Efficient Calculation of Exact Group Delay Sensitivities

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***Abstract*—This paper presents, explicitly, an approach to the exact calculation of group delay and its sensitivities with respect to component parameters based on the adjoint network concept and applicable to linear time-invariant circuits. In general, no more than four analyses are required and the computational effort is only moderately more than is necessary for a single analysis. The results presented are in a form particularly suited to the computer-aided design of microwave circuits and include useful tables of sensitivity expressions.**

I. Introduction

The applicability of the adjoint network concept [1], [2] to the evaluation of second-order network sensitivities [3], [4] has been known for several years. The computation of group delay [5], [6] and its sensitivities with respect to component parameters [5] using these ideas has also been suggested. The observation by Temes [5] about the use of perturbation in evaluating group delay sensitivities and the recent implementation by Bandler *et al.* [7] of perturbation techniques might suggest that exact computation is impractical.

It is the purpose of this paper, therefore, to present explicitly, with the aid of a microwave filter example, a suitable exact approach. In general, no more than four analyses are required and the computational effort is only moderately more than is necessary for a single analysis. The authors are not aware of any similar presentation in the literature.

II. Theory

The exact group delay of a linear time-invariant network N can be evaluated, using the adjoint network concept, and requires two network analyses, one of N and one of its adjoint network $\hat{N}$.

The group delay can be defined as [5], [6]

$$T_G = -\mathrm{Im}\left\{\frac{1}{V_0}\frac{\partial V_0}{\partial \omega}\right\} \qquad (1)$$

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where V_0 is the output voltage. The sensitivity of T_G with respect to the variable parameter ϕ_i will be given by

$$\frac{\partial T_G}{\partial \phi_i} = -\mathrm{Im}\left\{-\frac{1}{V_0^2}\frac{\partial V_0}{\partial \phi_i}\frac{\partial V_0}{\partial \omega} + \frac{1}{V_0}\frac{\partial^2 V_0}{\partial \phi_i \partial \omega}\right\}. \qquad (2)$$

In (2), $\partial V_0/\partial \phi_i$ and $\partial V_0/\partial \omega$ are known from the analyses of N and $\hat{N}$, where $\hat{N}$ is excited only by a current source $\hat{I}_0$ at the output port. Only $(\partial^2 V_0/\partial \phi_i \partial \omega)$ has to be evaluated.

Second-Order Sensitivities

Assume that the elements of N are described by a hybrid matrix, namely,

$$\begin{bmatrix} I_{aj} \\ V_{bj} \end{bmatrix} = \begin{bmatrix} Y_j & A_j \\ M_j & Z_j \end{bmatrix}\begin{bmatrix} V_{aj} \\ I_{bj} \end{bmatrix}, \qquad j = 1,2,\cdots,n. \qquad (3)$$

The elements of $\hat{N}$ will then be described by [2]

$$\begin{bmatrix} \hat{I}_{aj} \\ \hat{V}_{bj} \end{bmatrix} = \begin{bmatrix} Y_j^T & -M_j^T \\ -A_j^T & Z_j^T \end{bmatrix}\begin{bmatrix} \hat{V}_{aj} \\ \hat{I}_{bj} \end{bmatrix}, \qquad j = 1,2,\cdots,n. \qquad (4)$$

Using Tellegen's theorem we may write [2], [8]

$$[V_a^T \quad V_b^T]\begin{bmatrix} \hat{I}_a \\ \hat{I}_b \end{bmatrix} - [I_a^T \quad I_b^T]\begin{bmatrix} \hat{V}_a \\ \hat{V}_b \end{bmatrix} = \sum_{j=1}^{n}[V_{aj}^T \quad V_{bj}^T]\begin{bmatrix} \hat{I}_{aj} \\ \hat{I}_{bj} \end{bmatrix} - \sum_{j=1}^{n}[I_{aj}^T \quad I_{bj}^T]\begin{bmatrix} \hat{V}_{aj} \\ \hat{V}_{bj} \end{bmatrix} \qquad (5)$$

where the sign convention adopted is illustrated in Fig. 1 and where we assume an unindexed equation of the form of (3) describes the complete network with subscript a denoting voltage excited ports and b current excited ports. Applying the linear operator $\partial^2/\partial\phi\partial\psi$ to the voltages and currents of N, where ϕ and ψ are variable parameters, and taking V_a and I_b as discrete and independent, we have as an extension to (5) [3], [4], [8]

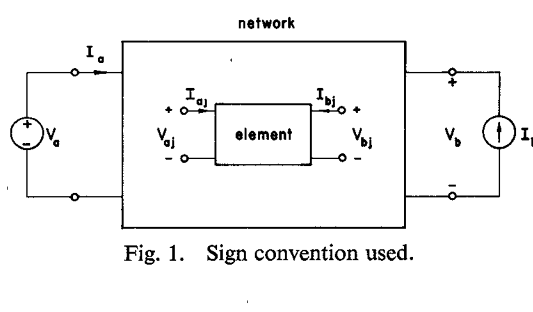


Fig. 1. Sign convention used.

$$\frac{\partial^2 V_b^T}{\partial\phi\partial\psi}\hat{I}_b - \frac{\partial^2 I_a^T}{\partial\phi\partial\psi}\hat{V}_a = \sum_{j=1}^{n}\begin{bmatrix}\frac{\partial^2 V_{aj}^T}{\partial\phi\partial\psi} & \frac{\partial^2 V_{bj}^T}{\partial\phi\partial\psi}\end{bmatrix}\begin{bmatrix}\hat{I}_{aj}\\ \hat{I}_{bj}\end{bmatrix} - \sum_{j=1}^{n}\begin{bmatrix}\frac{\partial^2 I_{aj}^T}{\partial\phi\partial\psi} & \frac{\partial^2 I_{bj}^T}{\partial\phi\partial\psi}\end{bmatrix}\begin{bmatrix}\hat{V}_{aj}\\ \hat{V}_{bj}\end{bmatrix}. \tag{6}$$

Differentiating (3) with respect to ψ and ϕ, we obtain

$$\begin{bmatrix}\frac{\partial^2 I_{aj}}{\partial\phi\partial\psi}\\ \frac{\partial^2 V_{bj}}{\partial\phi\partial\psi}\end{bmatrix} = \begin{bmatrix}\frac{\partial^2 Y_j}{\partial\phi\partial\psi} & \frac{\partial^2 A_j}{\partial\phi\partial\psi}\\ \frac{\partial^2 M_j}{\partial\phi\partial\psi} & \frac{\partial^2 Z_j}{\partial\phi\partial\psi}\end{bmatrix}\begin{bmatrix}V_{aj}\\ I_{bj}\end{bmatrix} + \begin{bmatrix}\frac{\partial Y_j}{\partial\psi} & \frac{\partial A_j}{\partial\psi}\\ \frac{\partial M_j}{\partial\psi} & \frac{\partial Z_j}{\partial\psi}\end{bmatrix}\begin{bmatrix}\frac{\partial V_{aj}}{\partial\phi}\\ \frac{\partial I_{bj}}{\partial\phi}\end{bmatrix} + \begin{bmatrix}\frac{\partial Y_j}{\partial\phi} & \frac{\partial A_j}{\partial\phi}\\ \frac{\partial M_j}{\partial\phi} & \frac{\partial Z_j}{\partial\phi}\end{bmatrix} \cdot \begin{bmatrix}\frac{\partial V_{aj}}{\partial\psi}\\ \frac{\partial I_{bj}}{\partial\psi}\end{bmatrix} + \begin{bmatrix}Y_j & A_j\\ M_j & Z_j\end{bmatrix}\begin{bmatrix}\frac{\partial^2 V_{aj}}{\partial\phi\partial\psi}\\ \frac{\partial^2 I_{bj}}{\partial\phi\partial\psi}\end{bmatrix}. \tag{7}$$

Rearranging (6) and substituting (7) in it we get

$$\frac{\partial^2 V_b^T}{\partial\phi\partial\psi}\hat{I}_b - \frac{\partial^2 I_a^T}{\partial\phi\partial\psi}\hat{V}_a = \sum_{j=1}^{n}\begin{bmatrix}\frac{\partial^2 V_{aj}^T}{\partial\phi\partial\psi} & \frac{\partial^2 I_{bj}^T}{\partial\phi\partial\psi}\end{bmatrix}\begin{bmatrix}\hat{I}_{aj}\\ -\hat{V}_{bj}\end{bmatrix} + \sum_{j=1}^{n}\left\{\begin{bmatrix}V_{aj}^T & I_{bj}^T\end{bmatrix}\begin{bmatrix}\frac{\partial^2 Y_j^T}{\partial\phi\partial\psi} & \frac{\partial^2 M_j^T}{\partial\phi\partial\psi}\\ \frac{\partial^2 A_j^T}{\partial\phi\partial\psi} & \frac{\partial^2 Z_j^T}{\partial\phi\partial\psi}\end{bmatrix} + \begin{bmatrix}\frac{\partial V_{aj}^T}{\partial\phi} & \frac{\partial I_{bj}^T}{\partial\phi}\end{bmatrix}\begin{bmatrix}\frac{\partial Y_j^T}{\partial\psi} & \frac{\partial M_j^T}{\partial\psi}\\ \frac{\partial A_j^T}{\partial\psi} & \frac{\partial Z_j^T}{\partial\psi}\end{bmatrix} + \begin{bmatrix}\frac{\partial V_{aj}^T}{\partial\psi} & \frac{\partial I_{bj}^T}{\partial\psi}\end{bmatrix}\begin{bmatrix}\frac{\partial Y_j^T}{\partial\phi} & \frac{\partial M_j^T}{\partial\phi}\\ \frac{\partial A_j^T}{\partial\phi} & \frac{\partial Z_j^T}{\partial\phi}\end{bmatrix} + \begin{bmatrix}\frac{\partial^2 V_{aj}^T}{\partial\phi\partial\psi} & \frac{\partial^2 I_{bj}^T}{\partial\phi\partial\psi}\end{bmatrix}\begin{bmatrix}Y_j^T & M_j^T\\ A_j^T & Z_j^T\end{bmatrix}\right\}\begin{bmatrix}-\hat{V}_{aj}\\ \hat{I}_{bj}\end{bmatrix}. \tag{8}$$

Using (4) which defines the adjoint network $\hat{N}$, (8) will be

$$\frac{\partial^2 V_b^T}{\partial\phi\partial\psi}\hat{I}_b - \frac{\partial^2 I_a^T}{\partial\phi\partial\psi}\hat{V}_a = \sum_{j=1}^{n}\left\{\begin{bmatrix}V_{aj}^T & I_{bj}^T\end{bmatrix}\begin{bmatrix}\frac{\partial^2 Y_j^T}{\partial\phi\partial\psi} & \frac{\partial^2 M_j^T}{\partial\phi\partial\psi}\\ \frac{\partial^2 A_j^T}{\partial\phi\partial\psi} & \frac{\partial^2 Z_j^T}{\partial\phi\partial\psi}\end{bmatrix} + \begin{bmatrix}\frac{\partial V_{aj}^T}{\partial\phi} & \frac{\partial I_{bj}^T}{\partial\phi}\end{bmatrix}\begin{bmatrix}\frac{\partial Y_j^T}{\partial\psi} & \frac{\partial M_j^T}{\partial\psi}\\ \frac{\partial A_j^T}{\partial\psi} & \frac{\partial Z_j^T}{\partial\psi}\end{bmatrix} + \begin{bmatrix}\frac{\partial V_{aj}^T}{\partial\psi} & \frac{\partial I_{bj}^T}{\partial\psi}\end{bmatrix}\begin{bmatrix}\frac{\partial Y_j^T}{\partial\phi} & \frac{\partial M_j^T}{\partial\phi}\\ \frac{\partial A_j^T}{\partial\phi} & \frac{\partial Z_j^T}{\partial\phi}\end{bmatrix}\right\}\begin{bmatrix}-\hat{V}_{aj}\\ \hat{I}_{bj}\end{bmatrix}. \tag{9}$$

To evaluate (9) first-order sensitivities of voltages and currents of element j with respect to ϕ and ψ have to be found. Considering the ports of the jth element as the ports of interest we can find the first-order sensitivities needed. Examining (9) we find that the hybrid matrix of the element j is differentiated with respect to ϕ and with respect to ψ. If the parameter ϕ does not belong to the jth element the derivative of the hybrid matrix with respect to ϕ will be zero. The same condition occurs with the parameter ψ. Assuming that each element has one parameter, k additional analyses of $\hat{N}$ will be needed to find (9), where k is the number of variable parameters [4], [9].

Group Delay Sensitivities

Considering a single output voltage V_0 and setting the adjoint voltage excitation vector $\hat{V}_a$ to zero, and replacing the parameter ϕ by ϕ_i and ψ by ω, (9) will be (for ϕ_i in the jth element)

$$\frac{\partial^2 V_0}{\partial\phi_i\partial\omega}\hat{I}_0 = \begin{bmatrix}V_{aj}^T & I_{bj}^T\end{bmatrix}\begin{bmatrix}\frac{\partial^2 Y_j^T}{\partial\phi_i\partial\omega} & \frac{\partial^2 M_j^T}{\partial\phi_i\partial\omega}\\ \frac{\partial^2 A_j^T}{\partial\phi_i\partial\omega} & \frac{\partial^2 Z_j^T}{\partial\phi_i\partial\omega}\end{bmatrix}\begin{bmatrix}-\hat{V}_{aj}\\ \hat{I}_{bj}\end{bmatrix} + \sum_{j=1}^{n}\begin{bmatrix}\frac{\partial V_{aj}^T}{\partial\phi_i} & \frac{\partial I_{bj}^T}{\partial\phi_i}\end{bmatrix}\begin{bmatrix}\frac{\partial Y_j^T}{\partial\omega} & \frac{\partial M_j^T}{\partial\omega}\\ \frac{\partial A_j^T}{\partial\omega} & \frac{\partial Z_j^T}{\partial\omega}\end{bmatrix}\begin{bmatrix}-\hat{V}_{aj}\\ \hat{I}_{bj}\end{bmatrix} + \begin{bmatrix}\frac{\partial V_{aj}^T}{\partial\omega} & \frac{\partial I_{bj}^T}{\partial\omega}\end{bmatrix}\begin{bmatrix}\frac{\partial Y_j^T}{\partial\phi_i} & \frac{\partial M_j^T}{\partial\phi_i}\\ \frac{\partial A_j^T}{\partial\phi_i} & \frac{\partial Z_j^T}{\partial\phi_i}\end{bmatrix}\begin{bmatrix}-\hat{V}_{aj}\\ \hat{I}_{bj}\end{bmatrix}. \tag{10}$$

The first term in (10) can be found from the currents and voltages in N and $\hat{N}$. Note that ω is common, in general, to all the elements.

Let us define E_j to be

$$E_j \triangleq \begin{bmatrix} \frac{\partial Y_j}{\partial \omega} & \frac{\partial A_j}{\partial \omega} \\ \frac{\partial M_j}{\partial \omega} & \frac{\partial Z_j}{\partial \omega} \end{bmatrix}. \tag{11}$$

Introduce now a new network $\hat{N}'$, which is the same as $\hat{N}$, but excited at the ports of each element by current and voltage sources $\hat{I}_{aj}'^s$ and $\hat{V}_{bj}'^s$, where

$$\begin{bmatrix} -\hat{I}_{aj}'^s \\ \hat{V}_{bj}'^s \end{bmatrix} \triangleq E_j^T \begin{bmatrix} -\hat{V}_{aj} \\ \hat{I}_{bj} \end{bmatrix}. \tag{12}$$

Consequently, by conventional adjoint network theory

$$\sum_{j=1}^{n} \begin{bmatrix} \frac{\partial V_{aj}^T}{\partial \phi_i} & \frac{\partial I_{bj}^T}{\partial \phi_i} \end{bmatrix} \begin{bmatrix} -\hat{I}_{aj}'^s \\ \hat{V}_{bj}'^s \end{bmatrix} = G_i' \tag{13}$$

where G_i' is the sensitivity component of N with respect to ϕ_i, known in terms of the voltages and currents in N and $\hat{N}'$. The second term in (10) is now G_i'.

Next, consider a network N', which is the same as N, but excited at the ports of each element by current and voltage sources $I_{aj}'^s$ and $V_{bj}'^s$, given by

$$\begin{bmatrix} I_{aj}'^s \\ V_{bj}'^s \end{bmatrix} = E_j \begin{bmatrix} V_{aj} \\ I_{bj} \end{bmatrix} \tag{14}$$

so that in N'

$$\begin{bmatrix} I_{aj}' \\ V_{bj}' \end{bmatrix} = \begin{bmatrix} I_{aj}'^s \\ V_{bj}'^s \end{bmatrix} + \begin{bmatrix} Y_j & A_j \\ M_j & Z_j \end{bmatrix} \begin{bmatrix} V_{aj}' \\ I_{bj}' \end{bmatrix}. \tag{15}$$

Differentiating (3) with respect to ω gives

$$\begin{bmatrix} \frac{\partial I_{aj}}{\partial \omega} \\ \frac{\partial V_{bj}}{\partial \omega} \end{bmatrix} = \begin{bmatrix} \frac{\partial Y_j}{\partial \omega} & \frac{\partial A_j}{\partial \omega} \\ \frac{\partial M_j}{\partial \omega} & \frac{\partial Z_j}{\partial \omega} \end{bmatrix} \begin{bmatrix} V_{aj} \\ I_{bj} \end{bmatrix} + \begin{bmatrix} Y_j & A_j \\ M_j & Z_j \end{bmatrix} \begin{bmatrix} \frac{\partial V_{aj}}{\partial \omega} \\ \frac{\partial I_{bj}}{\partial \omega} \end{bmatrix}. \tag{16}$$

Comparing (15) and (16), we see that

$$\begin{bmatrix} \frac{\partial V_{aj}}{\partial \omega} \\ \frac{\partial I_{bj}}{\partial \omega} \end{bmatrix} = \begin{bmatrix} V_{aj}' \\ I_{bj}' \end{bmatrix} \tag{17}$$

so that the third term in (10) can be calculated from the currents and voltages in N' and $\hat{N}$.

This means that the network response, its sensitivities, the group delay, and its sensitivities can be evaluated by at most four network analyses, independently of the number of variable parameters. This approach is useful for networks with a large number of parameters.

Table I shows E_j for some elements and for two possible hybrid-matrix formulations. Note that the first-order sensitivity components with respect to ω used in the calculation of the group delay can be found by multiplying E_j by appropriate vectors of the voltages and currents in N and $\hat{N}$. Table II shows second-order sensitivity expressions needed in evaluating the first term of (10). To avoid numerical problems when $\omega\tau = \pi/2$ Table III and Table IV have to be used when appropriate.

III. Interpretation

General Networks

If the nodal admittance matrix is used for the analysis, each two-terminal element will have a current source associated with it. This source for N' is

$$I_j'^s = \frac{\partial Y_j}{\partial \omega} V_j \tag{18}$$

where Y_j is the admittance of the element and V_j is the voltage across the element. The source for the second adjoint analysis is

$$\hat{I}_j'^s = \frac{\partial Y_j}{\partial \omega} \hat{V}_j. \tag{19}$$

Fig. 2 shows a two-terminal element and the current sources connected to it for the second analysis.

TABLE I
Element Derivatives with Respect to Frequency

Element	$\frac{\partial Y}{\partial \omega}$	$\frac{\partial Z}{\partial \omega}$
inductance	$-\frac{1}{j\omega^2 L}$	jL
capacitance	jC	$-\frac{1}{j\omega^2 C}$
short-circuited lossless transmission line[a]	$jY\tau \csc^2\omega\tau$	$jZ\tau \sec^2\omega\tau$
open-circuited lossless transmission line[a]	$jY\tau \sec^2\omega\tau$	$jZ\tau \csc^2\omega\tau$
lossless transmission line[a]	$-jY\tau \csc\omega\tau \begin{bmatrix} -\csc\omega\tau & \cot\omega\tau \\ \cot\omega\tau & -\csc\omega\tau \end{bmatrix}$	$jZ\tau \csc\omega\tau \begin{bmatrix} \csc\omega\tau & \cot\omega\tau \\ \cot\omega\tau & \csc\omega\tau \end{bmatrix}$

[a] For transmission lines, Z is the characteristic impedance, Y is the characteristic admittance, and τ is the delay time.

TABLE II
SECOND-DERIVATIVE SENSITIVITY EXPRESSIONS

Element	$\frac{\partial^2 Y}{\partial\phi_i \partial\omega}$	$\frac{\partial^2 Z}{\partial\phi_i \partial\omega}$	ϕ_i
inductance	$\frac{1}{j\omega^2 L^2}$	j	L
capacitance	j	$\frac{1}{j\omega^2 C^2}$	C
short-circuited lossless transmission line[a]	$-jY^2\tau\csc^2\omega\tau$	$j\tau\sec^2\omega\tau$	Z
	$j\tau\csc^2\omega\tau$	$-jZ^2\tau\sec^2\omega\tau$	Y
	$jY\csc^2\omega\tau\,(1-2\,\omega\tau\cot\omega\tau)$	$jZ\sec^2\omega\tau\,(1+2\,\omega\tau\tan\omega\tau)$	τ
open-circuited lossless transmission line[a]	$-jY^2\tau\sec^2\omega\tau$	$j\tau\csc^2\omega\tau$	Z
	$j\tau\sec^2\omega\tau$	$-jZ^2\tau\csc^2\omega\tau$	Y
	$jY\sec^2\omega\tau\,(1+2\,\omega\tau\tan\omega\tau)$	$jZ\csc^2\omega\tau\,(1-2\,\omega\tau\cot\omega\tau)$	τ
lossless transmission line[a]	$jY^2\tau\csc\omega\tau\begin{bmatrix}-\csc\omega\tau & \cot\omega\tau\\ \cot\omega\tau & -\csc\omega\tau\end{bmatrix}$	$j\tau\csc\omega\tau\begin{bmatrix}\csc\omega\tau & \cot\omega\tau\\ \cot\omega\tau & \csc\omega\tau\end{bmatrix}$	Z
	$-j\tau\csc\omega\tau\begin{bmatrix}-\csc\omega\tau & \cot\omega\tau\\ \cot\omega\tau & -\csc\omega\tau\end{bmatrix}$	$-jZ^2\tau\csc\omega\tau\begin{bmatrix}\csc\omega\tau & \cot\omega\tau\\ \cot\omega\tau & \csc\omega\tau\end{bmatrix}$	Y
	$-jY\csc\omega\tau\left\{\begin{bmatrix}-\csc\omega\tau & \cot\omega\tau\\ \cot\omega\tau & -\csc\omega\tau\end{bmatrix} -\omega\tau\begin{bmatrix}-2\csc\omega\tau\cot\omega\tau & \cot^2\omega\tau+\csc^2\omega\tau\\ \cot^2\omega\tau+\csc^2\omega\tau & -2\csc\omega\tau\cot\omega\tau\end{bmatrix}\right\}$	$jZ\csc\omega\tau\left\{\begin{bmatrix}\csc\omega\tau & \cot\omega\tau\\ \cot\omega\tau & \csc\omega\tau\end{bmatrix} -\omega\tau\begin{bmatrix}2\csc\omega\tau\cot\omega\tau & \csc^2\omega\tau+\cot^2\omega\tau\\ \csc^2\omega\tau+\cot^2\omega\tau & 2\csc\omega\tau\cot\omega\tau\end{bmatrix}\right\}$	τ

[a] For transmission lines, Z is the characteristic impedance, Y is the characteristic admittance, and τ is the delay time.

TABLE III
ELEMENT DERIVATIVES WITH RESPECT TO FREQUENCY WHEN $\omega\tau = \pi/2$

Element	$\frac{\partial Y}{\partial\omega}$	$\frac{\partial Z}{\partial\omega}$
short-circuited lossless transmission line	$jY\tau$	-
open-circuited lossless transmission line	-	$jZ\tau$
lossless transmission line	$jY\tau\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$jZ\tau\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

TABLE IV
SECOND-DERIVATIVE SENSITIVITY EXPRESSIONS WHEN $\omega\tau = \pi/2$

Element	$\frac{\partial^2 Y}{\partial\phi_i \partial\omega}$	$\frac{\partial^2 Z}{\partial\phi_i \partial\omega}$	ϕ_i
short-circuited lossless transmission line	$-jY^2\tau$	-	Z
	$j\tau$	-	Y
	jY	-	τ
open-circuited lossless transmission line	-	$j\tau$	Z
	-	$-jZ^2\tau$	Y
	-	jZ	τ
lossless transmission line	$\begin{bmatrix}-jY^2\tau & 0\\ 0 & -jY^2\tau\end{bmatrix}$	$\begin{bmatrix}j\tau & 0\\ 0 & j\tau\end{bmatrix}$	Z
	$\begin{bmatrix}j\tau & 0\\ 0 & j\tau\end{bmatrix}$	$\begin{bmatrix}-jZ^2\tau & 0\\ 0 & -jZ^2\tau\end{bmatrix}$	Y
	$jY\begin{bmatrix}1 & \frac{\pi}{2}\\ \frac{\pi}{2} & 1\end{bmatrix}$	$jZ\begin{bmatrix}1 & -\frac{\pi}{2}\\ -\frac{\pi}{2} & 1\end{bmatrix}$	τ

For a three-terminal element two current sources will be connected to its ports. These sources are of the form

$$I_1'^s = \frac{\partial Y_{11}}{\partial\omega} V_1 + \frac{\partial Y_{12}}{\partial\omega} V_2 \tag{20}$$

$$I_2'^s = \frac{\partial Y_{21}}{\partial\omega} V_1 + \frac{\partial Y_{22}}{\partial\omega} V_2.$$

The element and the two sources connected to it are shown in Fig. 3. The sources for the adjoint network are derived in a similar way.

In general we need the current excitation vector for the second two analyses, and using (18)–(20) these vectors can be expressed by

$$\boldsymbol{I}'^s = \frac{\partial \boldsymbol{Y}}{\partial\omega}\boldsymbol{V} \tag{21}$$

and

$$\hat{\boldsymbol{I}}'^s = \frac{\partial \boldsymbol{Y}^T}{\partial\omega}\hat{\boldsymbol{V}} \tag{22}$$

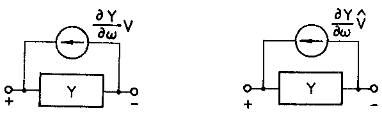


Fig. 2. Two-terminal element in the original and adjoint network and current sources associated with the element.

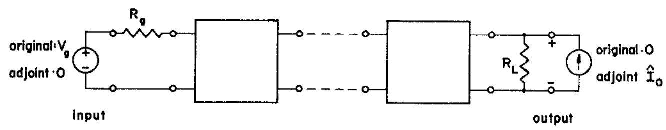


Fig. 4. Cascaded original and adjoint networks.

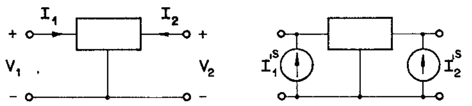


Fig. 3. A three-terminal element and the sources connected to it in the second original network analysis.

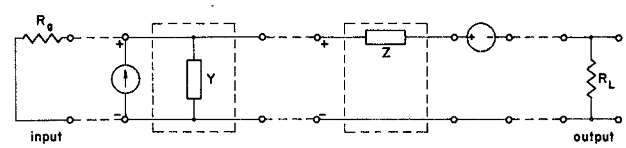


Fig. 5. Network representing N' or $\hat{N}'$, each with its appropriate sources.

where Y is the nodal admittance matrix, V is the node voltage vector in the first original network analysis, and $\hat{V}$ is the node voltage vector in the first adjoint network analysis.

The matrix $\partial Y/\partial\omega$ will have the same structure as the nodal admittance matrix, and we can take advantage of this fact and build up the $\partial Y/\partial\omega$ matrix at the same time and in a similar way as building the Y matrix. Hence the excitations are found by matrix multiplication. Using this approach only four analyses are needed to find the group delay and its sensitivities. Actually, LU factorization [10] is performed once and the forward and backward substitutions have to be repeated.

Cascaded Networks

In cascaded network analysis the network is considered to be a chain of two-port networks. Each two-port represents an element expressed by its hybrid matrix. Assume that the cascaded network has one input port and one output port as shown in Fig. 4. The group delay and its sensitivities can be found by the following steps.

Step 1: In the first original network analysis we assume that the current through the load has a certain value I_L. We carry out the analysis step by step starting from the load end. Suppose that the computed voltage at the generator end is V_{gc} and the actual generator voltage is V_{ga}. Since the network is linear, the actual values for all voltages and currents are found by multiplying the computed values by the factor V_{ga}/V_{gc}.

At this stage the sources for N' can be found. The shunt elements expressed by their admittances will have current sources connected across them which are evaluated as (18); on the other hand, the series elements expressed by their impedances will have voltage sources connected with them in series as shown in Fig. 5. For a certain element j connected in series the voltage source corresponding to this element is

$$V_j'^s = \frac{\partial Z_j}{\partial\omega} I_j \tag{23}$$

where Z_j is the impedance of the element and I_j is the current passing through the element.

Step 2: In the first adjoint network analysis, the generator end is short-circuited and a current source $\hat{I}_0$ is connected to the load end. Assuming a value for the current at the generator end, the analysis is carried out from the generator end to the load end. If $\hat{I}_{0c}$ is the computed value for the current source and $\hat{I}_{0a}$ is its actual value, the actual values of adjoint voltages and currents are found by multiplying the computed values by $\hat{I}_{0a}/\hat{I}_{0c}$. The sources for the next adjoint analysis can be found in similar way as the sources for the second original network analysis.

After the evaluation of Step 1 and Step 2 the group delay is computed.

Step 3: To get the group delay sensitivities, first we excite the original network as previously discussed. The generator end is short circuited and the sources connected to the corresponding elements are as shown in Fig. 5. As we can see there is no source at either end of the network, and in order to perform the analysis we have to find the Thévenin voltage at the output end [11]. Applying Tellegen's theorem we have

$$V_L'\hat{I}_0 = \sum_{i\in I_y} I_i'^s\hat{V}_i - \sum_{j\in I_z} V_j'^s\hat{I}_j \tag{24}$$

where

- V_L' Thévenin voltage at the output;
- $\hat{I}_0$ adjoint current excitation;
- $I_i'^s$ current source corresponding to shunt element i;
- $\hat{V}_i$ adjoint voltage across shunt element i;
- I_y index set for elements connected in shunt in the network;
- $V_j'^s$ voltage source corresponding to series element j;
- $\hat{I}_j$ adjoint current through series element j;
- I_z index set for elements connected in series in the network.

Recall that the prime superscript stands for the second analysis. We have to note that the adjoint network of the second original network will still be the adjoint of the first original network and all the $\hat{V}_i$ and $\hat{I}_j$ are known.

Since the output voltage is known at this stage we can carry out the second original network analysis, from the load end to the generator end, and hence the voltages and currents of this network are found.

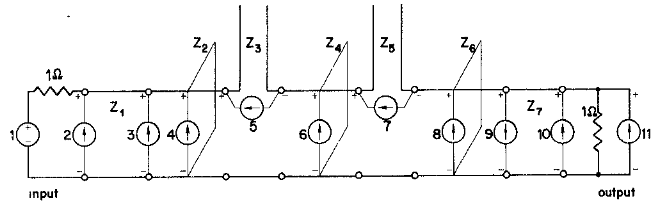


Fig. 6. Seven-section bandpass filter with sources required for the analyses. All sections are quarter-wave at 2.175 GHz. $Z_1 = Z_7 = 0.606595$, $Z_2 = Z_6 = 0.303547$, $Z_3 = Z_5 = 0.722287$, $Z_4 = 0.235183$.

TABLE V
GROUP DELAY AND ITS GRADIENTS

Normalized frequency	0.5	0.6	0.7	0.8	0.9	1.0
Group delay n sec	2.4864	1.18034	0.89524	0.78201	0.71711	0.71409
$\frac{\partial T_G}{\partial Z_1} = \frac{\partial T_G}{\partial Z_7}$	1.4736	0.29952	-0.11745	0.14144	0.13617	-0.09275
$\frac{\partial T_G}{\partial Z_2} = \frac{\partial T_G}{\partial Z_6}$	-5.9195	-1.0742	-0.38574	-0.33653	-0.29703	-0.22951
$\frac{\partial T_G}{\partial Z_3} = \frac{\partial T_G}{\partial Z_5}$	3.9441	0.45418	0.28131	0.14503	0.12701	0.15619
$\frac{\partial T_G}{\partial Z_4}$	-12.313	-1.2170	-0.52517	-0.72480	-0.49986	-0.38233
$\frac{\partial T_G}{\partial \tau_1} = \frac{\partial T_G}{\partial \tau_7}$	-1.4355	0.22932	1.5822	0.40020	1.5882	1.1276
$\frac{\partial T_G}{\partial \tau_2} = \frac{\partial T_G}{\partial \tau_6}$	-22.016	-3.2718	-0.35544	-0.11376	-0.55457	0.60610
$\frac{\partial T_G}{\partial \tau_3} = \frac{\partial T_G}{\partial \tau_5}$	-34.386	-2.8021	-1.7930	0.16148	1.6499	0.98148
$\frac{\partial T_G}{\partial \tau_4}$	-34.418	-2.3702	0.60009	-1.2475	-0.84414	0.78228

Step 4: For the second adjoint network analysis, the idea of using Tellegen's theorem to find the Norton current [11] at the generator end is applied. The second adjoint network is the one shown in Fig. 5, and applying Tellegen's theorem we have, analogously to (24)

$$V_g \hat{I}_g' = \sum_{j \in I_z} I_j \hat{V}_j'^s - \sum_{i \in I_y} V_i \hat{I}_i'^s. \tag{25}$$

Knowing the Norton current at the generator end, the second adjoint analysis is performed from the generator end to the load end, and hence the voltages and currents of the second adjoint network are known.

Consequently the group delay sensitivities can be evaluated after Step 4.

IV. EXAMPLE

Consider the filter shown in Fig. 6 with parameter values as shown [12]. Table V shows the group delay sensitivities with respect to characteristic impedances and delay times obtained by this method and perturbation. The agreement was to approximately seven significant figures. The parameters were perturbed by an absolute value of 0.5×10^{-7} both ways and using quadratic interpolation (central differences). The appropriate excitations of N, $\hat{N}$, N', and $\hat{N}'$ are shown in Table VI. Fig. 7 shows a plot versus frequency of the group delay and Fig. 8 of the group delay sensitivities with respect to characteristic impedances and delay times.

TABLE VI
EXCITATIONS OF THE CIRCUIT IN FIG. 6 NEEDED TO OBTAIN THE RESULTS OF TABLE V

Source	N	$\hat{N}$	N'	$\hat{N}'$
1	V_g	0	0	0
2	0	0	$(I'^s_{a1})_1$	$(\hat{I}'^s_{a1})_1$
3	0	0	$(I'^s_{a1})_2$	$(\hat{I}'^s_{a1})_2$
4	0	0	I'^s_{a2}	$\hat{I}'^s_{a2}$
5	0	0	I'^s_{a3}	$\hat{I}'^s_{a3}$
6	0	0	I'^s_{a4}	$\hat{I}'^s_{a4}$
7	0	0	I'^s_{a5}	$\hat{I}'^s_{a5}$
8	0	0	I'^s_{a6}	$\hat{I}'^s_{a6}$
9	0	0	$(I'^s_{a7})_1$	$(\hat{I}'^s_{a7})_1$
10	0	0	$(I'^s_{a7})_2$	$(\hat{I}'^s_{a7})_2$
11	0	$\hat{I}_0$	0	0

V. CONCLUSIONS

An efficient approach to the exact calculation of group delay sensitivities is presented, based on the adjoint network concept. A microwave filter example was used to demon-

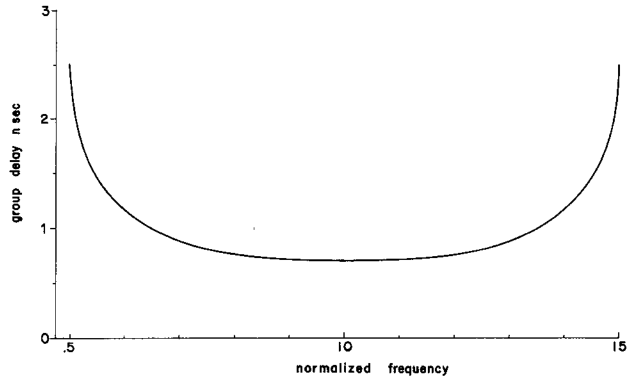
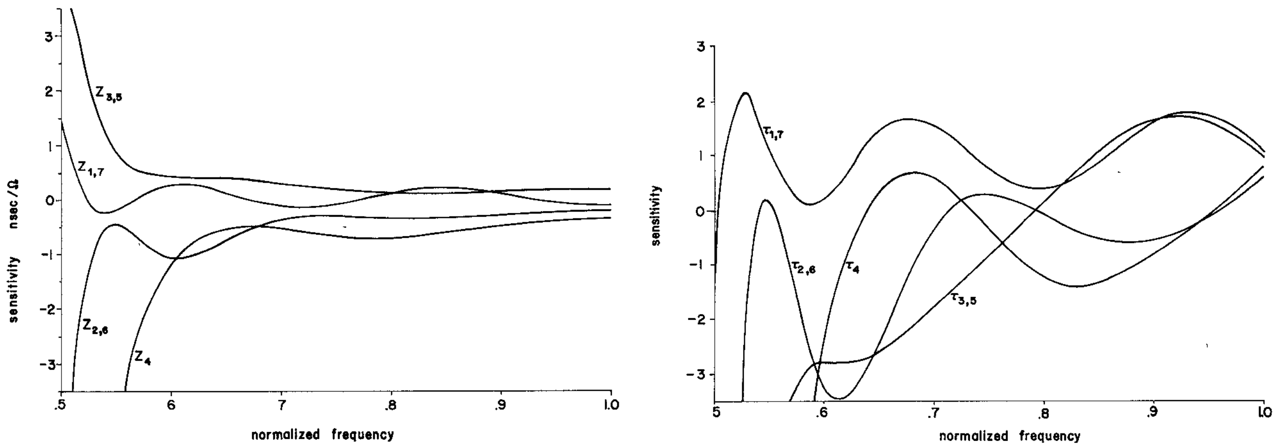


Fig. 7. Group delay of the filter.



(a) (b)

Fig. 8. (a) Group delay sensitivity with respect to characteristic impedances. (b) Group delay sensitivity with respect to delay times.

strate explicitly the analyses required. The approach is practical as well as exact, and should find use in circuit optimization involving group delay specifications. The results derived are particularly suited to microwave applications and include useful tables of sensitivity expressions. Further details of this work are available [13].

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