[2] J. A. Kong, "Reciprocity relationships for bianisotropic media," *Proc. IEEE* (Lett.), vol. 58, pp. 1966–1967, Dec. 1970.
[3] I. V. Lindell, "Some properties of lossless bianisotropic media," *Proc. IEEE* (Lett.), vol. 60, pp. 463–464, Apr. 1972.
[4] R. C. Costen and D. Adamson, "Three-dimensional derivation of the electrodynamic jump conditions and momentum-energy laws at a moving boundary," *Proc. IEEE*, vol. 53, pp. 1181–1196, Sept. 1965.
[5] H. C. Chen and D. K. Cheng, "Constitutive relations for a moving anisotropic medium," *Proc. IEEE* (Lett.), vol. 54, pp. 62–63, Jan. 1966.
[6] D. K. Cheng and J. A. Kong, "Convariant descriptions of bianisotropic media," *Proc. IEEE*, vol. 56, pp. 248–251, Mar. 1968.
[7] I. V. Lindel, "On the definiteness of the constitutive parameters of a moving anisotropic medium," *Proc. IEEE* (Lett.), vol. 60, pp. 638–639, May 1972.
[8] J. A. Arnaud and A. A. M. Saleh," Theorems for bianisotropic media," *Proc. IEEE* (Lett)., vol. 60, pp. 639–640, May 1972.
[9] J. A. Kong, "Theorems of bianisotropic media," *Proc. IEEE*, vol. 60, pp. 1036–1046, Sept. 1972.
[10] J. R. Collier and C. T. Tai, "Guided waves in moving media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 441–445, July 1965.
[11] L.J. Du and R. T. Compton, Jr., "Cutoff phenomena for guided waves in moving media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-14, pp. 358–363, Aug. 1966.
[12] P. Daly, "Guided waves in moving media," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-15, pp. 274–275, Apr. 1967.
[13] J. A. Kong and D. K. Cheng, "On guided waves in moving anisotropic media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 99–103, Feb. 1968.
[14] K. Kurokawa, "Electromagnetic waves in waveguides with wall impedance," *IRE Trans. Microwave Theory Tech.*, vol. MTT-10, pp. 314–320, Sept. 1962.
[15] R. B. Dybdal, L. Peters, Jr., and W. H. Peake, "Rectangular waveguides with impedance walls," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 2–9, Jan. 1971.
[16] V. H. Rumsey, "Reaction concept in electromagnetic theory," *Phys. Rev.*, vol. 94, pp. 1483–1491, 1954; also, *errata, ibid.*, vol. 95, p. 1705, 1954.
[17] R. F. Harrington, *Time-Harmonic Electromagnetic Fields.* New York: McGraw-Hill, 1961, p. 118.
[18] M. Kobayashi, "Comments on some properties of lossless bianisotropic media" and ——, "On the definiteness of the constitutive parameters of a moving anisotropic medium," to be published.

New Results in the Least pth Approach to Minimax Design

J. W. BANDLER, SENIOR MEMBER, IEEE,
C. CHARALAMBOUS, STUDENT MEMBER, IEEE,
J. H. K. CHEN, AND W. Y. CHU, STUDENT MEMBER, IEEE

***Abstract*—We present results of two general approaches for obtaining minimax designs through a sequence of least pth approximations demonstrating increased efficiency over previous least pth algorithms. Documented computer programs are available.**

INTRODUCTION

This short paper demonstrates the acceleration of convergence to minimax solutions by extrapolation on a sequence of least pth solutions [1] with geometrically increasing values of p, and compares the results with an efficient extension of work by Charalambous and Bandler [2], [3], in which a sequence of least pth solutions with finite values of p are obtained in an effort to reach a minimax solution. Documented computer programs are available [4], [5], as well as the theoretical background [6]–[8].

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J. W. Bandler is with the Group on Simulation, Optimization and Control and the Department of Electrical Engineering, McMaster University, Hamilton, Ont., Canada.
C. Charalambous is with the Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ont., Canada.
J. H. K. Chen and W. Y. Chu are with Bell-Northern Research, Ottawa, Ont., Canada.

THEORY

We minimize, with respect to $\boldsymbol{\phi}$ for given ξ and $p > 1$, the function

$$U(\boldsymbol{\phi},\xi,p) \triangleq \begin{cases} M(\boldsymbol{\phi},\xi)\left(\sum_{i\epsilon K}\left(\dfrac{f_i(\boldsymbol{\phi}) - \xi}{M(\boldsymbol{\phi},\xi)}\right)^q\right)^{1/q}, & M(\boldsymbol{\phi},\xi) \neq 0 \\ 0, & M(\boldsymbol{\phi},\xi) = 0 \end{cases} \quad (1)$$

where

$$M_f(\boldsymbol{\phi}) \triangleq \max_{i\epsilon I} f_i(\boldsymbol{\phi}), \qquad M(\boldsymbol{\phi},\xi) \triangleq M_f(\boldsymbol{\phi}) - \xi,$$
$$q \triangleq p \operatorname{sgn} M(\boldsymbol{\phi},\xi)$$

and

$$K = \begin{cases} I \subset \{1,2,\cdots,m\}, & M(\boldsymbol{\phi},\xi) < 0 \\ J \triangleq \{i \mid f_i(\boldsymbol{\phi}) - \xi \geq 0,\, i \in I\}, & M(\boldsymbol{\phi},\xi) > 0 \end{cases} \quad (2)$$

and where $\boldsymbol{\phi} \triangleq [\phi_1\phi_2\cdots\phi_k]^T$ is the design parameter vector, and $f_1(\boldsymbol{\phi}), f_2(\boldsymbol{\phi}),\cdots, f_m(\boldsymbol{\phi})$ are m linear or nonlinear functions directly related to the response error functions such that if $M_f(\boldsymbol{\phi}) > 0$ the specifications are violated and if $M_f(\boldsymbol{\phi}) < 0$ the specifications are satisfied.

Charalambous has shown [6] that if we have $\boldsymbol{u}$ and $\boldsymbol{\phi}$ such that

$$\sum_{i=1}^{m} u_i \nabla f_i(\boldsymbol{\phi}) = \mathbf{0}$$

$$\sum_{1=1}^{m} u_i = 1, \qquad u_i \geq 0, \qquad i = 1,2,\cdots,m \quad (3)$$

then, if $\sum_{i=1}^{m} u_i f_i(\boldsymbol{\phi})$ is convex with respect to $\boldsymbol{\phi}$,

$$\sum_{i=1}^{m} u_i f_i(\boldsymbol{\phi}) \leq M_f(\check{\boldsymbol{\phi}}) \leq M_f(\boldsymbol{\phi}) \quad (4)$$

where $\check{\boldsymbol{\phi}}$ is the minimax optimum which is being sought,

$$\boldsymbol{u} \triangleq [u_1u_2\cdots u_m]^T$$

and

$$\boldsymbol{\nabla} \triangleq [\partial/\partial\phi_1 \partial/\partial\phi_2 \cdots \partial/\partial\phi_k]^T.$$

The conditions (3) are satisfied at each optimum point $\check{\boldsymbol{\phi}}(p,\xi)$ for a least pth objective function, yielding

$$\sum_{i=1}^{m} u_i f_i(\check{\boldsymbol{\phi}}(p,\xi)) \leq M_f(\check{\boldsymbol{\phi}}) \leq M_f(\check{\boldsymbol{\phi}}(p,\xi)) \quad (5)$$

where, assuming K contains all critical sample points,

$$u_i = \frac{v_i}{\sum_{i\epsilon K} v_i} \quad (6)$$

$$v_i = \begin{cases} \left(\dfrac{f_i(\check{\boldsymbol{\phi}}(p,\xi)) - \xi}{M(\check{\boldsymbol{\phi}},\xi)}\right)^{q-1}, & i \in K \\ 0, & i \notin K. \end{cases} \quad (7)$$

The first term of (5), under the stated conditions, is a lower bound on $M_f(\check{\boldsymbol{\phi}})$. It is, at any least pth solution, an optimistic indication of the ultimate minimax error to be expected for a particular design.

THE ξ ALGORITHM

An Implementation of Work by Charalambous [6]

ξ^{r+1} is set to the lower bound from the rth optimization. p is kept constant. We start at ϕ^0 by setting ξ^1 to an initial guess of a lower bound and setting $\xi^0 < \xi^1$ to a margin for discarding all functions f_i for which $f_i(\phi^0) < \xi^0$, which are considered unlikely to be active at $\check{\phi}$. ξ^r is used as a level for discarding functions, so that

$$I^{r+1} = \{i \mid f_i(\check{\phi}^r) \geq \xi^r\}.$$

The functions specifically required for the least pth objective and its gradients are successively reduced as the optimum is approached enabling a saving of effort in gradient computations. The program, called MINOPT [4], can also be restarted efficiently from any point instead of having to repeat the entire process. Convergence of the basic algorithm has been proved [6].

THE p ALGORITHM

An Implementation of Work by Chu [8]

p_{r+1} is set to cp_r, where $c > 1$. ξ is kept constant. We let $\phi_0{}^r = \check{\phi}^r$, and for $r > 1$

$$\phi_j{}^r = \frac{c^j \phi_{j-1}{}^r - \phi_{j-1}{}^{r-1}}{c^j - 1}, \qquad j = 1,\cdots,n_0 \tag{8}$$

where $n_0 = \min\,[r-1,n]$, $\phi_j{}^r$ signifies the jth-order estimate of $\check{\phi}$ after r minima have been obtained, and n is the highest order of extrapolation.

The next minimum is estimated by solving for $n_0 \geq 1$

$$\phi_{j-1}{}^{r+1} = \frac{(c^j - 1)\phi_j{}^{r+1} + \phi_{j-1}{}^r}{c^j} \tag{9}$$

for $j = n_0,\cdots,1$, using $\phi_{n_0}{}^{r+1} = \phi_{n_0}{}^r$. The next starting point is $\phi_0{}^{r+1}$.

We take $I^{r+1} = \{i \mid \mu_i > \eta\}$ where η is preassigned and

$$\mu_i = \frac{v_i}{\sum_{i \in K} v_i} \tag{10}$$

$$v_i = \begin{cases} \left(\dfrac{f_i(\phi_0{}^{r+1}) - \xi}{M(\phi_0{}^{r+1},\xi)}\right)^{q_{r+1}}, & i \in K \\ 0, & i \notin K \end{cases} \tag{11}$$

and K is chosen according to (2) with I set to I^r.

p is not increased when $r = 1$ in determining μ_i. No extrapolation can be made at this stage, so the starting point for the next minimization is the solution to the first one. In the program, called FLOPT2 [5], past solutions may be retained for future runs permitting extrapolation to be implemented immediately. Theoretical justification of the algorithm has been demonstrated [7], [8].

EXAMPLES

The unconstrained minimization method throughout was a recent quasi-Newton method [9], and the computer used was a CDC 6400.

We consider the same two starting points (leading to Problems 1 and 2, see Table I) and sample points for the optimization in the minimax sense of a three-section 100-percent relative-bandwidth 10:1-transmission-line transformer as in Charalambous and Bandler [2, table I]. The effort to reach or exceed a reflection coefficient of 0.197 29 (optimal to 5 figures) is compared taking f_i as the modulus of the reflection coefficient. We take as variables the lengths l_i and characteristic impedances Z_i. About $\frac{1}{3}$ to $\frac{1}{2}$ of the response evaluations used by the Charalambous–Bandler algorithms [2], tabulated in Table I, are required. Increasing p in the ξ algorithm gave poorer results. The p algorithm appears relatively insensitive to the sequence of p used.

Table II shows details of the progress of the p algorithm on Problem 1 and Table III shows the corresponding progress of the ξ algorithm.

A bandpass filter of symmetrical structure [10], the first four components of which are a unit element followed by a shunt shorted stub, a series open stub, and a shunt shorted stub, is considered next, with specifications of 0.1 dB from 1.0875 to 3.2625 GHz (passband) and 50 dB at 0.6 and 3.75 GHz. The initial normalized characteristic impedances were taken as 0.63, 0.33, 1.27, 0.26, 1.27, 0.33, and 0.63. The lengths were fixed at normalized values of 1. The functions f_i were set to $\pm$ the difference in decibels between the response and specifications, the positive sign corresponding to the

TABLE I
OPTIMIZATION OF A THREE-SECTION 10:1 TRANSFORMER OVER 100-PERCENT RELATIVE BANDWIDTH

Method	Effort[a] required to reach or exceed a reflection coefficient of 0.19729 (optimal to 5 figures)						
		Problem 1			Problem 2		
	Parameter ξ or p	Function Evaluations	Sample Points	Response Evaluations	Function Evaluations	Sample Points	Response Evaluations
ξ-algorithm	0.1	28	11	308	19	11	209
(lower bound)	0.18846	16	7	112	16	7	112
p = 2	0.19730	45	4	180	53	4	212
$\xi^0 = 0$	0.19729	89		600	88		533
p-algorithm	8	39	11	429	29	11	319
3rd order	48	17	8	136	18	8	144
extrapolation	288	15	4	60	14	4	56
$\xi = 0$	1728	12	4	48	11	4	44
$\eta = 0.001$		83		673	72		563
Charalambous -	Alg.1	165	11	1815	105	11	1155
Bandler [2]	Alg.2	155	11	1705	95	11	1045

[a] Does not include response evaluations to determine sample points to be used.

TABLE II
PROGRESS OF THE p ALGORITHM ON PROBLEM 1

r	Value of p_r	rth optimum[a]	Extrapolated Solution	Max. reflection coefficient	
				rth optimum	at extrapolated solution
1	8	.98828 1.62868 1.00004 3.16228 .98828 6.13993	same	.21017	.21017
2	48	.99833 1.63478 .99991 3.16228 .99833 6.11703	1.00035 1.63600 .99988 3.16228 1.00035 6.11246	.19838	.19863
3	288	.99973 1.63472 .99999 3.16228 .99973 6.11726	1.00000 1.63467 1.00000 3.16228 1.00000 6.11744	.19747	.19732
4	1728	.99995 1.63471 1.00000 3.16228 .99995 6.11730	1.00000 1.63471 1.00000 3.16228 1.00000 6.11730	.19732	.19729

[a] Parameter vector $[l_1/l_q\ Z_1\ l_2/l_q\ Z_2\ l_3/l_q\ Z_3]^T$ where l_q is the quarter-wavelength at center frequency.

TABLE III
PROGRESS OF THE ξ ALGORITHM ON PROBLEM 1

r	Value of ξ^r	rth optimum[a]	Max. reflection coefficient
1	.1	.97238 1.59720 .98791 3.16228 .97238 6.26097	.25530
2	.18846	.99709 1.63451 1.00013 3.16228 .99709 6.11804	.19929
3	.19730	1.00000 1.63471 1.00000 3.16228 1.00000 6.11730	.19729

[a] Parameter vector $[l_1/l_q\ Z_1\ l_2/l_q\ Z_2\ l_3/l_q\ Z_3]^T$ where l_q is the quarter-wavelength at center frequency.

TABLE IV
LOWER BOUNDS FOR THE SEVEN-SECTION FILTER

Passband Specification 0.1 dB
Value of p^0 2
Value of ξ^0 0

Stopband Specification (dB)	First Maximum Error (dB)	Predicted Lower Bound (dB)	Next Maximum Error (dB)
50	-0.0256	-0.0283	-0.0282
55	0.1430	0.1154	0.1160
60	0.6211	0.4954	0.4986
65	1.5486	1.3148	1.3195

passband, the negative sign to the stopband. Twenty-one uniformly spaced passband points were considered but, due to symmetry, only the first ten were actually used.

We let $\xi = 0$, $\eta = 0.0001$, $p_1 = 2$, $p_2 = 12$, $p_3 = 72$, and $p_4 = 432$. Third-order extrapolation was used leading to 73, 16, 14, and 12 function evaluations per optimization. The sample points were, respectively, 11, 11, 7, and 6 for a total of 1149 response evaluations. The final extrapolated solution gives characteristic impedances of 0.606 458, 0.303 062, 0.722 085, 0.235 612, 0.722 085, 0.303 062, and 0.606 458 (symmetrical as expected). The 8 passband ripples as evaluated at the 21 points fell within 0.065 30–0.065 31 dB. The stopband responses were 50.0347 dB. Central processing unit (CPU) time on the CDC 6400 was 5 s. About half this time would be expected if characteristic impedance symmetry were exploited.

The ξ algorithm generated the results of Table IV employing the actual ripple maxima in the passband as found by quadratic approximations. The lower bound estimation process works whether the specifications are violated or satisfied, yielding an immediate indication of how good a design in the minimax sense one can expect from the results of only one least squares approximation. This table also shows, by comparing columns 3 and 4, that only two optimizations with $p = 2$ of the ξ algorithm yield, for engineering purposes, essentially equal-ripple responses. For a 50-dB specification the final solution has characteristic impedances[1] of 0.606 595, 0.303 547, 0.722 287, 0.235 183, 0.722 287, 0.303 547, and 0.606 595 with deviations from specifications of $-0.028\,245$ (equal to at least 5 figures).

CONCLUSIONS

Two new algorithms and related results for the least pth approach to minimax design have been presented. Documented computer programs are available from J. W. Bandler at a nominal charge. A more detailed presentation of this material is also available [11].

REFERENCES

[1] J. W. Bandler and C. Charalambous, "Practical least pth optimization of networks," *IEEE Trans. Microwave Theory Tech.* (*1972 Symp. Issue*), vol. MTT-20, pp. 834–840, Dec. 1972.

[2] ——, "New algorithms for network optimization," *IEEE Trans. Microwave Theory Tech.* (*1973 Symp. Issue*), vol. MTT-21, pp. 815–818, Dec. 1973.

[3] ——, "Nonlinear minimax optimization as a sequence of least pth optimization with finite values of p," *Int. J. Syst. Sci.*, to be published.

[4] J. W. Bandler, C. Charalambous, and J. H. K. Chen, "MINOPT—An optimization program based on recent minimax results," Faculty Eng., McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-70, Dec. 1974.

[5] J. W. Bandler and W. Y. Chu, "FLOPT2—A program for least pth optimization with extrapolation to minimax solutions," Faculty Eng., McMaster Univ. Hamilton, Ont., Canada, Rep. SOC-84, Apr. 1975.

[6] C. Charalambous, "Minimax optimization of recursive digital filters using recent minimax results," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 333–345, Aug. 1975.

[7] A. V. Fiacco and G. P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*. New York: Wiley, 1968.

[8] W. Y. Chu, "Extrapolation in least pth approximation and nonlinear programming," Faculty Eng., McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-71, Dec. 1974.

[9] R. Fletcher, "FORTRAN subroutines for minimization by quasi-Newton methods," Atomic Energy Research Establishment, Harwell, Berks., England, Rep. AERE-R7125, 1972.

[10] M. C. Horton and R. J. Wenzel, "General theory and design of optimum quarter-wave TEM filters", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 316–327, May 1965.

[11] J. W. Bandler, C. Charalambous, J. H. K. Chen, and W. Y. Chu, "New results in the least pth approach to minimax design," Faculty Eng., McMaster Univ., Hamilton, Ont., Canada, Rep. SOC-80, Mar. 1975.

[1] Actually symmetrical to at least the accuracy of the CDC 6400.

Analysis of the Transient Temperature Distribution in a Stripline with Triple-Layer Dielectric

MASAKI SASAKI, NORINOBU YOSHIDA, ICHIRO FUKAI, MEMBER, IEEE, AND JUN-ICHI FUKUOKA

***Abstract*—The transient temperature distributions in the cross section of a stripline with triple-layer dielectric substrate are found by employing the finite element method. The calculations for three cases of different depths of center conductor considered as heat source are shown.**

For each case, the calculated temperature distributions are shown at $t = 10$ s when the temperature variation has a large gradient in time and at the steady state.

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The authors are with the Faculty of Engineering, Hokkaido University, Sapporo, Japan.

With the integralization of electric devices and the appearance of high-power semiconductor devices, miniaturization of transmission systems has become necessary. In dealing with such systems, it is of considerable importance to confirm the rise and distribution of temperatures appearing in the operation of the devices in as much as they are in a risk of thermal destruction and thermal degeneration. For the analysis of these problems, some theoretical methods and numerical procedures have been proposed. There are, however, various difficulties in the analysis of such theoretical methods.

In the present analysis, the finite element method based on the variational method was used because of its advantage in dealing with the complicated contours as well as composite media. In relation to the heat conduction equation in a two-dimensional case, the functional is defined as [1]

$$\chi = \iint \left\{ \tfrac{1}{2}k \left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 \right] - \left(q - C\frac{\partial T}{\partial t} \right) T \right\} dx\, dy \tag{1}$$

where T is the temperature, C the heat capacity, k heat conductivity, and q the rate of heat generation. The temperature rise and distribution can be obtained by finding the function T by which the functional χ is made stationary. To carry out the preceding method, the domain is divided into many triangular elements, χ is differentiated with respect to T, the derivative is set equal to zero. The resulting equation is thus given by

$$\left\{\frac{\partial x}{\partial T_n}\right\} = \sum \left\{\frac{\partial x}{\partial T_n}\right\}^e = [H]\{T\} + [P]\left\{\frac{\partial T}{\partial t}\right\} - \{K\} = 0 \tag{2}$$

where $[H]$ is the heat conductivity matrix, $[P]$ is the heat capacity matrix, and $\{K\}$ is a vector which expresses the distribution of heat sources. Applying the trapezoidal approximation for the derivative with respect to time, the following difference equation was obtained for all nodal temperatures in a matrix form

$$\left([H] + \frac{2}{\Delta t}[P]\right)\{T\}_t = [P]\left(\left\{\frac{\partial T}{\partial t}\right\}_{t-\Delta t} + \frac{2}{\Delta t}\{T\}_{t-\Delta t}\right) + \{K\}_t. \tag{3}$$

To illustrate the correctness of the method, we employed a simple problem where heat source q ($=100$ W/cm^3) distributes uniformly in the square column of alumina with infinite length under the Newton cooling condition. The temperature rise at the center of the column obtained by both the exact analytical solution and by the finite element method are shown in Fig. 1, and difference between the two methods is within one percent.

Then the temperature characteristics are calculated for striplines with triple-layer dielectric media. The analytical model is shown in Fig. 2 where $H = W_2 = 0.1$ cm, $a_1 = a_2 = a_3 = H/3$, $W_1 = 10a_1$, and $b = 0.001$ cm. For symmetry, the right half-plane is considered. The center medium is alumina, the heat source material is copper, and other media are glass. The respective thermal constants are shown in Table I. The rate of heat generation per unit volume q is 10^4 W/cm^3. The boundary condition at the surface where $x = 0$ is

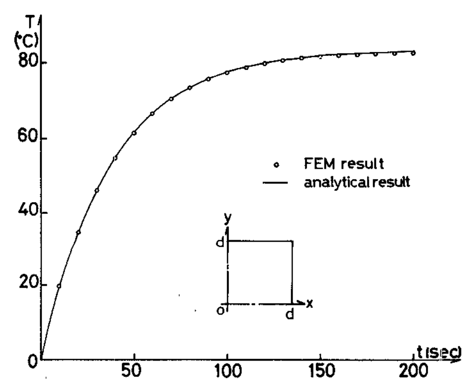

Fig. 1. Example 1: Geometry of the problem and the temperature rise at center of heat source. $d = 0.5$ mm.