YIELD ESTIMATION FOR EFFICIENT DESIGN CENTERING ASSUMING ARBITRARY STATISTICAL DISTRIBUTIONS

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Summary: Based upon a uniform distribution inside an orthocell in the toleranced parameter space, it is shown how production yield and yield sensitivities can be evaluated for arbitrary statistical distributions. Formulas for yield and yield sensitivities in the case of a uniform distribution of outcomes between the tolerance extremes are given. A general formula for the yield, which is applicable to any arbitrary statistical distribution, is presented.

1. Introduction:

The yield problem has usually been treated through the Monte Carlo method of analysis. Elias [1] presented an approach which applies the Monte Carlo analysis directly to the nonlinear constraints. In an effort to reduce computational time Director and Hachtel [2] suggested applying the Monte Carlo method in conjunction with a polytope describing the constraint region. This polytope (a simplex being a special case [3]) might be defined by quite a large number of hyperplanes. For example, for a space of k dimensions, as described by the algorithm, this number may initially be 2^k. Scott and Walker [4] suggested an efficient technique using Monte Carlo analysis with space regionalization. However, the number of required analyses increases exponentially with the number of variables in order to get the response at the center of each region. Regionalization was later used by Leung and Spence [5] exploiting the technique of systematic exploration. This technique is only applicable to linear circuits.

Karafin [6] used a different approach. The yield was estimated according to truncated Taylor series approximations for the constraints. In the approach presented here we assume a reasonable nominal point and reasonable linear approximations to the constraints. These will usually be available if a centering or a worst-case tolerance assignment problem is solved first. The assumption of a reasonable nominal point was also required by Karafin [6].

The approach is based upon partitioning the region under consideration into a collection of orthotopic cells (orthocells). A weight is assigned to each orthocell and a uniform distribution is assumed inside it. The weights are obtained from tabulated values for known distributions or obtained according to sampling the components used. The freedom in choosing the sizes of the orthocells allows the use of previous information about the problem. A formula for the yield is derived according to these assumptions and it is applicable to any statistical distribution, whether we have independent parameters or correlated parameters with discrete or continuous tolerances.

2. Yield with a Uniform Distribution:

Define the tolerance region R_ε by

$$R_\varepsilon \triangleq \left\{ \underset{\sim}{\phi} \mid \phi_i^0 - \varepsilon_i \leq \phi_i \leq \phi_i^0 + \varepsilon_i \ , \ i = 1, 2, \ldots, k \right\} \quad , \tag{1}$$

where k is the number of designable parameters, ϕ^0 is the nominal parameter vector and ε is the vector of absolute tolerances of the corresponding parameters. Now, define the function V(R) as the hypervolume of the set R. Thus, for the case of independent parameters and assuming a uniform distribution of outcomes between the tolerance extremes, the yield is given by

$$Y = \frac{V(R_\varepsilon \cap R_c)}{V(R_\varepsilon)} \quad , \tag{2}$$

where

$$R_c \triangleq \left\{ \underset{\sim}{\phi} \mid g_\ell(\underset{\sim}{\phi}) \geq 0 \ , \ \ell = 1, 2, \ldots, m \right\} \tag{3}$$

is the constraint region defined by m linearized constraints

$$g_\ell(\underset{\sim}{\phi}) = \underset{\sim}{\phi}^T \underset{\sim}{q}^\ell - c^\ell \ , \ \ell = 1, 2, \ldots, m \ . \tag{4}$$

Assuming no overlapping of nonfeasible regions defined by different constraints inside the orthotope R_ε, i.e.,

$$R_i \underset{i \neq j}{\cap} R_j = \emptyset \quad , \tag{5}$$

where

$$R_\ell \triangleq \left\{ \underset{\sim}{\phi} \mid g_\ell(\underset{\sim}{\phi}) < 0 \right\} \cap R_\varepsilon \ , \tag{6}$$

the yield can be expressed as

$$Y = 1 - \frac{\sum_{\ell=1}^{m} V(R_\ell)}{V(R_\varepsilon)} \quad . \tag{7}$$

Define the set of all vertices of the orthotope R_ε by [7]

$$R_V \triangleq \left\{ \underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E} \, \underset{\sim}{\mu}, \ \mu_i \in \{-1, 1\}, \ i = 1, 2, \ldots, k \right\} , \tag{8}$$

where E is a k x k diagonal matrix with ε_i, i = 1, 2, ..., k along the diagonal and using the following vertex enumeration scheme:

$$r = 1 + \sum_{i=1}^{k} \frac{\mu_i^r + 1}{2} 2^{i-1} \quad . \tag{9}$$

Corresponding to each constraint $g_\ell(\underset{\sim}{\phi}) \geq 0$, let us define a reference vertex

$$\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^0 + \underset{\sim}{E} \, \underset{\sim}{\mu}^r \quad , \tag{10}$$

where

$$\mu_i^r = - \text{sign } (q_i^\ell) \ , \ i = 1, 2, \ldots, k \ . \tag{11}$$

If $g_\ell(\underset{\sim}{\phi}^r) \geq 0$, then $V(R_\ell) = 0$. Otherwise we find the distance between the intersection of the hyperplane $g_\ell(\underset{\sim}{\phi}) = 0$ and the reference vertex $\underset{\sim}{\phi}^r$ along an edge of R_ε in the ith direction given by

$$\alpha_i^\ell = \mu_i^r \, g_\ell(\underset{\sim}{\phi}^r) \ / \ q_i^\ell \quad . \tag{12}$$

The general formula for $V^\ell = V(R_\ell)$ is

$$V^\ell = \left\{ \frac{1}{k!} \prod_{j=1}^{k} \alpha_j^\ell \right\} \left\{ \sum_{s \in S_\ell} (-1)^{\nu^s} \, (\delta_\ell^s)^k \right\} \quad , \tag{13}$$

where

$$\delta_\ell^s = 1 - \sum_{j=1}^{k} \frac{\varepsilon_j}{\alpha_j^\ell} \left|\mu_j^s - \mu_j^r\right| \quad , \tag{14}$$

$$S_\ell \overset{\Delta}{=} \left\{ s \mid g_\ell(\underset{\sim}{\phi}^s) < 0,\ \underset{\sim}{\phi}^s = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\ \underset{\sim}{\mu}^s \right\} \quad , \tag{15}$$

$$\nu^s = \sum_{i=1}^{k} \left|\mu_i^s - \mu_i^r\right| \ / \ 2 \quad . \tag{16}$$

Hence,

$$\frac{\partial Y}{\partial \phi_i^0} = - \sum_{\ell=1}^{m} \frac{\partial V^\ell}{\partial \phi_i^0} \bigg/ V(R_\varepsilon) \quad , \tag{17}$$

$$\frac{\partial Y}{\partial \varepsilon_i} = \left(\frac{1}{\varepsilon_i} \sum_{\ell=1}^{m} V^\ell - \sum_{\ell=1}^{m} \frac{\partial V^\ell}{\partial \varepsilon_i} \right) \bigg/ \left(2^k \prod_{j=1}^{k} \varepsilon_j \right) \quad . \tag{18}$$

If $g_\ell(\underset{\sim}{\phi}^r) < 0$

$$\frac{\partial V^\ell}{\partial \phi_i^0} = \left\{ \frac{q_i^\ell}{k!} \sum_{p=1}^{k} \left(\frac{\mu_p^r}{q_p^\ell} \prod_{\substack{j=1 \\ j \neq p}}^{k} \alpha_j^\ell \right) \right\} A$$

$$+ B \left\{ k\ q_i^\ell \sum_{s \epsilon S_\ell} (-1)^{\nu^s} (\delta_\ell^s)^{k-1} \left(\sum_{j=1}^{k} \frac{\mu_j^r}{q_j^\ell} \frac{\varepsilon_j}{(\alpha_j^\ell)^2} \left|\mu_j^s - \mu_j^r\right| \right) \right\} \quad , \tag{19}$$

$$\frac{\partial V^\ell}{\partial \varepsilon_i} = \mu_i^r \frac{\partial V^\ell}{\partial \phi_i^0} - B \left\{ \frac{k}{\alpha_i^\ell} \sum_{s \epsilon S_\ell} \left|\mu_i^s - \mu_i^r\right| (-1)^{\nu^s} (\delta_\ell^s)^{k-1} \right\} \quad , \tag{20}$$

where

$$A = \sum_{s \epsilon S_\ell} (-1)^{\nu^s} (\delta_\ell^s)^k \quad , \tag{21}$$

$$B = \frac{1}{k!} \prod_{j=1}^{k} \alpha_j^\ell \quad . \tag{22}$$

It is to be noted that the yield sensitivities are discontinuous whenever a vertex $\underset{\sim}{\phi}^s$ satisfies the equation $g_\ell(\underset{\sim}{\phi}^s) = 0$ for any $\ell = 1, 2, \ldots, m$.

3. Yield with Statistical Distributions:

The probability distribution function (PDF) might extend as far as $(-\infty, \infty)$, however, for all practical cases we consider a tolerance region R_ε such that

$$\int_{R_\varepsilon} F(\underset{\sim}{\phi})\ d\phi_1\ d\phi_2 \ldots d\phi_k \simeq 1 \quad , \tag{23}$$

where $F(\underset{\sim}{\phi})$ is the PDF.

The orthotope R_ε is now partitioned into a set of orthocells $R(i_1, i_2, \ldots, i_k)$, where $i_j = 1, 2, \ldots, n_j$, n_j is the number of intervals in the jth direction and j = 1, 2, ..., k. A weighting factor $W(i_1, i_2, \ldots, i_k)$ is assigned to each orthocell and is given by

$$W(i_1, i_2, \ldots, i_k) = w(i_1, i_2, \ldots, i_k) \,/\, V(R(i_1, i_2, \ldots, i_k)), \tag{24}$$

where

$$w(i_1, i_2, \ldots, i_k) = \int_{R(i_1,i_2,\ldots,i_k)} F(\underset{\sim}{\phi})\, dv \quad , \tag{25}$$

$$V(R(i_1, i_2, \ldots, i_k)) = \int_{R(i_1,i_2,\ldots,i_k)} dv = \prod_{j=1}^{k} \varepsilon_{j,i_j} \quad , \tag{26}$$

$$dv = d\phi_1\, d\phi_2 \ldots d\phi_k \tag{27}$$

and $\varepsilon_{1,i_1}, \varepsilon_{2,i_2}, \ldots, \varepsilon_{k,i_k}$ are the dimensions of the orthocell.

By exploiting the way (13) is constructed, a formula for the weighted nonfeasible hypervolume with respect to the ℓth constraint is constructed and is given by

$$V^{\ell} = \left[\frac{1}{k!} \prod_{j=1}^{k} \alpha_j^{\ell}\right] \left[\sum_{i_1=1}^{n_1+1} \sum_{i_2=1}^{n_2+1} \cdots \sum_{i_k=1}^{n_k+1} \Delta W(i_1,i_2,\ldots,i_k)\, (\delta(i_1,i_2,\ldots,i_k))^k\right], \tag{28}$$

where, for <u>indexing with respect to</u> $\underset{\sim}{\phi}^r$, i.e., numbering starts at this vertex, α_j^{ℓ} = the distance from the reference vertex to the point of intersection in the jth direction,

$$\delta(i_1, i_2, \ldots, i_k) = \max\left(0, \left(1 - \sum_{j=1}^{k} \frac{1}{\alpha_j} \sum_{p=1}^{i_j} \varepsilon_{j,p-1}\right)\right) \quad , \tag{29}$$

$$\varepsilon_{j,0} = 0 \quad , \; j = 1, 2, \ldots, k \quad , \tag{30}$$

$$\begin{aligned} \Delta W(i_1,i_2,\ldots,i_k) = {} & W(i_1,i_2,\ldots,i_k) - \sum_{j=1}^{k} W(i_1,i_2,\ldots,i_{j-1},i_j-1,i_{j+1},\ldots,i_k) \\ & + \sum_{\substack{j,p=1 \\ j\neq p}}^{k} W(i_1, i_2,\ldots,i_j-1,\ldots,i_p-1,\ldots,i_k) - \ldots \\ & + (-1)^k\, W(i_1-1,i_2-1,\ldots,i_k-1) \end{aligned} \tag{31}$$

and

$$W(i_1,i_2,\ldots,i_k) = 0 \;\text{ if }\; i_j = 0 \;\text{ or }\; i_j = n_j+1 \text{ for any } j. \tag{32}$$

For the case of independent parameters we use

$$\Delta W(i_1,i_2,\ldots,i_k) = \prod_{j=1}^{k} (W_j(i_j) - W_j(i_j-1)) \quad , \tag{33}$$

where

$$W_j(0) = W_j(n_j+1) = 0 \quad , \tag{34}$$

$$W_j(i_j) = \int_{R_j(i_j)} f_j(\phi_j)\, d\phi_j \Big/ (\varepsilon_{j,i_j}) \;,\; i_j=1,2,\ldots,n_j \;, \tag{35}$$

$f_j(\phi_j)$ is the PDF of the jth parameter and $R_j(i_j)$ is the ith interval for that parameter.

Again, assuming nonoverlapping, nonfeasible regions defined by different constraints inside the orthotope R_ε, the yield can be expressed as

$$Y = 1 - \sum_{\ell=1}^{m} V^{\ell} \quad . \tag{36}$$

4. Examples:

The Karafin bandpass filter [6, 8], was used for verification of the yield formula using the first set of nominal values obtained by Bandler and Liu [8]. All inductors have the same Q at the nominal value given in [8] as the corresponding inductors in [6]. The results given in [8] as indicated by the authors violates the specifications at unconsidered frequency points. The adjoint network technique was used for evaluating the sensitivities and, hence, linearizing the constraints at these frequency points. The linearization was done at the worst violating vertex, i.e., the vertex which gives the most negative value for that particular constraint.

Taking a 10% tolerance on all 8 parameters, assuming uniform distributions, considering the sample points {190, 240, 360, 480, 490, 700, 860} Hz, the present approach indicates a yield of 92.6% in 1.7s computer time as compared with 93.0% in 51s computer time using 2000 Monte Carlo points with the nonlinear constraints.

In addition, the same uniform distribution of outcomes was considered but with the (more accurate) 5% tolerance components removed. For the aforementioned sample points, the present approach indicates 68.9% in 4.9s and the Monte Carlo method 71.0% in 45.6s.

Next, we consider the case of normal distributions with uncorrelated parameters [9]. An interval of 4 times the standard deviation was divided into 3 subintervals, the weights being in the ratio 0.2298 : 0.4950 : 0.2298 [10]. For a standard deviation on all 8 parameters of 6% the present approach indicates a yield of 88.4 % in 7.4s compared with 87.0% in 68s by the Monte Carlo method.

5. Conclusions:

It has been shown how yield may be estimated for arbitrary statistical distributions in an efficient way without recourse to the Monte Carlo method. Examples involving a number of distributions have been presented and the results contrasted with those given by the Monte Carlo method. It is felt that this work should be useful in optimization [11]. A full version of this paper with more theory, examples and illustrations is available [12].

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