the voltage and the current at reference plane i in the multiplexer. (a) Reference plane i in the main cascade. (b) Reference plane i in a channel.

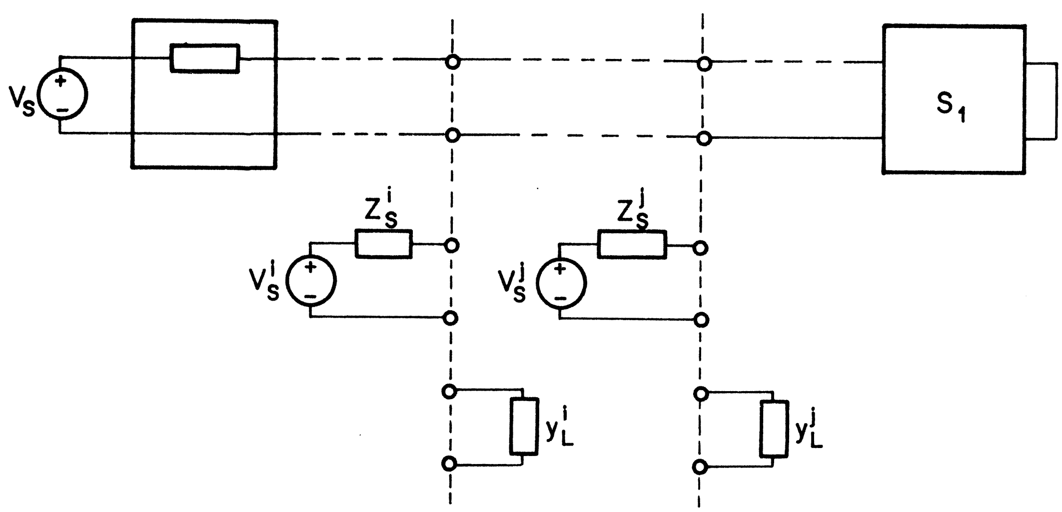


Fig. 6(a) Thevenin and Norton equivalents at reference plane i and j, where reference plane i is towards the source port w.r.t. reference plane j and reference plane j is in the main cascade.

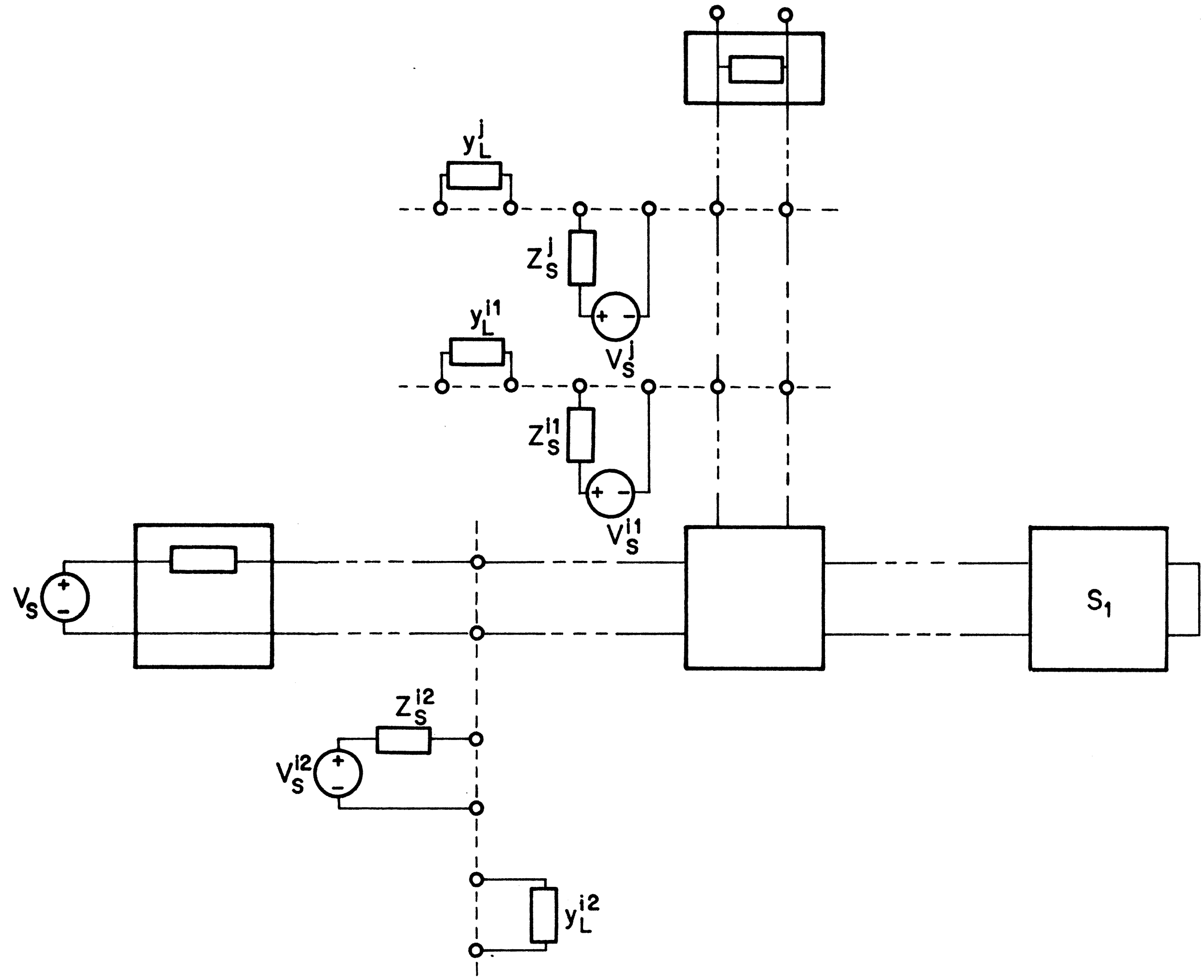


Fig. 6(b) Thevenin and Norton equivalents at reference plane i and j, where reference plane i is towards the source port w.r.t. reference plane j and reference plane j is in a channel.

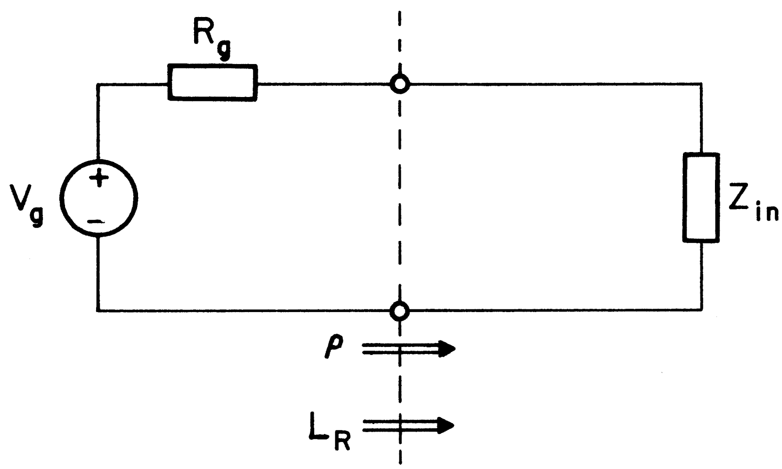


Fig. 7(a) Illustration of reflection coefficient and return loss for a general network.

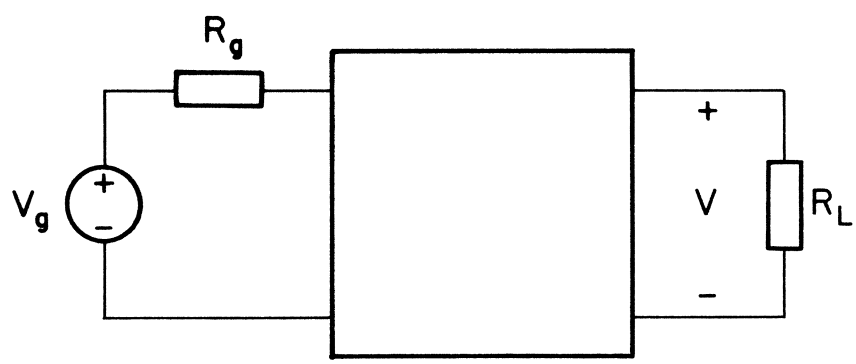


Fig. 7(b) Illustration of a general network for the definition of insertion loss.

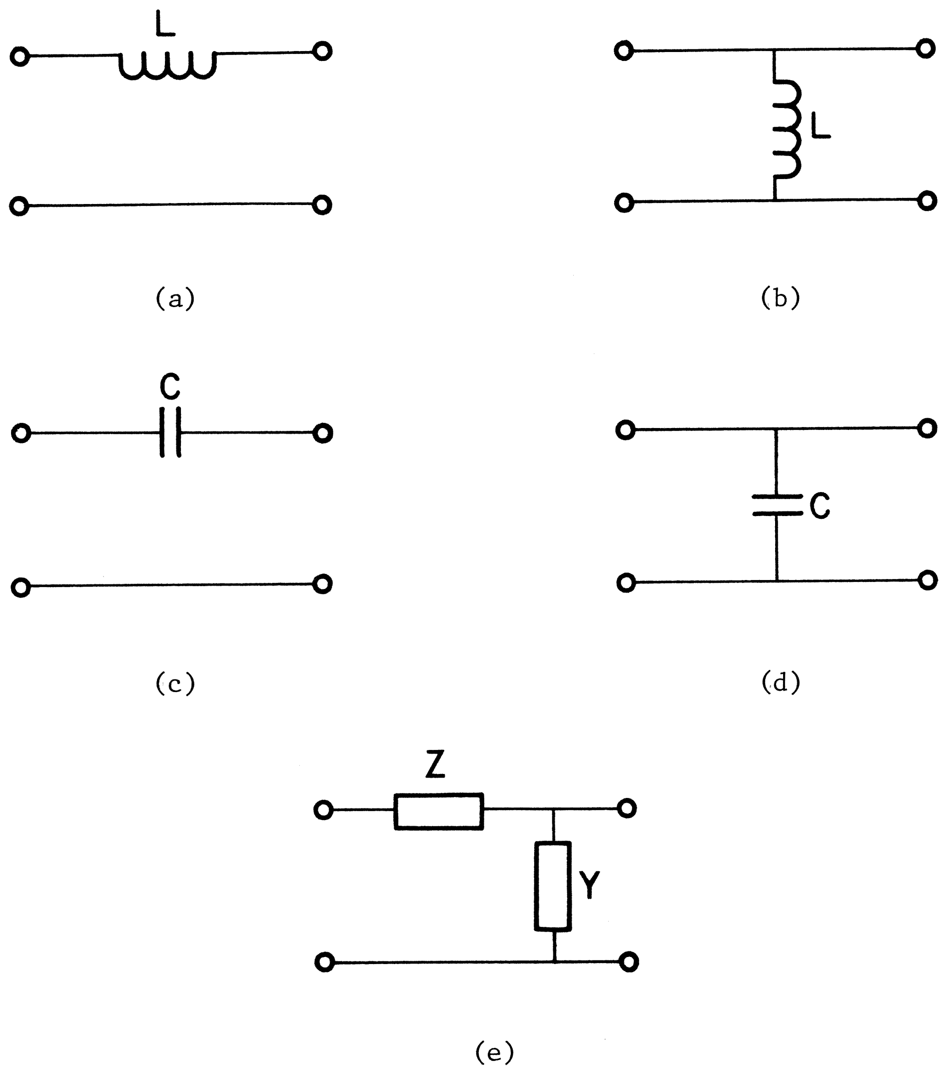


Fig. 8 Elements considered. (a) Series inductor. (b) Shunt inductor. (c) Series capacitor. (d) Shunt capacitor. (e) Ladder section.

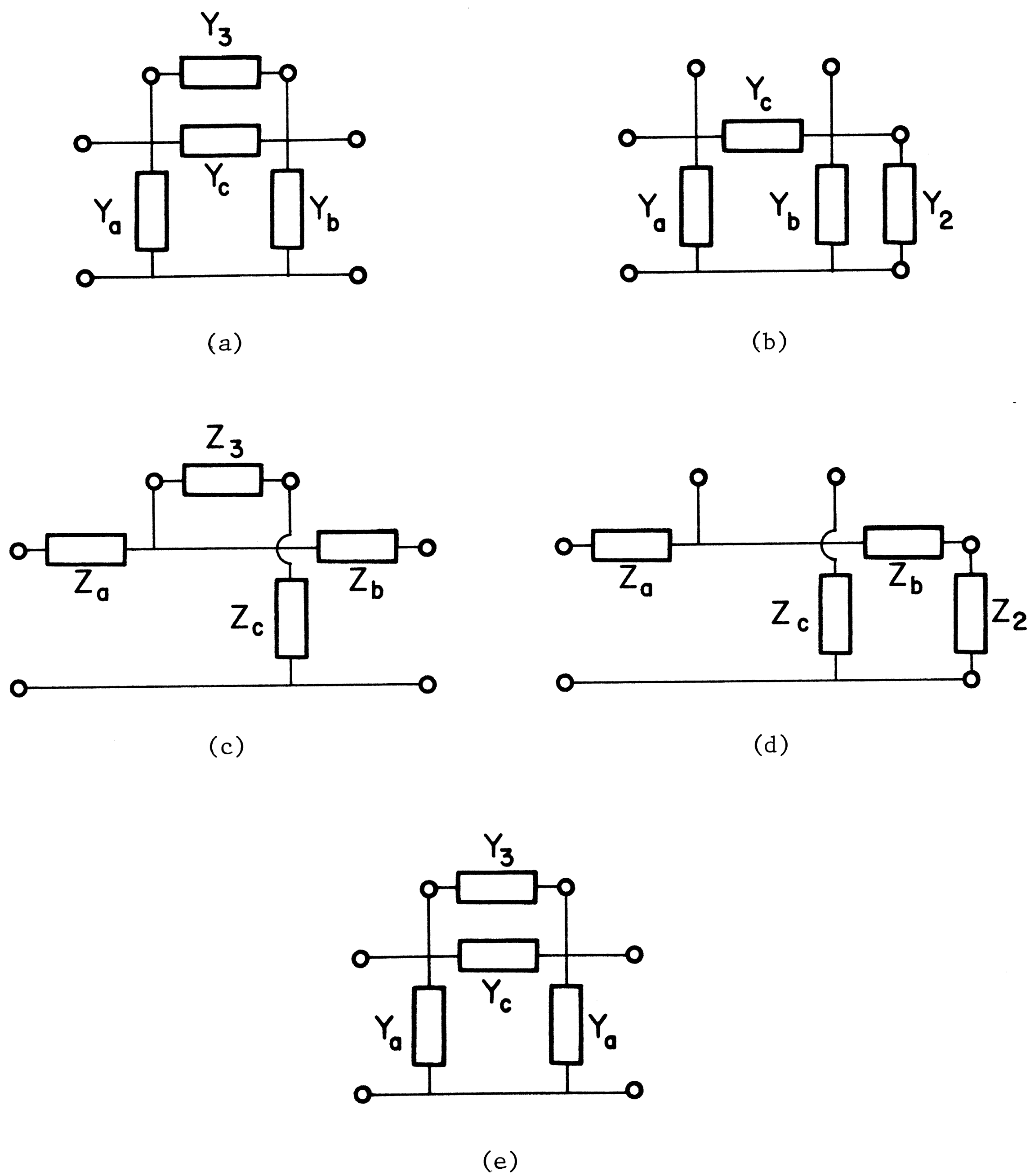


Fig. 9 Junction subnetworks. (a) Series 3-port (terminating port 3). (b) Series 3-port (terminating port 2). (c) Parallel 3-port (terminating port 3). (d) Parallel 3-port (terminating port 2).

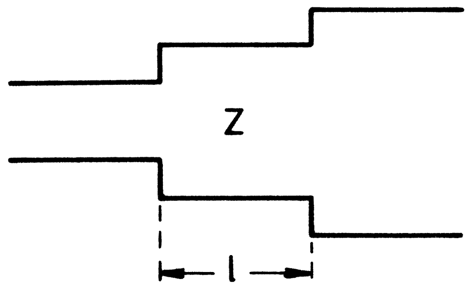


Fig. 10 Waveguide section.

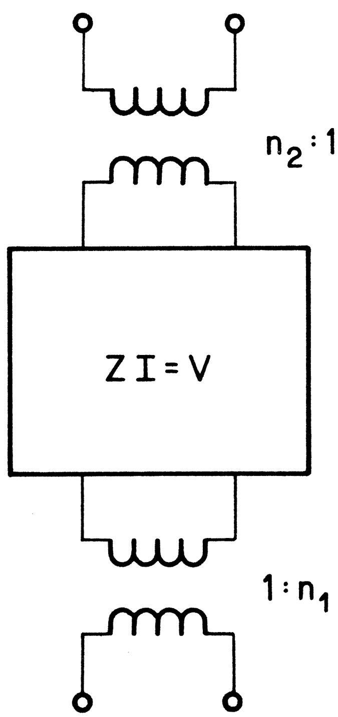


Fig. 11 Unterminated narrow band coupled cavity filter.