LARGE CHANGE SENSITIVITY ANALYSIS VIA A MINIMUM ORDER REDUCED SYSTEM

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Abstract

The reduced system of equations commonly employed in large change sensitivity computations in linear systems is expressed in nodal form as compared to the branch relations conventionally used in the literature. A systematic approach formulating a minimum order reduced system is developed for variables of RCL types. The involvement of active elements is also discussed. Besides its theoretical interest, this approach can improve computational efficiency over conventional methods when many parameters undergo large changes with few nodes affected.

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I. INTRODUCTION

The increasing demand for highly efficient techniques capable of solving circuit equations repeatedly with different parameter values has been responsible for the development of large change sensitivity analysis methods. The common approach to this problem in linear networks is to formulate a reduced system whose solutions are then used to update the system responses [1]. Various techniques, e.g., current source substitution [2], adjoint network [3], Householder formula [1,2,4,8], partitioning [1,5] and scattering theory [6,7] have been employed. Recently, Haley and Current gave an overview of this area and presented general approaches encompassing most of the previous methods [7].

In conventional approaches, the reduced system is formulated as a pxp system, where p is the number of variables. Therefore, in a case where p is not very small compared to the order of the original system, the efficiency of large change algorithms is greatly degenerated. Such a case occured in Example 8.1.1 of Vlach and Singhal where a 3x3 system had to be solved in order to update the response of a 2x2 system, merely because 3 variables exist [5].

A further reduction of the reduced system is made possible by the discovery that the order of such a system can be as low as the rank of the original system deviation matrix [6,7]. This manifests itself as a minimum system [8].

The case of $p > \text{rank}(\Delta \mathbf{A})$ occurs when some variables do not change and/or, when the variable locations are "structurally degenerate", e.g., if loops exist. The first situation has been fairly treated in the literature [1,5]. The next logical step is to develop a systematic approach to realize the minimum reduced system by employing topological relations among variables. Such an effective method is developed and presented in this paper, valid when variables are of RCL types. The involvement of active components as variables is also discussed. We concentrate on the "structural degenerate" problem by assuming that all variables have been sorted such that ϕ_i, $i = 1,2,\ldots,p$ are the variables whose values actually change. Relevant terminologies from graph theory are defined in Appendix I for readers' convenience.

II. LARGE CHANGE COMPUTATION

Suppose a linearized network is represented by

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}\,, \tag{1}$$

where $\mathbf{A}$ is a nxn matrix characterizing the network, $\mathbf{b}$ is a n-vector representing the excitation and $\mathbf{x}$ is a n-vector containing system responses.

When system parameters $\phi_1, \phi_2, \ldots, \phi_p$ are changed, causing the change of $\mathbf{A}$ by $\Delta\mathbf{A}$, response changes can be calculated by large change formulas. The commonly used method is to express $\Delta\mathbf{A}$ as a triple product as [1]

$$\Delta\mathbf{A} = \mathbf{V}\,\mathbf{D}\,\mathbf{W}^T \tag{2}$$

or using parameter matrix decomposition of $\Delta\mathbf{A}$ as [6,7]

$$\Delta\mathbf{A} = \sum_{i=1}^{r} \mathbf{v}_i\,\Delta\phi_i\,\mathbf{w}_i^T,\ \ r \leq p\,. \tag{3}$$

The responses are then calculated using the Householder formula [4] or its various equivalents [1-8] as

$$\mathbf{x} + \Delta\mathbf{x} = \mathbf{x} - \mathbf{A}^{-1}\mathbf{V}(\mathbf{D}^{-1} + \mathbf{W}^T\mathbf{A}^{-1}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{x}\,. \tag{4}$$

The calculation of (4) involves the solution of a reduced system whose size is determined from the formulation of $\mathbf{V}$, $\mathbf{D}$ and $\mathbf{W}$ as appeared in (2) and (3). We focus on this formulation. Subsequent calculations leading to $\Delta\mathbf{x}$ can be performed according to the literature [1-8].

Using the well-established methods [1-7], one can generate a pxp reduced system for an arbitrary linear network by choosing

$$\mathbf{D} = \mathrm{diag}\,\{\Delta\phi_1, \Delta\phi_2, \ldots, \Delta\phi_p\}\,, \tag{5}$$

$$\mathbf{V} = [\mathbf{v}_1\ \ \mathbf{v}_2 \cdots \mathbf{v}_p] \tag{6}$$

and

$$\mathbf{W} = [\mathbf{w}_1\ \ \mathbf{w}_2 \cdots \mathbf{w}_p]\ , \tag{7}$$

where $\mathbf{v}_i$ and $\mathbf{w}_i$ $i=1, \ldots, p$ are n-vectors containing ± 1 and 0. ϕ_i, $i=1, \ldots, p$ represents the value of variable i, being of the type that enter the tableau or modified nodal equations in the form $\mathbf{v}_i \phi_i \mathbf{w}_i^T$ [5].

It can be seen that this formulation gives each variable an equal treatment and no consideration regarding topological relations of these variables is taken into account. In order to distinguish with the new method, we refer to the method of (5)-(7) as a formulation based on branch relations.

A thorough exploitation of the topological relations among variables is vital for the order of the reduced system to be decreased from p to rank $(\Delta\mathbf{A})$ which is the minimum. A promising formulation is the one based on nodal relations.

III. FORMULATION OF V, D AND W BASED ON NODAL RELATIONS FOR VARIABLES OF RCL TYPES

Let the network topology be represented by graph G and the edge set of G be represented by E, respectively. Let E' be a subset of E such that an edge in E corresponding to a variable is classified in E'. The induced subgraph of G on edge set E' is denoted as G'. Separate G' into blocks G_1', G_2' ..., G_b', $b \geq 1$, such that $G' = G_1' \cup G_2' \cup \ldots G_b'$ and $G_i \cap G_j$ is either null or empty containing only a cut-vertex of G' for all $i,j = 1,2,\ldots,b$ and $i \neq j$.

Formulation of V, D and W

For RCL type variables in a linear network, **V**, **D** and **W** can be formulated using nodal relations instead of the conventional branch relations so as to achieve a minimum order reduced system. **V**, **D** and **W** are nxr, rxr and nxr matrices, respectively and are decomposed such that

$$\mathbf{V} = [\mathbf{V}_1 \ \mathbf{V}_2 \ldots \mathbf{V}_b], \tag{8}$$

$$\mathbf{W} = [\mathbf{W}_1 \ \mathbf{W}_2 \ldots \mathbf{W}_b] \tag{9}$$

and

$$\mathbf{D} = \text{diag}\,\{\mathbf{D}_1\,,\mathbf{D}_2\,,\ldots\,,\mathbf{D}_b\}\ , \tag{10}$$

where $\mathbf{D}_i$ is the nodal admittance matrix of G_i' using the $\Delta\phi$ as parameter and $\mathbf{V}_i$ and $\mathbf{W}_i$ are incidence matrices of G_i' indicating vertex locations of G_i' as seen from G. Suppose G_i' has m vertices, m $\geq$ 2. $\mathbf{D}_i$ is (m − 1) x (m − 1) since one vertex can be considered as "ground" and is taken as a reference vertex. $\mathbf{V}_i$ and $\mathbf{W}_i$ are both nx(m–1) where each column vector corresponds to a non-reference vertex of G_i'. If this non-reference vertex and the reference vertex of G_i' appear in G as the kth and ℓth vertices, respectively, the corresponding columns of $\mathbf{V}_i$ and $\mathbf{W}_i$ are equal to $\mathbf{e}_k - \mathbf{e}_\ell$ or $\mathbf{e}_k$, if the ℓth vertex corresponds to the ground, $\mathbf{e}_k$ being a unit n-vector with 1 in its kth position and zeros everywhere else. For mathematical simplicity, a vertex in G_i' is taken as a reference vertex if it corresponds to the ground of the overall circuit.

Verification of the Reduced System as Minimal Order

It can be found that **V** and **W** both have full column rank (see Appendix II). Therefore, there exist nonsingular nxn matrices **P** and **Q** such that

$$\mathbf{P\,V} = \mathbf{Q\,W} = [\mathbf{e}_1\ \mathbf{e}_2 \ldots \mathbf{e}_r]\,. \tag{11}$$

Consequently,

$$\begin{aligned}\mathbf{P}\,(\Delta\mathbf{A})\,\mathbf{Q}^T &= \mathbf{P\,V\,D\,W}^T\,\mathbf{Q}^T \\ &= \text{diag}\,\{\mathbf{D}, \mathbf{0}\}\ .\end{aligned} \tag{12}$$

We obtain

$$\text{rank}\,(\Delta\mathbf{A}) = \text{rank}\,(\mathbf{D})\,. \tag{13}$$

When $\Delta\phi_i \neq 0$, for all i, i = 1, 2, ..., p, **D** is full rank since each G_i' is a connected graph [9]. Thus the order of the reduced system, i.e., r is equal to rank (Δ**A**), the minimum value being achieved.

Discussion 1: Relations Between the Branch and the Nodal Based Methods

If there are no loops in G', each block G_i', i = 1,2,...,b contains one edge only. Therefore, the conventional branch based formulation becomes a special case of the nodal based one when no loops exist in G'. In other cases, the reduced system is further reduced to its minimum by using the nodal approach.

As a different interpretation of relations between the two formulations, each block G_i', i = 1,2,...,b can be considered as a multi-terminal element with $\mathbf{D}_i$ as its $\mathbf{Y}$ matrix representation. Each element enters the overall system deviation matrix $\Delta\mathbf{A}$ in the form $\mathbf{V}_i\mathbf{D}_i\mathbf{W}_i^T$. Apply the conventional method and treat the block elements $\mathbf{D}_1$, $\mathbf{D}_2$, ..., $\mathbf{D}_b$ as if treating actual elements $\Delta\phi_1$, $\Delta\phi_2$, ..., $\Delta\phi_p$. Topological relations of G are used in sorting variables into blocks G_i', i = 1,2,...,b.

Discussion 2: Active Elements Involved as Variables

The existence of active elements often yields nonsymmetry in the nodal admittance matrix. Consider the voltage-controlled current source with variable g_m as an example. When neither the controller nor the source is looped with other variables, the corresponding $\mathbf{V}_i$, $\mathbf{D}_i$ and $\mathbf{W}_i$ are the same as those obtained using branch based methods. Otherwise, if the controller (source) is looped with other variables, a vertical (horizontal) rectangular matrix $\mathbf{D}$ can be formulated to ensure the reduced system to be in its minimum order and to free $\mathbf{V}$ and $\mathbf{W}$ from values of variables. The current (voltage) graph [10] can be used to form $\mathbf{V}$ ($\mathbf{W}$) and to reflect the vertical (horizontal) property of $\mathbf{D}$. Generalized Householder formulas [8] are utilized. This phenomenon is illustrated in Example 2 of Section IV.

Another method is to introduce a composite vector as suggested by Haley and Current [7]. If the controller (source) is in a loop, $\mathbf{V}$ ($\mathbf{W}$) becomes composite. A minimum system is achieved with a square $\mathbf{D}$. $\mathbf{V}$ or $\mathbf{W}$ now contain the variable g_m and need to be reprocessed for each change of variable value g_m.

IV. EXAMPLES

Example 1: Variables Being of RCL Type

Consider the 10 node circuit of Fig. 1 with ϕ_i, i = 1, 2, ..., 7 as variables. Topological relations showing the network graph G, the induced subgraph of G on edge set E' and the blocks are given in Fig. 2. G' is divided into G_1' and G_2'. Thus we have

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \\ & \mathbf{D}_2 \end{bmatrix}$$
$$= \begin{bmatrix} \Delta\phi_7 & 0 & 0 & 0 \\ 0 & \Delta\phi_1 + \Delta\phi_2 + \Delta\phi_4 & -\Delta\phi_4 & -\Delta\phi_2 \\ 0 & -\Delta\phi_4 & \Delta\phi_4 + \Delta\phi_5 + \Delta\phi_6 & -\Delta\phi_5 \\ 0 & -\Delta\phi_2 & -\Delta\phi_5 & \Delta\phi_2 + \Delta\phi_3 + \Delta\phi_5 \end{bmatrix}, \tag{14}$$

$$\mathbf{V} = [\mathbf{V}_1 \quad \mathbf{V}_2] = [\mathbf{e}_4 - \mathbf{e}_5 \quad \mathbf{e}_3 - \mathbf{e}_9 \quad \mathbf{e}_4 - \mathbf{e}_9 \quad \mathbf{e}_8 - \mathbf{e}_9] \ , \tag{15}$$

and

$$\mathbf{W} = \mathbf{V} . \tag{16}$$

Notice that nodes 5 and 9 have been taken as references for G_1' and G_2', respectively. The reduced system is 4x4.

The computational effort of our approach compared with the existing methods of [1-5] is shown in Table I.

Example 2: Active Component Involved as Variables

Again, we consider the circuit of Fig. 1. Suppose in case 1, G_1 and g_m are variables. We choose node 10 as the reference node. A vertical rectangular **D** is formed as

$$\mathbf{D} = [\Delta G_1 \ \ \Delta g_m]^T . \tag{17}$$

Also, $\mathbf{V} = [\mathbf{e}_5 - \mathbf{e}_{10} \ \ \mathbf{e}_6]$ and $\mathbf{W} = \mathbf{e}_5 - \mathbf{e}_{10}$, which can be observed from the current graph and the voltage graph, respectively (see Fig. 3). The reduced system is a scalar equation obtained by employing the Vertical-Rectangular-Householder-Formula (VRF) [8]

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\,\mathbf{V}\,\mathbf{D}(\mathbf{1} + \mathbf{W}^{T}\mathbf{A}^{-1}\,\mathbf{V}\,\mathbf{D})^{-1}\,\mathbf{W}^{T}\mathbf{A}^{-1}, \tag{18}$$

where **1** denotes an identity matrix. The response changes are calculated as

$$\Delta\mathbf{x} = -\frac{\mathbf{P}_{V}\,\mathbf{D}(\mathbf{W}^{T}\mathbf{x})}{1 + (\mathbf{W}^{T}\mathbf{P}_{V})\,\mathbf{D}}, \tag{19}$$

where $\mathbf{P}_V$ is obtained by solving $\mathbf{A}\mathbf{P}_V = \mathbf{V}$ in the preparatory calculations.

Suppose in case 2, G_2 and g_m are variables. A horizontal rectangular **D** is formed as

$$\mathbf{D} = [\Delta G_2 \quad \Delta g_m]\,. \tag{20}$$

Also, $\mathbf{V}=\mathbf{e}_6$ and $\mathbf{W}=[\mathbf{e}_6 \quad \mathbf{e}_5-\mathbf{e}_{10}]$, using the current and voltage graphs, respectively, as shown in Fig. 4. The corresponding Horizontal-Rectangular-Householder-Formula (HRF) is [8]

$$\Delta(\mathbf{A}^{-1}) = -\,\mathbf{A}^{-1}\,\mathbf{V}(\mathbf{1} + \mathbf{D}\,\mathbf{W}^{T}\mathbf{A}^{-1}\,\mathbf{V})^{-1}\,\mathbf{D}\,\mathbf{W}^{T}\mathbf{A}^{-1}. \tag{21}$$

The response changes are calculated as

$$\Delta\mathbf{x} = -\frac{\mathbf{P}_{V}\,\mathbf{D}(\mathbf{W}^{T}\mathbf{x})}{1 + \mathbf{D}(\mathbf{W}^{T}\mathbf{P}_{V})}. \tag{22}$$

If we perform the multiplication **VD** and $\mathbf{DW}^T$ first and then substitute into (18) and (21), respectively, the computations become the same as that of the composite vector approach [7].

Computational efforts using the Rectangular Householder Formulas, composite vector approaches, conventional approaches and the direct method are compared in Table II. Our major concern is the operational count required for each new set of variable values. It is observed that using the Rectangular Householder Formulas is most efficient in both cases of this example.

Example 3: The Case of Example 8.1.1 of Vlach and Singhal [5]

Consider the circuit of Fig. 5 where G_1, G_2 and G_3 are all variables. Evidently, the "reduced" system is of order 2 using the nodal based approach which gives

$$\mathbf{D} = \Delta\mathbf{A} = \begin{bmatrix} \Delta G_1 + \Delta G_2 & -\Delta G_2 \\ -\Delta G_2 & \Delta G_2 + \Delta G_3 \end{bmatrix} \tag{23}$$

and

$$\mathbf{V} = \mathbf{W} = \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{24}$$

Compared with the branch based method which yields a 3x3 system, the operational count is reduced from 23 to 16 for each set of values of ΔG_i, $i=1,2,3$.

Although for this circuit, one would rather solve the original network equations than use large change formulas, such a variable structure can exist in a large system where an efficient large change algorithm is extremely important.

V. CONCLUSIONS

We have discussed an approach to formulating a minimal reduced system for efficient application of existing large change formulas. In practical Monte-Carlo analysis, network optimization, identification and tuning, cases may exist where a cluster of variables exist in a subnetwork which has been decomposed from the overall network. In this case, structural degeneracy often exists, thus, as shown in the examples, using our nodal based approach to formulate the reduced system, one can save considerable computational effort in calculating overall response changes. The cost of the new approach is that **D** becomes block diagonal instead of simple diagonal as in the conventional approach. One should either use sophisticated software exploiting the structural property of **D**, resulting computational savings over the conventional method in most cases, or employ simple matrix inversion subroutines to perform $\mathbf{D}^{-1}$ as required in large change formulas, resulting in improved efficiency when rank $(\Delta\mathbf{A}) \leq 0.63$ p. Further research is being directed to a formulation where elements other than RCL can also be accommodated and arbitrarily distributed active components can be handled.

APPENDIX I [11,12]

Let $G = (V,E)$ denote a graph where V and E is the vertex set and the edge set, respectively.

Definition 1: $G' = (V', E')$ is an edge-induced subgraph of G if every vertex in V' is the end vertex of some edge in E'.

Definition 2: A vertex v is a cut vertex of a connected graph G if and only if there exist two vertices u and w distinct from v such that v is on every u-w path.

Definition 3: A block of a separable graph G is a maximal nonseparable subgraph of G.

APPENDIX II

It is evident that the $\mathbf{V}_i$, $i = 1,2,\ldots,b$ are of full column rank. If a vertex of G, say the ℓth vertex, is a cut-vertex in G' and appears in G_i', we add all rows of $\mathbf{V}$ except the ℓth, to the ℓth row, resulting in all elements in the ℓth row of $\mathbf{V}_i$ being zero. This operation is performed for every G_i' whose reference vertex does not correspond to the ground of the overall network. We start with each block which has no more than one cut-vertex of G'. Since G_i' and G_j', $i \neq j$ have at most one vertex (the cut-vertex) in common, the above operations in $\mathbf{V}$ produce no effects between $\mathbf{V}_i$ and $\mathbf{V}_j$. It can be seen that after this operation, any non-zero row,say the kth row, of $\mathbf{V}$ becomes a row having the form

$$\begin{bmatrix} 0 \ldots 0 & \mathbf{v}_i^T & 0 \ldots 0 \end{bmatrix}, \tag{A-1}$$

where $\mathbf{v}_i = \mathbf{e}_k{}^T \mathbf{V}_i$, $i \in \{1, 2, \ldots, b\}$. Thus we have

$$\text{rank}\,(\mathbf{V}) = \sum_{i=1}^{b} \text{rank}\,(\mathbf{V}_i) = r, \tag{A-2}$$

where r is the number of columns of $\mathbf{V}$.

Similarly,

$$\text{rank}\,(\mathbf{W}) = \sum_{i=1}^{b} \text{rank}\,(\mathbf{W}_i) = r. \tag{A-3}$$

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TABLE I

COMPARISON OF OPERATIONAL COUNTS USING

DIFFERENT METHODS FOR EXAMPLE 1

Steps	Nodal Based Approach (r=4, n=10)	Branch Based Approach (p=7, n=10)	Direct Method (n=10)
LU factorization of **A**	$\frac{n^3 - n}{3}$	$\frac{n^3 - n}{3}$	$\frac{n^3 - n}{3}$
solve $\mathbf{A}\,\mathbf{x} = \mathbf{b}$	n^2	n^2	n^2
solve $\mathbf{A}\,\mathbf{P}_V = \mathbf{V}$	rn^2	pn^2	
form and solve reduced system	$\leq \frac{4r^3 - r}{3} + r^2$	$\frac{p^3 + 2p}{3} + p^2$	
update response **x**	nr	np	
total number of multiplications and divisions for 200 sets of variable values for Example 1	21,630	48,730	86,430

TABLE II

COMPARISON OF OPERATIONAL COUNTS USING

DIFFERENT METHODS FOR EXAMPLE 2

Steps	Using VRF or HRF	Composite Vector Approach	Conventional Method	Direct Method
LU factorization of **A**	330	330	330	330
solution of $\mathbf{Ax} = \mathbf{b}$	100	100	100	100
solution of $\mathbf{AP_V} = \mathbf{V}$	200, 100*	–,100	200	–
subtotal (1)†	630, 530	430, 530	630	430
form and solve reduced system	3, 5	101, 7	8	–
update response	22, 10	10, 10	20	–
subtotal (2)†	25, 15	111, 17	28	430

† Subtotals (1) and (2) correspond to the preparatory calculation and the calculation for each new set of variable values, respectively.

* Two numbers appearing in the same slot correspond to case 1 and case 2, respectively.

– indicates that the corresponding step is not required.

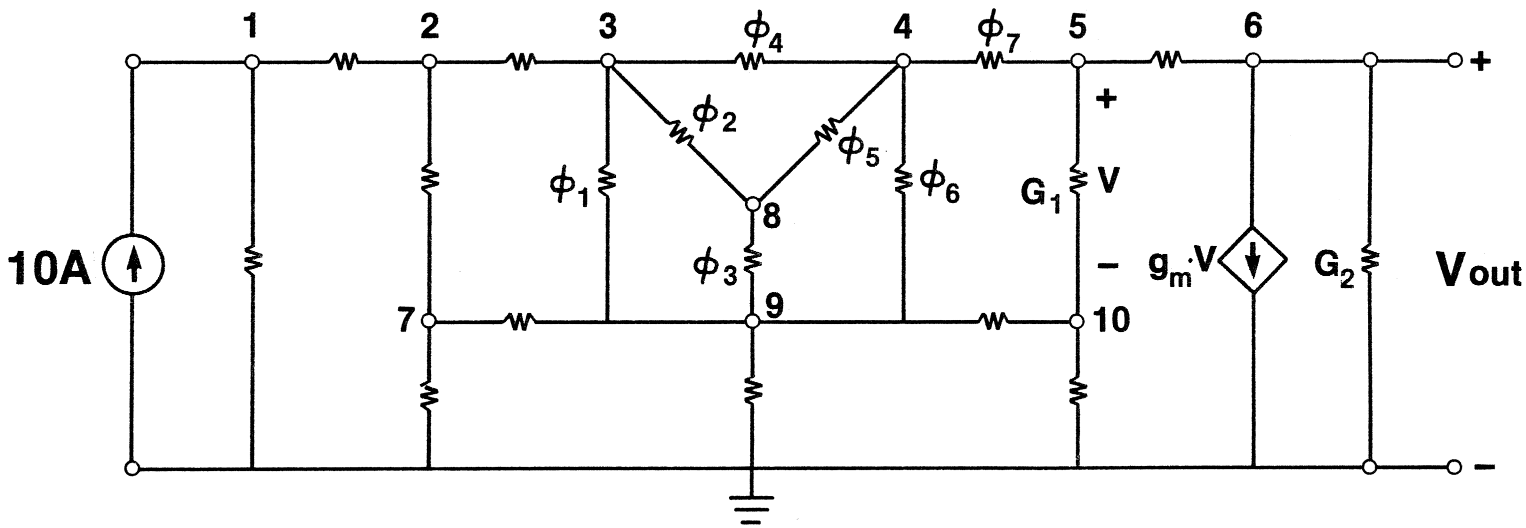

Fig. 1. An arbitrary 10 node network. All variables ϕ_1, ϕ_2, ..., ϕ_7 and G_1, G_2 are conductances of the associated components. The ground is referred to as node 0.

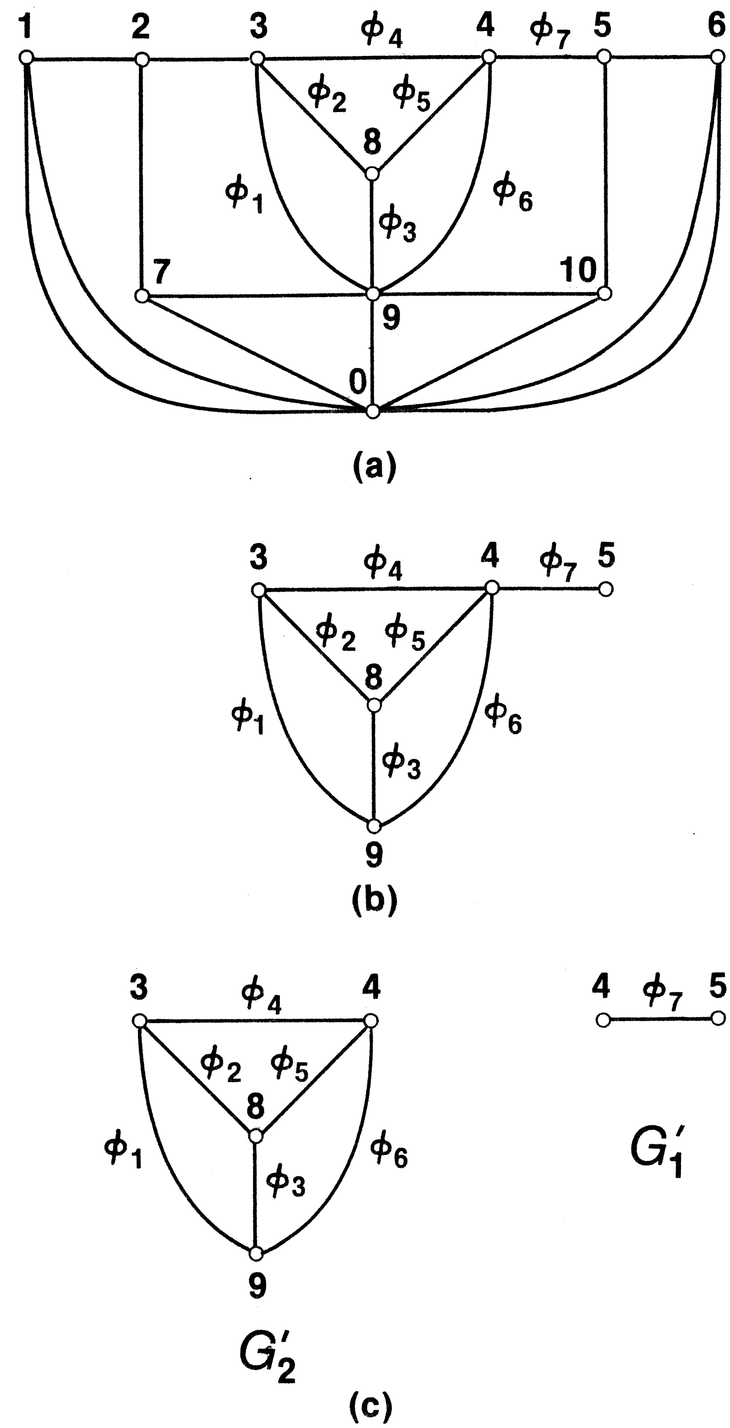

Fig. 2. Topological relations for the circuit of Fig. 1. (a) Graph G, (b) Edge induced subgraph G' for Example 1 and, (c) Blocks $G_1{}'$ and $G_2{}'$.

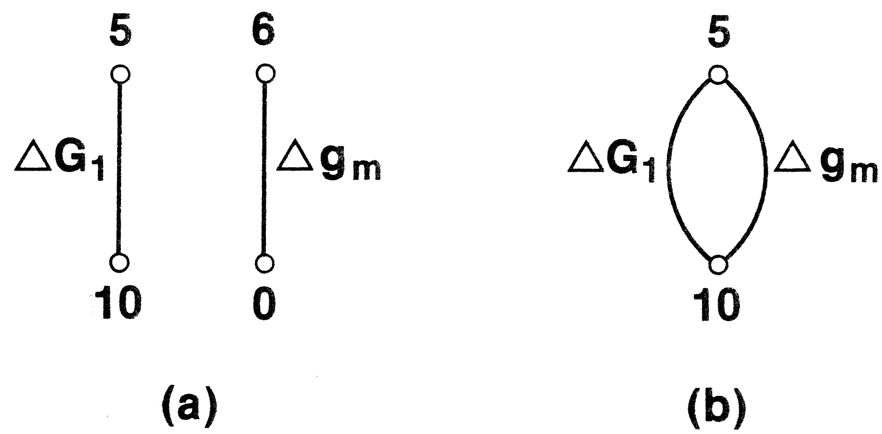

Fig. 3. Topological relation for Case 1 of Example 2. (a) Edge induced current subgraph, (b) Edge induced volgage subgraph.

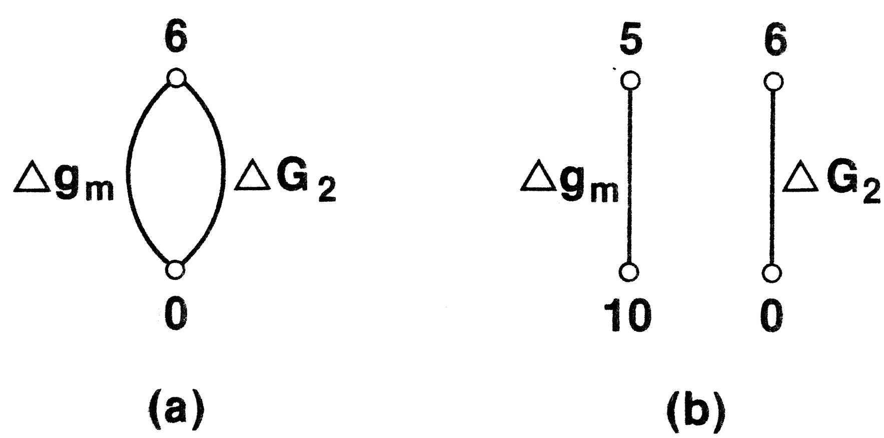

Fig. 4. Topological relation for Case 2 of Example 2. (a) Edge induced current subgraph, (b) Edge induced voltage subgraph.

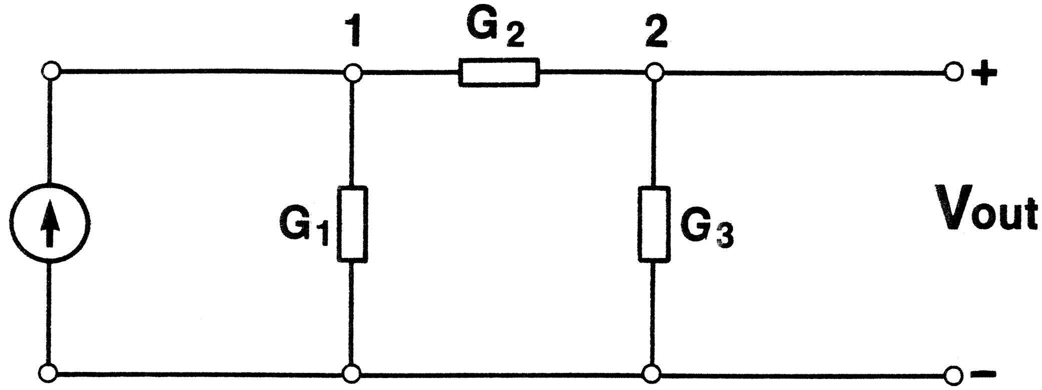

Fig. 5. Network for Example 3.