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A SURVEY OF ASYMPTOTIC WAVEFORM EVALUATION (AWE) AND COMPLEX FREQUENCY HOPPING (CFH)

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Abstract This report discusses two powerful techniques to approximate the transfer function of linear networks. The first technique is called Asymptotic Waveform Evaluation (AWE) to approximate the transfer function of both lumped and distributed linear element networks by calculating the approximate poles and their corresponding residues of the transfer function. The second technique Complex Frequency Hopping (CFH) applies AWE several times in a specific frequency range to extract the accurate dominant poles of the network.

I. INTRODUCTION

Moment matching techniques have been proved to be efficient methods to approximate the transfer function of linear networks. The Asymptotic Waveform Evaluation (AWE) first introduced by Pillage and Rohrer in [1] is one of this techniques. It uses Padé approximation [2] to approximate the transfer function of a linear network as a rational function (a ratio of two polynomials). Then, it matches this rational function with the moments of the output response. These moments are defined as the coefficients of Taylor series expansion of the output response around a fixed point s_0 in the complex frequency plane. AWE can be used to approximate the transfer function of both linear lumped element networks [1] and networks with distributed elements [3]. In spite of being a powerful technique to

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approximate the transfer function, it sometimes generates unstable poles (poles in the positive half of the complex frequency plane) for well known stable networks. Besides AWE generates only approximate poles not accurate ones. Complex Frequency Hopping (CFH) on the other hand overcomes some of AWE's problems in the sense that it generates only stable accurate dominant poles in a specific frequency range. It applies AWE at different expansion points (called hops) inside a range of frequency from 0 to f_{max} to extract the dominant poles of the transfer function. This report is organised as two main parts. The first part surveys AWE technique while the second part surveys CFH. We will consider the application of AWE and CFH in linear lumped element networks and leave the application in distributed elements networks as a future work.

II. ASYMPTOTIC WAVEFORM EVALUATION (AWE)

The AWE is a technique used to approximate the transfer function of a linear network around an expansion point s_0 in the complex frequency plane. It was originally introduced to evaluate the transfer function of lumped element networks [1] and later extended to deal with networks with distributed elements [3] such as transmission lines. This technique gives good approximation around the expansion point s_0 and its accuracy degrades as we move far from s_0 . It can be used for both transient and steady state analysis.

Consider a system with transfer function $H(s)$. The Taylor expansion around $s = 0$ is given by [3]

$$H(s) = m_0 + m_1 s + m_2 s^2 + \dots, \quad (1)$$

and the Laurent expansion is [3]

$$H(s) = \dots + m_{-2} s^{-2} + m_{-1} s^{-1} + m_0 + m_1 s + m_2 s^2 + \dots \quad (2)$$

The coefficients m_{-2} , m_{-1} , m_0 , m_1 and m_2 are called the moments of the transfer function [1,2]. It is an essential step in the AWE technique to generate these moments. Once these moments have been

generated, they are matched using the Padé approximation [2]. Therefore, in the AWE technique, the following two processes should be considered, moment generation and moment matching.

Moment Generation

A moment generation algorithm for lumped elements networks was first introduced in [1]. The algorithm was extended to evaluate the moments for networks containing distributed elements such as lossy coupled transmission lines in [3]. Suppose that we have a lumped element network. The modified nodal admittance network equations can be written as [4]

$$\mathbf{Y}(s)\mathbf{X}(s) = \mathbf{E} \quad (3)$$

where $\mathbf{Y}(s)$ is the nodal admittance matrix, $\mathbf{X}(s)$ is the vector of output variables and \mathbf{E} is the vector of input sources. Therefore, $\mathbf{X}(s)$ can be found by

$$\mathbf{X}(s) = \mathbf{Y}^{-1}(s) \mathbf{E}. \quad (4)$$

The Taylor expansion of $\mathbf{X}(s)$ around $s = 0$ is given by [3]

$$\mathbf{X}(s) = \sum_{n=0}^{\infty} \mathbf{M}_n s^n, \quad (5)$$

where the vector of moments \mathbf{M}_n is defined by [3]

$$\mathbf{M}_n = \frac{\left[\frac{d^n}{ds^n} \mathbf{Y}^{-1}(s) \right]_{s=0} \mathbf{E}}{n!}. \quad (6)$$

Suppose that a single output i is selected from the set of output variables vector $\mathbf{X}(s)$. The transfer function for this variable is given by,

$$H_i(s) = m_{i,0} + m_{i,1} s + m_{i,2} s^2 + \dots \quad (7)$$

where $m_{i,0}$, $m_{i,1}$ and $m_{i,2}$ are the i th elements of the moments vectors \mathbf{M}_0 , \mathbf{M}_1 and \mathbf{M}_2 . For simplicity the moments $m_{i,0}$, $m_{i,1}$ and $m_{i,2}$ of the output variable i will be referred to as m_0 , m_1 and m_2 , respectively.

It is not easy to compute the moments from (6) because of the derivatives of the inverse of the matrix $\mathbf{Y}(s)$, so a recursive formula [3] has been used to evaluate the moments vector \mathbf{M}_n . This recursive

formula is given by

$$[\mathbf{Y}(s)]_{s=0} \mathbf{M}_n = - \sum_{k=1}^n \frac{[\frac{d^k}{ds^k} \mathbf{Y}(s)]_{s=0}}{k!} \mathbf{M}_{n-k}, \quad (8)$$

with

$$[\mathbf{Y}(s)]_{s=0} \mathbf{M}_0 = \mathbf{E}. \quad (9)$$

A simplification of equations (8) and (9) for some lumped linear element networks is presented in Appendix A. The results obtained in Appendix A will be used in the examples considered in this report.

Moment Matching

Once the moments have been generated, they are matched using the Padé approximation [2]. Suppose that we have an input source(s) and a specified output i , the frequency response of the output i of a linear network can be written as

$$H_i(s) = c_i + \sum_{j=1}^{N_p} \frac{k_j}{s - p_j}, \quad (10)$$

where $\{p_j\}$ are the complex frequency N_p poles and $\{k_j\}$ are the corresponding residues. The constant c_i is the direct coupling constant between the input and the output and is oftentimes zero (c_i is zero in all examples presented in this report). This response can be formulated using Padé approximation as

$$H_i(s) \approx \frac{P_L(s)}{Q_M(s)}. \quad (11)$$

The numerator polynomial $P_L(s)$ and the denominator polynomial $Q_M(s)$ are of order L and M , respectively, and given by

$$P_L(s) = a_0 + a_1 s + a_2 s^2 + \dots + a_L s^L \quad (12)$$

$$Q_M(s) = 1 + b_1 s + b_2 s^2 + \dots + b_M s^M. \quad (13)$$

In the Padé approximation there is no restriction on L and M , however a choice of $L = M-1$ or $L = M$ is recommended [3]. The coefficients b_1, b_2, \dots, b_M are derived from the moments [2] by solving the system

of linear equations (14).

Hence, the approximate poles of the transfer function are found by calculating the roots of the denominator polynomial $Q_M(s)$ in (15).

$$\begin{bmatrix} m_{L-M+1} & m_{L-M+2} & \cdots & m_L \\ m_{L-M+2} & m_{L-M+3} & \cdots & m_{L+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m_L & m_{L+1} & \cdots & m_{L+M-1} \end{bmatrix} \begin{bmatrix} b_M \\ b_{M-1} \\ \cdot \\ \cdot \\ \cdot \\ b_1 \end{bmatrix} = \begin{bmatrix} m_{L+1} \\ m_{L+2} \\ \cdot \\ \cdot \\ \cdot \\ m_{L+M} \end{bmatrix} \quad (14)$$

$$1 + b_1 s + b_2 s^2 + \dots + b_M s^M = 0. \quad (15)$$

As mentioned in [1] it is much better to generate a transient response in terms of q poles than evaluating the denominator and numerator polynomials in the Padé approximation. Thus, the approximated transfer function is defined as

$$H_i(s) \approx c_i^{\wedge} + \sum_{j=1}^q \frac{k_j^{\wedge}}{s - p_j^{\wedge}}, \quad (16)$$

where q is the number of poles p_j^{\wedge} (notice that q is less than or equal to the total number of poles N_p).

The time domain response is simply

$$h_i(t) = c_i^{\wedge} \mathbf{d}(t) + \sum_{i=1}^q k_j^{\wedge} \exp(p_j^{\wedge} t). \quad (17)$$

By matching the moments in (7) with the approximated transfer function in (16) the approximated poles $\{p_j^{\wedge}\}$ and their corresponding residues are found from the moments [3] as follows:

The reduced characteristic polynomial is given by

$$a_0 + a_1 p^{-1} + a_2 p^{-2} + a_3 p^{-3} + \dots + a_{q-1} p^{-q+1} + p^{-q} = 0 \quad (18)$$

where the coefficients a_0, a_1, \dots, a_{q-1} are related to the moments by [3]

$$\begin{bmatrix} m_0 & m_1 & \dots & m_{q-1} \\ m_1 & m_2 & \dots & m_q \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ m_{q-1} & m_q & \dots & m_{2q-2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ \cdot \\ \cdot \\ a_{q-1} \end{bmatrix} = - \begin{bmatrix} m_q \\ m_{q+1} \\ \cdot \\ \cdot \\ \cdot \\ m_{2q-1} \end{bmatrix}. \quad (19)$$

After solving for a_0, a_1, \dots, a_{q-1} the reciprocal of the poles are calculated from equation (18). Having the approximated poles $p_1^{\wedge}, p_2^{\wedge}, \dots, p_q^{\wedge}$ the corresponding residues are related to the moments [3] by the system of linear equations

$$\begin{bmatrix} p_1^{-1} & p_2^{-1} & \dots & p_q^{-1} \\ p_1^{-2} & p_2^{-2} & \dots & p_q^{-2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_1^{-q} & p_2^{-q} & \dots & p_q^{-q} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ \cdot \\ \cdot \\ k_q \end{bmatrix} = - \begin{bmatrix} m_0 \\ m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_{q-1} \end{bmatrix}. \quad (20)$$

Therefore, the AWE technique around the expansion point $s = 0$ uses equations (18), (19) and (20) to compute the approximated poles and their corresponding residues. Appendix B shows the modification required to use the AWE technique around any arbitrary point $s = \mathbf{a} + j\mathbf{b}$ in the complex frequency plane.

The following steps summarize the AWE technique for evaluating the frequency response of an electric network.

- (1) Formulate the nodal admittance matrix $\mathbf{Y}(s)$.
- (2) Get the moments vector \mathbf{M}_n using (8) and (9) around $s = 0$ or $s = \mathbf{a} + j\mathbf{b}$.
- (3) Use the Padé approximation (16) to approximate the frequency response of the system.
- (4) Get the approximate poles and their corresponding residues using (18), (19) and (20) or the results obtained in Appendix B.

III. EXAMPLES OF AWE

The AWE technique has been implemented as a MATLAB [5] program in this report. The results obtained by the program are compared with either the analytical solution (as in example 1) or by the response obtained by using the OSA90/hope [6] simulator (as in example 2).

Example 1

Consider the g - l - c parallel resonant circuit shown in Fig. 1, where $g = 0.002 \Omega^{-1}$, $l = 1$ mH and $c = 2500 \mu\text{F}$. The transfer function of this circuit is defined by

$$H(s) = \frac{V_o(s)}{I_i(s)}.$$

The nodal admittance equation for this network is given by

$$\left[g + \frac{1}{sl} + sc \right] V_o(s) = I_i(s),$$

which means that the admittance matrix $\mathbf{Y}(s)$ is simply a scalar. The analytical solution for this problem is given by the expression

$$H(s) = \frac{V_o(s)}{I_i(s)} = \frac{1}{g + \frac{1}{sl} + sc} = \frac{sl}{s^2 lc + slg + 1}.$$

The actual poles of $H(s)$ can be calculated by solving the 2nd order polynomial $s^2 lc + slg + 1 = 0$ and their values are $p_{1,2} = -0.3947 \pm j628.32$.

By using the AWE technique around the expansion point $s_0 = j2\pi f_0$, where $f_0 = 100$ Hz is the resonant frequency, the values of the approximated poles obtained in the 1st (using only one approximated pole) and the 2nd (using two approximated poles) order approximations, respectively, as well as the actual poles are shown in Table I. The values of the approximated poles in the 2nd order approximation (in which the number of approximated poles is the same as the total number of poles) are the same as the values of the actual poles. This is always true as long as there is no error involved in

computing the moments of the transfer function. Fig. 2 shows the approximated transfer function obtained by using one approximated pole versus the actual transfer function $H(s = j\omega)$ in dB. From this figure we notice that the AWE technique gives good results around the expansion point s_0 and its accuracy degrades as we move far from s_0 . Fig. 3 shows the approximated transfer function in case of using two approximated poles versus the actual transfer function in dB.

Example 2

The second example is a 5th order ladder-prototype LPF [7] shown in Fig. 4. The total number of poles is 5 (the same number as the order of the filter). The nodal admittance equation for this circuit is given by

$$\mathbf{Y}(s) \mathbf{X}(s) = \mathbf{E},$$

where

$$\mathbf{Y}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -g_s & (g_s + \frac{1}{sL_1} + sC_1) & -\frac{1}{sL_1} & 0 \\ 0 & -\frac{1}{sL_1} & (\frac{1}{sL_1} + \frac{1}{sL_2} + sC_2) & -\frac{1}{sL_2} \\ 0 & 0 & -\frac{1}{sL_2} & (g_0 + \frac{1}{sL_2} + sC_3) \end{bmatrix},$$

$$\mathbf{X}(s) = [V_1 \ V_2 \ V_3 \ V_4]^T \text{ and } \mathbf{E} = [1 \ 0 \ 0 \ 0]^T.$$

It is clear that the matrix $\mathbf{Y}(s)$ can be decomposed into a sum of three matrices as given by (A1) in Appendix A, so the moments of this system can be generated simply using equations (A5), (A6) and (A7) in Appendix A. The AWE technique has been used around the expansion point $s_0 = j2\pi f_0$, $f_0 = 0.5$ Hz to analyse this circuit. The approximated poles obtained in the 2nd, 3rd and 5th order approximations, respectively, are shown in Table II.

Figs. 5, 6 and 7 show the transfer function obtained by using OSA90/hope [6] simulator in dB versus the approximated transfer function obtained by using the 2nd, 3rd and 5th order AWE

approximation, respectively.

In Fig. 5 and 6 we notice that the approximated transfer function obtained by AWE gives precise results around the expansion point s_0 (which corresponds to $f_0 = 0.5$ Hz in this example) and poor results as we move far from s_0 . In Fig. 7 the approximated transfer function matches the simulated one (obtained by the OSA90/hope simulator) which means that the approximated poles in the 5th order approximation are the same as, or at least very close to, the actual poles of the transfer function.

V. COMPLEX FREQUENCY HOPPING (CFH)

Complex Frequency Hoping (CFH) is a technique to approximate the transfer function of lumped and distributed networks [10] and it has been extended to solve electromagnetic (EM) fields problems [11]. It exploits the AWE technique by applying it many times around different expansion points (called hops) within a specific frequency range (usually from $\mathbf{w} = 0$ to $\mathbf{w} = \mathbf{w}_{max} = j2\mathbf{f}_{max}$) to extract the accurate dominant poles (and their corresponding residues) of a network in this range of frequency.

The expansion point s_0 is called “hop” and must lie within the specified frequency range (i.e., from $\mathbf{w} = 0$ to $\mathbf{w} = \mathbf{w}_{max}$). CFH uses two hops at a time, starting with the hop at $s = j\mathbf{w} = 0$ and the hop at $s = j\mathbf{w}_{max} = j2\pi\mathbf{f}_{max}$ then finds out if any accurate dominant pole(s) can be extracted from these two hops (this process is called pole(s) selection). If no dominant poles can be extracted from these two hops, it searches for another two hops within the specified frequency range and repeats the whole procedure. CFH chooses the next two hops using a binary search strategy [10] (this will be explained in the search strategy section). Therefore CFH performs two processes: The first process is “pole(s) selection” to check and extract the dominant poles between two hops; and the second process is to choose the most adequate search strategy (to search for the next hop) that minimizes the total number of hops used. These two processes will be examined in the next two sections.

Pole Selection

Suppose that two hops have been generated. The question is that if these two hops do have

accurate dominant pole(s) how can these poles be detected and extracted? Recall that the AWE gives accurate results around the expansion point and inaccurate results for points far from the expansion point. In other words, the poles very close to the expansion point s_0 are more accurate than the poles far from s_0 . Therefore, if a pole p is confirmed to be accurate, all poles closer than p to s_0 are also tagged as accurate [10]. The outermost pole (the farthest pole from the expansion point s_0) which has been confirmed to be accurate defines the radius of accuracy $R_{accuracy}$ and consequently all poles within the circle centred at s_0 and with radius $R_{accuracy}$ are marked as accurate dominant poles. If a pole is detected in two hops then this pole is confirmed accurate and the radius of accuracy $R_{accuracy}$ is defined as its distance from the expansion point s_0 .

The following steps summarise the Pole Selection algorithm which is used to extract the accurate dominant poles from two hops [10].

Step 1 Generate poles and residues for each hop using (18), (19) and (20).

Step 2 If the same pole (with error tolerance ϵ_{err}) is detected in two different hops, it is marked as accurate.

Step 3 The distance between a hop (i.e., the expansion point) and its farthest confirmed accurate pole defines a radius of accuracy $R_{accuracy}$. All poles within $R_{accuracy}$ are marked as accurate and their corresponding residues are also marked as accurate.

Step 4 Poles/residues not marked by the steps above are rejected since there is no confirmation of their accuracy.

Fig. 8 illustrates the Pole Selection algorithm.

Search Strategy

CFH begins by applying AWE around two hops (one at $s = j\omega = 0$ and the other at $s = j\omega = j\omega_{max}$) to find two expansions then finds out whether there are common poles between the two expansions or not. If there are some common poles, it collects the accurate dominant poles from the two expansions

buy using the Pole Selection algorithm above and terminates. If not, it applies the AWE a round a new hope (a new expansion point) within the specified frequency range (from $\omega = 0$ to $\omega = \omega_{max}$). Therefore, there must be an efficient search strategy to select this new hop in order to minimize the total number of hops required to generate all dominant poles within the frequency range.

For any search strategy to be efficient it must be subjected to several constraints [10].

(1) The search should be confined to the upper left half of the complex frequency plane. Since stable systems (networks) have conjugate poles in the left half plane or in the negative part of the real axis. One of the two left half quadrants is chosen as a search area. In CFH [10] the upper left quadrant has been chosen (arbitrary) as the search area. Poles in the lower left quadrant can be duplicated from the poles in the upper left quadrant.

(2) The search should be confined to or near the imaginary axis. Poles near the imaginary axis ($j\omega$) are critical to both the transient and frequency response. The contribution of poles far from ($j\omega$) is very transient in the time domain and has no much influence [10] in the frequency domain.

(3) The search should be confined to points lower than the maximum frequency f_{max} [10]. Poles much higher than f_{max} have no influence in the response of the network within the working frequency range (from $f = 0$ to $f = f_{max}$).

The above constraints reduce the search area in the CFH and consequently minimize the number of hops required to generate all accurate dominant poles in the frequency range.

Using this search area an algorithm was developed in [10] using a binary search strategy to extract accurate dominant poles. The algorithm can be summarised in the following steps.

Step 1 Apply AWE around $s = 0$ and $s = j\omega_{max} = j2\pi f_{max}$, respectively and compute the poles and their corresponding residues using equations (18), (19) and (20).

Step 2. If there is a common pole(s) between the two hops $s = 0$ and $s = j\omega_{max} = j2\pi f_{max}$, terminate. Define a radius of accuracy for each hop and collect poles and residues between the two hops using the

Pole Selection algorithm. Else apply the AWE at the midpoint $s = j\mathbf{w}_{max}/2$ between the two end points.

Step 3 Continue the search in a similar binary manner between the hops at $s = 0$ and $s = j\mathbf{w}_{max}/2$ and at $s = j\mathbf{w}_{max}/2$ and $s = j\mathbf{w}_{max}$. If no common poles between $s = 0$ and $s = j\mathbf{w}_{max}/2$ were found define a new hop at $s = j\mathbf{w}_{max}/4$ and continue the search between the two hops at $s = 0$ and $s = j\mathbf{w}_{max}/4$ and the two hops at $s = j\mathbf{w}_{max}/4$ and $s = j\mathbf{w}_{max}/2$. The same search strategy is repeated if no common poles were found between the two hops $s = j\mathbf{w}_{max}/2$ and $s = j\mathbf{w}_{max}$.

Step 4 Stop the search between any two hops if they are nearly equal with respect to an error tolerance ϵ_{err} .

The flow chart of this algorithm is shown in Fig. 9.

VI. EXAMPLES OF CFH

In this report a MATLAB [5] program has been written to implement the CFH technique. The program was tested in the two examples presented in Section III in this report. In each example the results obtained by CFH technique and AWE technique are compared.

Example 1

This example is the same as Example 1 mentioned in Section III. The dominant poles of the transfer function are computed using CFH in the frequency range from $f = 0$ Hz to $f = 200$ Hz and the total number of moments used is 4. The program terminated after 2 hops and produced 2 dominant accurate poles. Table III shows the dominant poles produced by CFH, the poles produced by AWE around the expansion point $s_0 = j2\pi f_0$, where $f_0 = 100$ Hz and using 4 moments and the actual poles obtained from the analytical solution.

Example 2

The second example is the circuit shown in Fig. 4. This example was analyzed before using AWE technique in Section III of this report. The nodal admittance equation of this circuit is mentioned

in Example 2 in Section III. The CFH in the frequency range from $f = 0$ Hz to $f = 1$ Hz is used to evaluate the accurate dominant poles of the transfer function of this network. The number of moments used in each hop is 8. The program converged after 22 hops and produced 5 dominant poles. Table IV shows the dominant poles obtained by CFH and the poles obtained by using AWE around the expansion point $s_0 = j2\pi f_0$, where $f_0 = 0.5$ Hz. It is clear that the result obtained by CFH is much more accurate than the result obtained by AWE. Fig. 10 shows the transfer function by CFH, AWE and the OSA90/hope [6] simulator in dB in the frequency range from $f = 0$ Hz to $f = 1$ Hz. When applying the CFH technique in the same frequency range but using 10 moments instead of 8, the technique produces 5 dominant poles after only 2 hops. The dominant poles obtained by the CFH and those obtained by the AWE using 10 moments around the expansion point $s_0 = j2\pi f_0$, where $f_0 = 0.5$ Hz, are shown in Table V. It is clear that the poles obtained by CFH are the same as the poles obtained by AWE in this case. This because AWE produces the actual poles if the total number of poles of the network (5 in this example) is used (assuming that the moments are calculated precisely).

CONCLUSIONS

This report surveyed two powerful techniques to approximate the transfer function of linear networks. The first technique (AWE) matches the moments of the linear network using Padé approximation to generate the approximate poles and their corresponding residues of the transfer function. It gives good results for some linear networks, but it suffers from instability problems, that is it may generate unstable poles for well known stable networks. The second technique (CFH) on the other hand applies AWE at different expansion points (hops) within a specific frequency range to generate the accurate dominant poles inside this range. All poles generated by CFH are guaranteed to be stable since unstable poles are rejected during the running of the algorithm. The application of AWE and CFH in lumped element networks is only considered here while the application in networks with distributed elements has been left as a future work.

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APPENDIX A

A Simplification of Formulas (8) and (9)

For some linear lumped elements networks (the networks analyzed in this report) the modified admittance matrix can be decomposed into the sum of three matrices as follows:

$$\mathbf{Y}(s) = \mathbf{G} + \frac{\mathbf{L}}{s} + s \mathbf{C}, \quad (\text{A1})$$

where \mathbf{G} , \mathbf{L} and \mathbf{C} are constant matrices. The elements of \mathbf{G} are function of the conductive elements in the network, the elements of \mathbf{L} are function of the inductive elements in the network and the elements of \mathbf{C} are function of the capacitive elements in the network and, of course, the values of the elements of \mathbf{G} , \mathbf{L} and \mathbf{C} depends on the network topology. It is clear that $\mathbf{Y}(s)$ and its derivatives are not defined at $s = 0$ unless $\mathbf{L} = \mathbf{0}$. Therefore, the expansion point used in the AWE should not be zero unless $\mathbf{L} = \mathbf{0}$. The values of $\mathbf{Y}(s)$ and its derivatives at s_0 are given by

$$\mathbf{Y}(s_0) = \mathbf{G} + \frac{\mathbf{L}}{s_0} + s_0 \mathbf{C} \quad (\text{A2})$$

$$\mathbf{Y}^{(1)}(s_0) = \mathbf{C} - \frac{\mathbf{L}}{s_0^2} \quad (\text{A3})$$

$$\mathbf{Y}^{(n)}(s_0) = (-1)^n \frac{n!}{s_0^{n+1}} \mathbf{L}, \quad n \geq 2. \quad (\text{A4})$$

Therefore, after substituting by $\mathbf{Y}(s_0)$, $\mathbf{Y}^{(1)}(s_0)$, and $\mathbf{Y}^{(n)}(s_0)$ in the recursive relations (8) and (9) they become

$$\mathbf{Y}(s_0) \mathbf{M}_0 = \mathbf{E} \quad (\text{A5})$$

$$\mathbf{Y}(s_0) \mathbf{M}_1 = -\left[\mathbf{C} - \frac{\mathbf{L}}{s_0^2}\right] \mathbf{M}_0 \quad (\text{A6})$$

$$\mathbf{Y}(s_0) \mathbf{M}_n = -\left(\mathbf{C} - \frac{\mathbf{L}}{s_0^2}\right) \mathbf{M}_{n-1} - \mathbf{L} \sum_{r=1}^n \frac{(-1)^r}{s_0^{r+1}} \mathbf{M}_{n-r}, \quad n \geq 2. \quad (\text{A7})$$

Equations (A5), (A6) and (A7) will be used to generate the moments in the examples presented in this report.

APPENDIX B

AWE around an arbitrary point in the complex frequency plane [10]

The Taylor expansion of the vector of output variables $\mathbf{X}(s)$ around a complex frequency point $s = s_0 = \mathbf{a} + j\mathbf{b}$ is given by

$$\mathbf{X}(s) = \sum_{n=0}^{\infty} \mathbf{M}'_n (s - s_0)^n, \quad (\text{B1})$$

where the vector of moments \mathbf{M}'_n is the same as \mathbf{M}_n in equation (6) but after substituting with $s = s_0$ in the derivatives of $\mathbf{Y}^{-1}(s)$ and it can also be computed by using the recursive formulas (8) and (9) at $s = s_0$.

The transfer function corresponding to the output variable i in terms of the moments of the network is given by

$$H_i(s) = m'_{i,0} + m'_{i,1} (s - s_0) + m'_{i,2} (s - s_0)^2 + \dots, \quad (\text{B2})$$

where $m'_{i,0}$, $m'_{i,1}$ and $m'_{i,2}$ are the moments of the output variable i and, for simplicity, they will be referred to as m_0 , m_1 and m_2 . After replacing $(s - s_0)$ by a new variable s' (B2) becomes

$$H_i(s') = m_0 + m_1 s' + m_2 s'^2 + \dots. \quad (\text{B3})$$

Also the approximated transfer function in terms of q poles is given by

$$H_i(s) \approx c_i + \sum_{j=1}^q \frac{k_j}{s - p_j}. \quad (\text{B4})$$

Using the substitution $s' = s - s_0$ and denoting $(p_j - s_0)$ by p'_j (where the set $\{p'_j\}$ is called the set of shifted poles) equation (B4) becomes

$$H_i(s') \approx c_i + \sum_{j=1}^q \frac{k_j}{s' - p'_j}. \quad (\text{B5})$$

Therefore, equations (B3) and (B5) are matched to compute the set of shifted poles $\{p'_j\}$ and their corresponding residues $\{k_j\}$. It is clear that equations (B3) and (B5) are similar to equations (7)

and (16). Thus, the shifted poles and their corresponding residues can be computed from (18), (19) and (20). Finally, the poles of transfer function $\{\hat{p}_j\}$ are simply obtained by adding s_0 to each element of the set of shifted poles $\{p'_j\}$.

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TABLE I
ACTUAL AND APPROXIMATED POLES OF THE
PARALLEL RESONANT CIRCUIT IN FIG. 1

1 st order poles	2 nd order poles	Actual poles
$-0.39478 + j628.31$	$-0.39478 + j628.31$	$-0.39478 + j628.31$
	$-0.39478 - j628.31$	$-0.39478 - j628.31$

TABLE II
APPROXIMATED POLES OF THE TRANSFER
FUNCTION IN FIG. 4

2nd order approximation	3rd order approximation	5th order approximation
$-0.6242 + j1.9657$	$0.9047 + j1.2235$	$-0.0549 + j0.9659$
$0.5308 + j1.9210$	$0.0645 + j1.4404$	$-0.0549 - j0.9659$
	$-1.0529 + j1.3028$	$-0.1436 + j0.5970$
		$-0.1436 - j0.5970$
		-0.1775

TABLE III
POLES OBTAINED BY CFH, AWE AND THE ANALYTICAL
SOLUTION OF THE PARALLEL RESONANT CIRCUIT IN FIG. 1

Poles obtained by CFH	Poles obtained by AWE	Actual poles
$-0.39478 + j628.31$	$-0.39478 + j628.31$	$-0.39478 + j628.31$
$-0.39478 - j628.31$	$-0.39478 - j628.31$	$-0.39478 - j628.31$

TABLE IV
 POLES OBTAINED BY CFH (USING 8 MOMENTS IN EACH HOP) AND
 BY AWE (USING 8 MOMENTS), OF THE TRANSFER FUNCTION IN FIG. 4

Poles obtained by CFH	Poles obtained by AWE
-0.1781	$0.9692 + j0.3051$
$-0.1436 + j0.5970$	$0.1304 + j0.9955$
$-0.1436 - j0.5970$	$-0.2978 + j0.9910$
$-0.0543 + j0.9626$	$-1.1787 + j0.4117$
$-0.0543 - j0.9626$	

TABLE V
 POLES OBTAINED BY CFH (USING 10 MOMENTS IN EACH HOP) AND
 BY AWE (USING 10 MOMENTS) OF THE TRANSFER FUNCTION IN FIG. 4

Poles obtained by CFH	Poles obtained by AWE
-0.1775	-0.1775
$-0.1436 + j0.5970$	$-0.1436 + j0.5970$
$-0.1436 - j0.5970$	$-0.1436 - j0.5970$
$-0.0549 + j0.9659$	$-0.0549 + j0.9659$
$-0.0549 - j0.9659$	$-0.0549 - j0.9659$

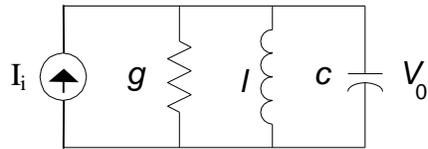


Fig. 1. Parallel resonant circuit.

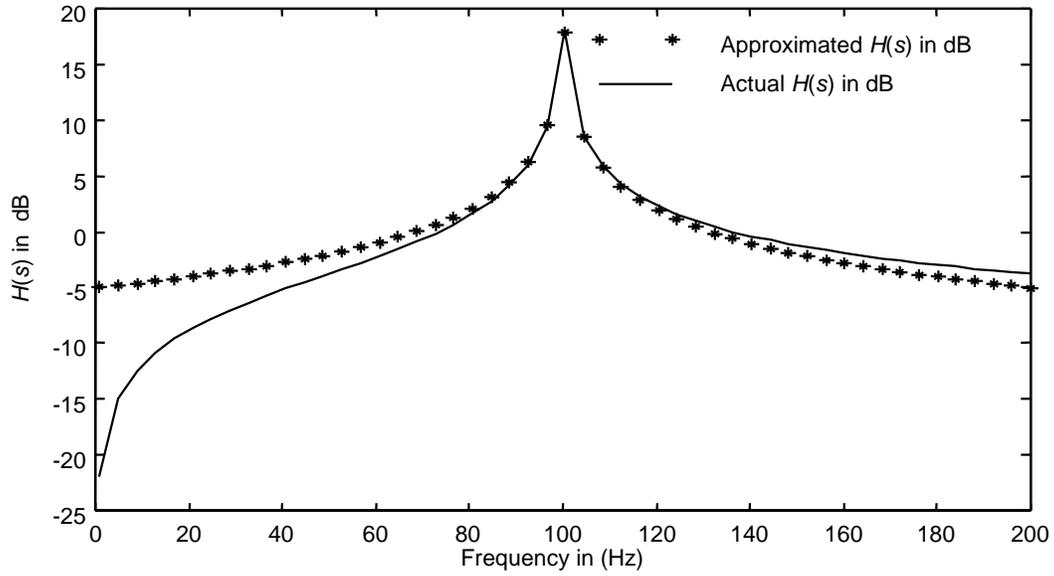


Fig. 2. AWE 1st order approximation response versus the actual response.

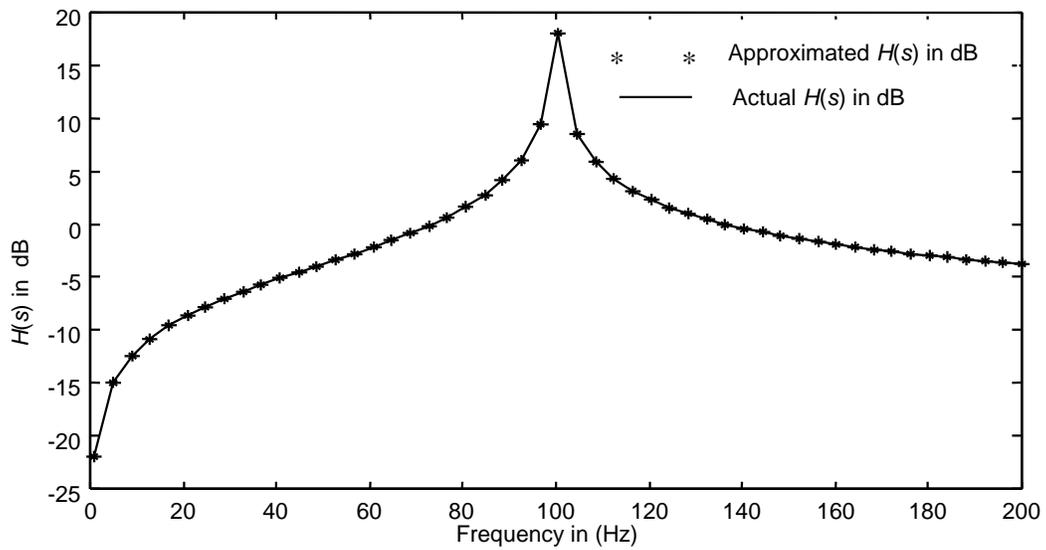
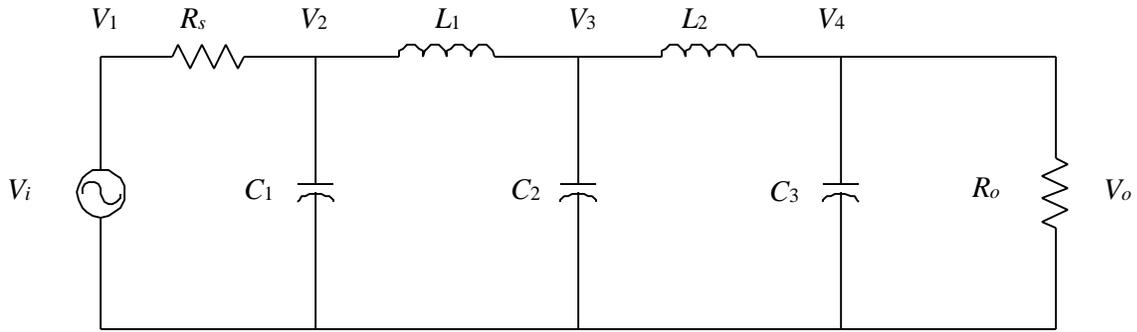


Fig. 3. AWE 2nd order approximation response versus the actual response.



$R_s = R_o = 1 \text{ ohm}, L_1 = L_2 = 0.7618 \text{ H}, C_1 = C_3 = 3.482 \text{ F}, C_2 = 4.538 \text{ F}$

Fig. 4. 5th order ladder-prototype LPF.

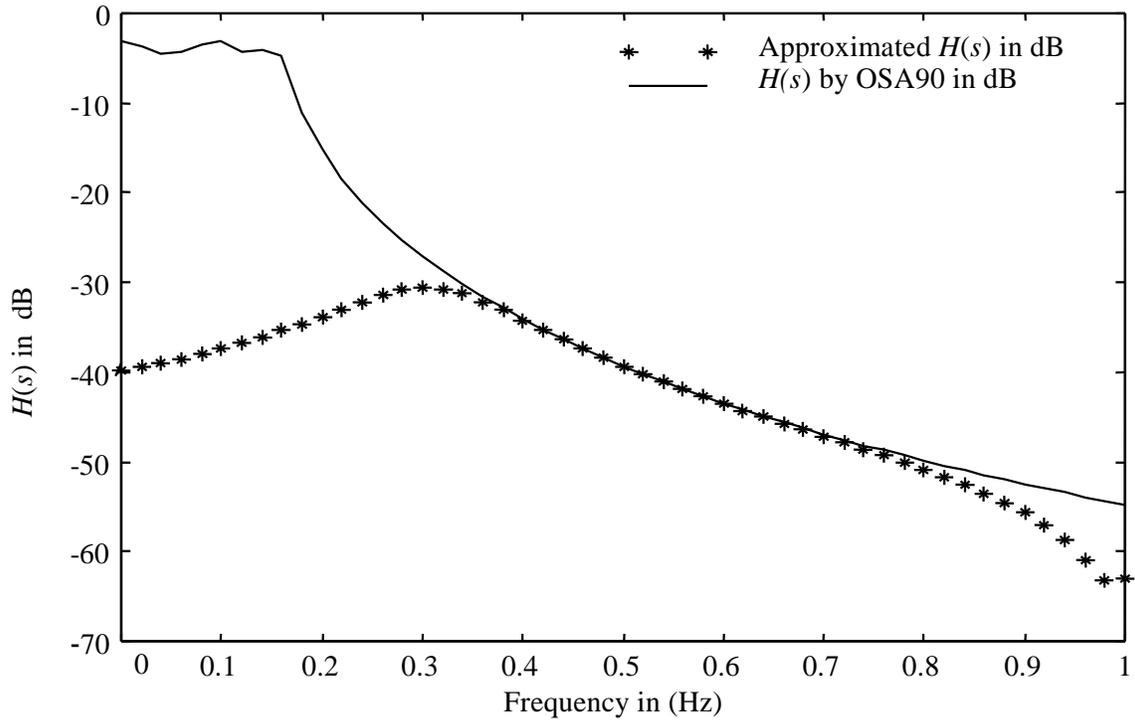


Fig. 5. 2nd order approximation response versus the response obtained by the OSA90/hope simulator.

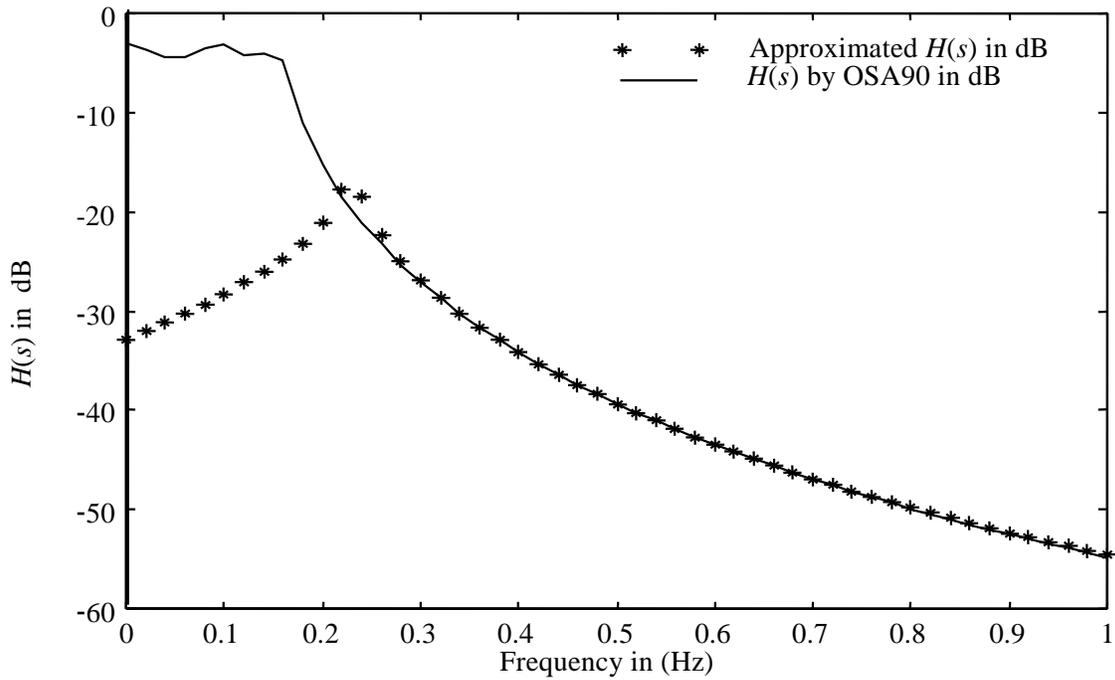


Fig. 6. 3rd order approximation response versus the response obtained by the OSA90/hope [6] simulator.

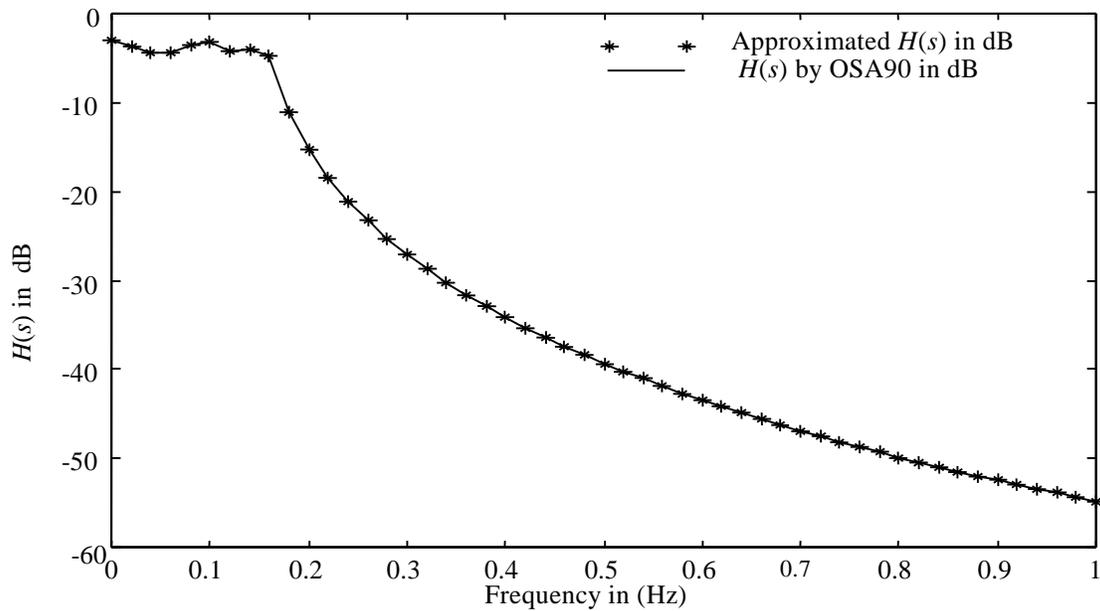


Fig. 7. 5th order approximation response versus the response obtained by the OSA90/hope [6] simulator.

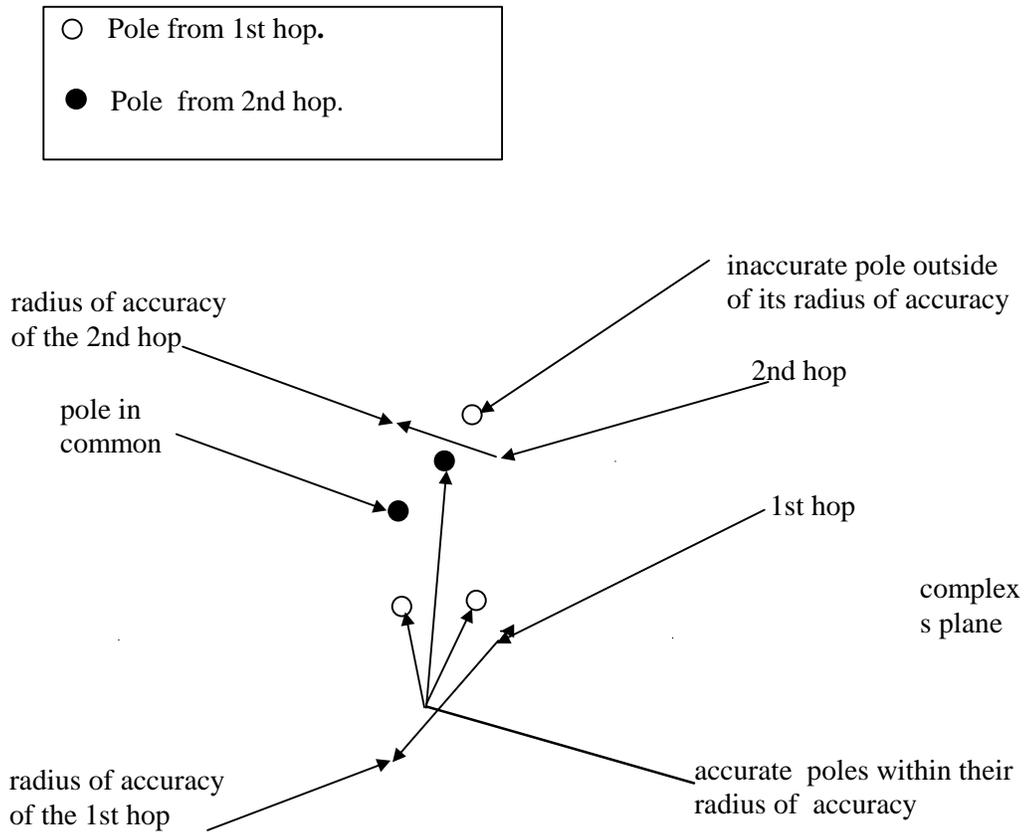


Fig. 8. Pole selection algorithm for two hops [10].

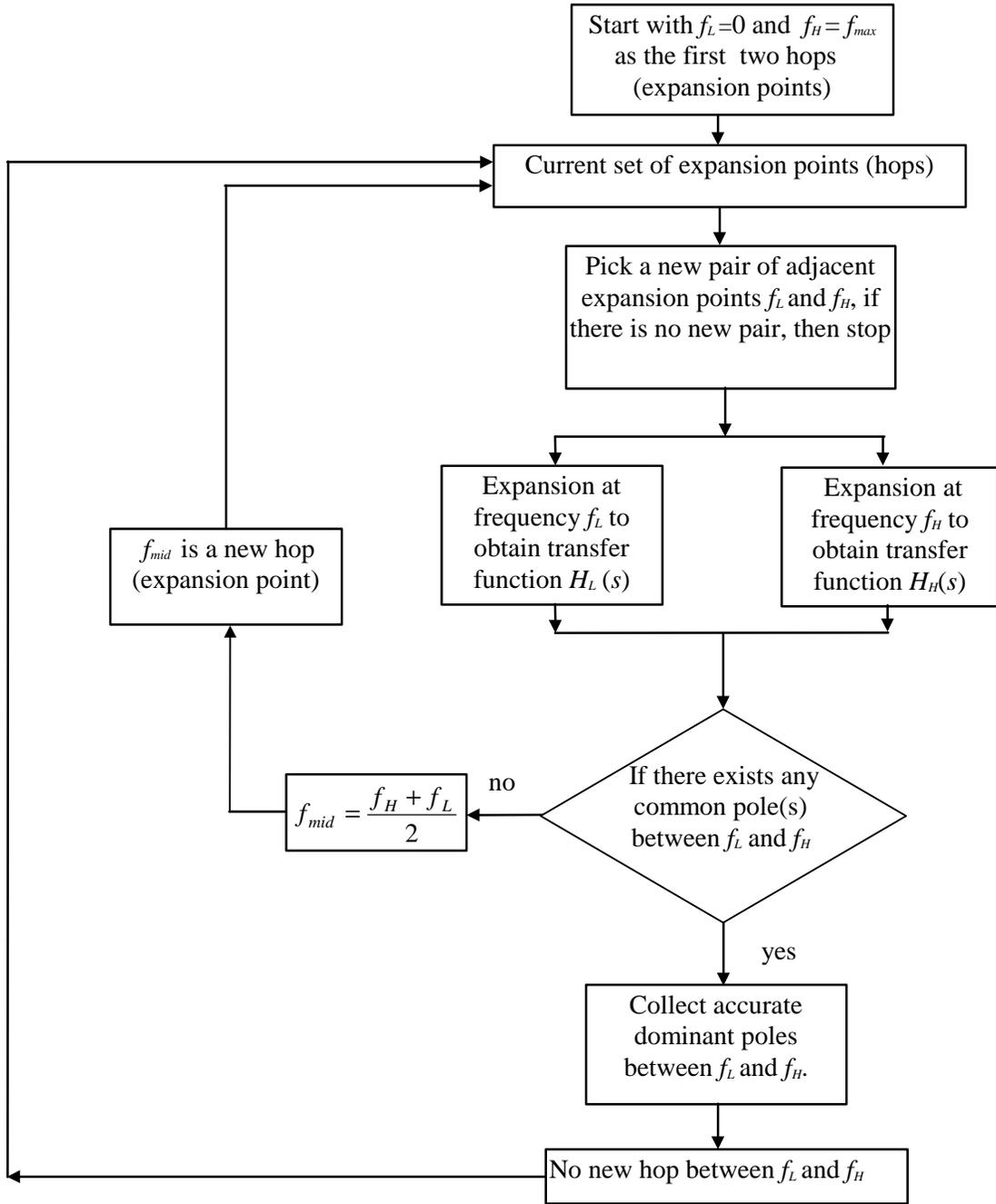


Fig. 9. CFH flow chart.

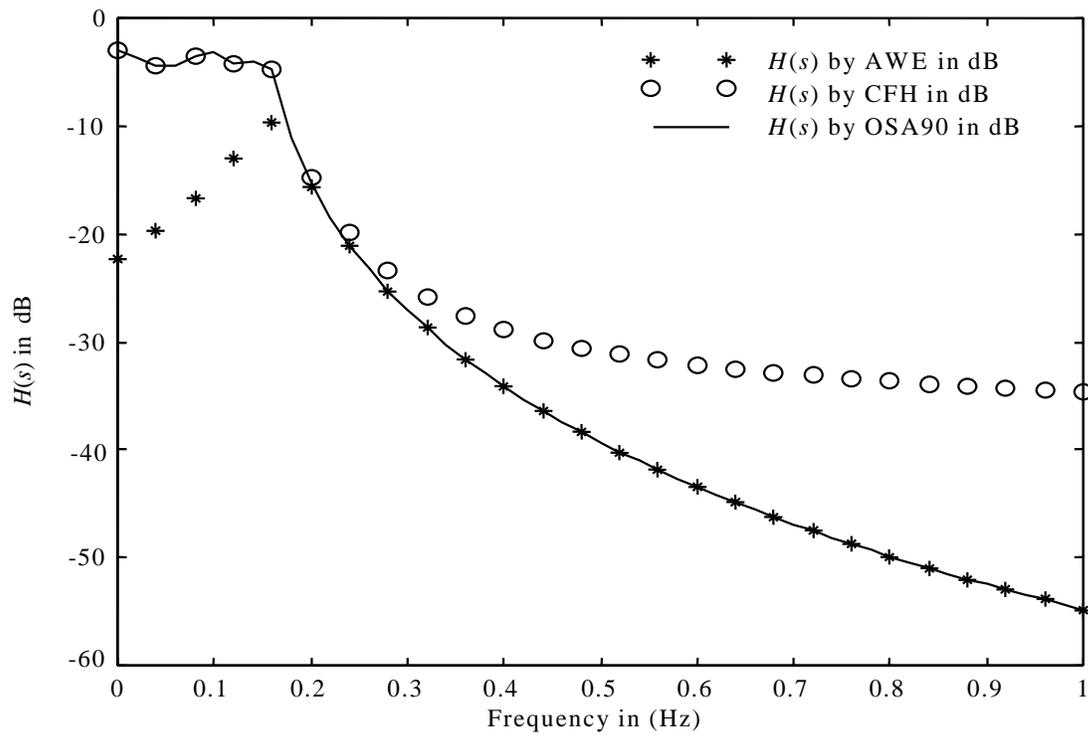


Fig. 10. The response obtained by CFH, AWE and OSA90/hope simulator for the transfer function in Fig. 4.

