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YIELD ESTIMATION FOR EFFICIENT DESIGN CENTRING ASSUMING ARBITRARY STATISTICAL DISTRIBUTIONS*

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SUMMARY

Based upon a uniform distribution inside an orthocell in the toleranced parameter space, it is shown how production yield and yield sensitivities can be evaluated for arbitrary statistical distributions. Formulae for yield and yield sensitivities in the case of a uniform distribution of outcomes between the tolerance extremes are given. A general formula for the yield, which is applicable to any arbitrary statistical distribution, is presented. An illustrative example for verifying the formulae is given. Karafin's bandpass filter has been used for applying the yield formula for a number of different statistical distributions. Uniformly distributed parameters between tolerance extremes, uniformly distributed parameters with accurate components removed and normally distributed parameters were considered. Comparisons with Monte Carlo analysis were made to contrast efficiency.

1. INTRODUCTION

Design centring and enlarging parameter tolerances, particularly for mass-produced designs such as integrated circuits, is a requirement for cost reduction. It is this aim which emphasizes the problem of yield estimation and makes it an integral part of the design process.

The yield problem has usually been treated through the Monte Carlo method of analysis. Elias[1] presented an approach which applies the Monte Carlo analysis directly to the nonlinear contraints. In an effort to reduce computational time Director and Hachtel[2] suggested applying the Monte Carlo method in conjunction with a polytope describing the constraint region. This polytope (a simplex being a special case[3]) might be defined by quite a large number of hyperplanes. For example, for a space of k dimensions, as described by the algorithm, this number may initially be 2^k. Scott and Walker[4] suggested an efficient technique using Monte Carlo analysis with space regionalization. However, the number of required analyses increases exponentially with the number of variables in order to get the response at the centre of each region. Regionalization was later used by Leung and Spence[5] exploiting the technique of systematic exploration. This technique is only applicable to linear circuits.

Karafin[6] used a different approach. The yield was estimated according to truncated Taylor series approximations for the constraints. In the approach presented here we assume a reasonable nominal point and reasonable linear approximations to the constraints. These will usually be available if a centring or a worst-case tolerance assignment problem is solved first. The assumption of a reasonable nominal point was also required by Karafin.[6]

The approach is based upon partitioning the region under consideration into a collection of orthotopic cells (orthocells). A weight is assigned to each orthocell and a uniform distribution is assumed inside it. The weights are obtained from tabulated values for known distributions or obtained according to sampling the components used. The freedom in choosing the sizes of the orthocells allows the use of previous information about the problem. A formula for the yield is derived according to these assumptions

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and it is applicable to any statistical distribution, whether we have independent parameters or correlated parameters with discrete or continuous tolerances.

An illustrative example was used to verify the yield and the yield sensitivity formulae for the uniform case. A comparison with the Monte Carlo analysis method as applied to Karafin's bandpass filter[6] is given for the following statistical distributions:

(a) A uniform distribution of outcomes between tolerance extremes using different values for the tolerances.
(b) A uniform distribution of outcomes between tolerance extremes, but with more accurate components selected out.
(c) Parameters with normal distributions for different values of the standard deviation.

Since the uniform distribution is basic to the presentation, we solve the problem of a uniform distribution first and generalize it for any distribution later.

2. YIELD WITH A UNIFORM DISTRIBUTION

The yield is simply defined by

$$Y \triangleq N/M \tag{1}$$

where N is the number of outcomes which satisfy the specifications and M is the total number of outcomes.

Define the tolerance region R_ε by

$$R_\varepsilon \triangleq \{\boldsymbol{\phi} | \phi_i^0 - \varepsilon_i \leq \phi_i \leq \phi_i^0 + \varepsilon_i, \quad i = 1, 2, \ldots, k\} \tag{2}$$

where k is the number of designable parameters, $\boldsymbol{\phi}^0$ is the nominal parameter vector and $\boldsymbol{\varepsilon}$ is the vector of absolute tolerances of the corresponding parameters.

Now, define the function $V(R)$ as the hypervolume of the set R. Thus, for the case of independent parameters and assuming a uniform distribution of outcomes between the tolerance extremes, (1) reduces to

$$Y = \frac{V(R_\varepsilon \cap R_c)}{V(R_\varepsilon)} \tag{3}$$

where

$$R_c \triangleq \{\boldsymbol{\phi} | g_l(\boldsymbol{\phi}) \geq 0, \quad l = 1, 2, \ldots, m\} \tag{4}$$

is the constraint region defined by m linearized constraints

$$g_l(\boldsymbol{\phi}) = \boldsymbol{\phi}^T \mathbf{q}^l - c^l, \quad l = 1, 2, \ldots, m \tag{5}$$

Assuming no overlapping of nonfeasible regions defined by different constraints inside the orthotope R_ε, i.e.

$$R_i \underset{i \neq j}{\cap} R_j = \varnothing \tag{6}$$

where

$$R_l \triangleq \{\boldsymbol{\phi} \,|\, g_l(\boldsymbol{\phi}) < 0\} \cap R_\varepsilon \tag{7}$$

the yield can be expressed as

$$Y = 1 - \frac{\sum_{l=1}^{m} V(R_l)}{V(R_\varepsilon)} \tag{8}$$

Define the set of all vertices of the orthotope R_ε by[7]

$$R_v \triangleq \{\boldsymbol{\phi} | \boldsymbol{\phi} = \boldsymbol{\phi}^0 + \mathbf{E}\boldsymbol{\mu}, \quad \mu_i \in \{-1, 1\}, i = 1, 2, \ldots, k\} \tag{9}$$

where $\mathbf{E}$ is a $k \times k$ diagonal matrix with ε_i, $i = 1, 2, \ldots, k$ along the diagonal and using the following vertex enumeration scheme:

$$r = 1 + \sum_{i=1}^{k} \frac{\mu_i^r + 1}{2} 2^{i-1} \tag{10}$$

Corresponding to each constraint $g_l(\boldsymbol{\phi}) \geq 0$, let us define a reference vertex

$$\boldsymbol{\phi}^r = \boldsymbol{\phi}^0 + \mathbf{E}\boldsymbol{\mu}^r \tag{11}$$

where

$$\mu_i^r = -\operatorname{sign}(q_i^l), \qquad i = 1, 2, \ldots, k \tag{12}$$

If $g_l(\boldsymbol{\phi}^r) \geq 0$, then $V(R_l) = 0$. Otherwise we find the distance between the intersection of the hyperplane $g_l(\boldsymbol{\phi}) = 0$ and the reference vertex $\boldsymbol{\phi}^r$ along an edge of R_ε in the ith direction given by

$$\begin{aligned} \alpha_i^l &= \mu_i^r g_l(\boldsymbol{\phi}^r)/q_i^l \\ &= \mu_i^r \left\{ \phi_i^0 + \mu_i^r \varepsilon_i - \frac{1}{q_i^l}\left[c^l - \sum_{\substack{j=1 \\ j \neq i}}^{k} q_j^l (\phi_j^0 + \mu_j^r \varepsilon_j) \right] \right\}, \qquad i = 1, 2, \ldots, k \end{aligned} \tag{13}$$

In order to derive an expression for $V^l = V(R_l)$, consider the two-dimensional examples shown in Figure 1. The nonfeasible area in Figure 1(a) is given by

$$\begin{aligned} V &= \Delta\phi^r ab - \Delta\phi^4 ac - \Delta\phi^1 bd \\ &= \frac{1}{2}\alpha_1\alpha_2 - \frac{1}{2}\left[\alpha_1\left(1 - \frac{2\varepsilon_1}{\alpha_1}\right)\right]\left[\alpha_2\left(1 - \frac{2\varepsilon_1}{\alpha_1}\right)\right] - \frac{1}{2}\left[\alpha_1\left(1 - \frac{2\varepsilon_2}{\alpha_2}\right)\right]\left[\alpha_2\left(1 - \frac{2\varepsilon_2}{\alpha_2}\right)\right] \\ &= \frac{1}{2}\alpha_1\alpha_2\left[1 - \left(1 - \frac{2\varepsilon_1}{\alpha_1}\right)^2 - \left(1 - \frac{2\varepsilon_2}{\alpha_2}\right)^2\right] \end{aligned}$$

Also, in Figure 1(b), the nonfeasible area is given by

$$\begin{aligned} V &= \Delta\phi^r ab - \Delta\phi^4 ac - \Delta\phi^1 bd + \Delta\phi^2 cd \\ &= \frac{1}{2}\alpha_1\alpha_2\left[1 - \left(1 - \frac{2\varepsilon_1}{\alpha_1}\right)^2 - \left(1 - \frac{2\varepsilon_2}{\alpha_2}\right)^2 + \left(1 - \frac{2\varepsilon_1}{\alpha_1} - \frac{2\varepsilon_2}{\alpha_2}\right)^2\right] \end{aligned}$$

A three-dimensional example is shown in Figure 2. In that example the linear constraint cuts the orthotope at the polygon $a\,b\,c\,d\,e$ and the volume is given by

$$\begin{aligned} V = {} & \frac{1}{6}\alpha_1\alpha_2\alpha_3 - \frac{1}{6}\alpha_1\alpha_2\alpha_3\left(1 - \frac{2\varepsilon_1}{\alpha_1}\right)^3 - \frac{1}{6}\alpha_1\alpha_2\alpha_3\left(1 - \frac{2\varepsilon_2}{\alpha_2}\right)^3 \\ & - \frac{1}{6}\alpha_1\alpha_2\alpha_3\left(1 - \frac{2\varepsilon_3}{\alpha_3}\right)^3 + \frac{1}{6}\alpha_1\alpha_2\alpha_3\left(1 - \frac{2\varepsilon_1}{\alpha_1} - \frac{2\varepsilon_2}{\alpha_2}\right)^3 \end{aligned}$$

Hence, the general formula can be written as

$$V(R_l) = \left\{ \frac{1}{k!} \prod_{j=1}^{k} \alpha_j^l \right\} \left\{ \sum_{s \in S_l} (-1)^{\nu^s} (\delta_l^s)^k \right\} \tag{14}$$

where

$$\delta_l^s = 1 - \sum_{j=1}^{k} \frac{\varepsilon_j}{\alpha_j^l} |\mu_j^s - \mu_j^r| \tag{15}$$

$$S_l \triangleq \{ s \mid g_l(\boldsymbol{\phi}^s) < 0, \quad \boldsymbol{\phi}^s = \boldsymbol{\phi}^0 + \mathbf{E}\boldsymbol{\mu}^s,\ \mu_i^s \in \{-1, 1\}, \quad i = 1, 2, \ldots, k \} \tag{16}$$

$$\nu^s = \sum_{i=1}^{k} |\mu_i^s - \mu_i^r|/2 \tag{17}$$

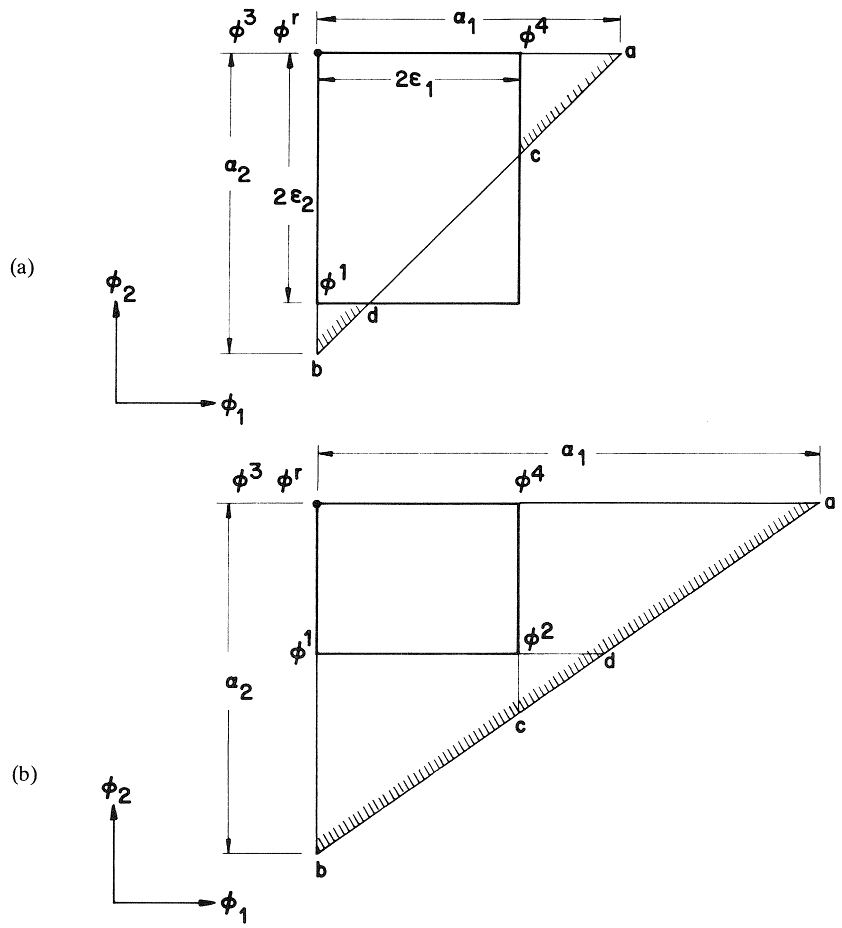

Figure 1. Two-dimensional examples illustrating the calculation of the nonfeasible hypervolumes, (a) Tolerance region partially feasible, (b) Tolerance region nonfeasible

An illustration of (14) for the case of $k=3$ is shown in Figure 2. Since

$$V(R_\varepsilon)=2^k \prod_{j=1}^{k} \varepsilon_j \tag{18}$$

the yield sensitivities can be expressed as

$$\frac{\partial Y}{\partial \phi_i^0}=-\sum_{l=1}^{m} \frac{\partial V^l}{\partial \phi_i^0} \Big/ V(R_\varepsilon) \tag{19}$$

$$\frac{\partial Y}{\partial \varepsilon_i}=\left(\frac{1}{\varepsilon_i}\sum_{l=1}^{m} V^l-\sum_{l=1}^{m}\frac{\partial V^l}{\partial \varepsilon_i}\right) \Big/ V(R_\varepsilon) \tag{20}$$

We take

$$\frac{\partial V^l}{\partial \phi_i^0}=\frac{\partial V^l}{\partial \varepsilon_i}=0 \quad \text{if } g_l(\boldsymbol{\phi}^r)\geq 0$$

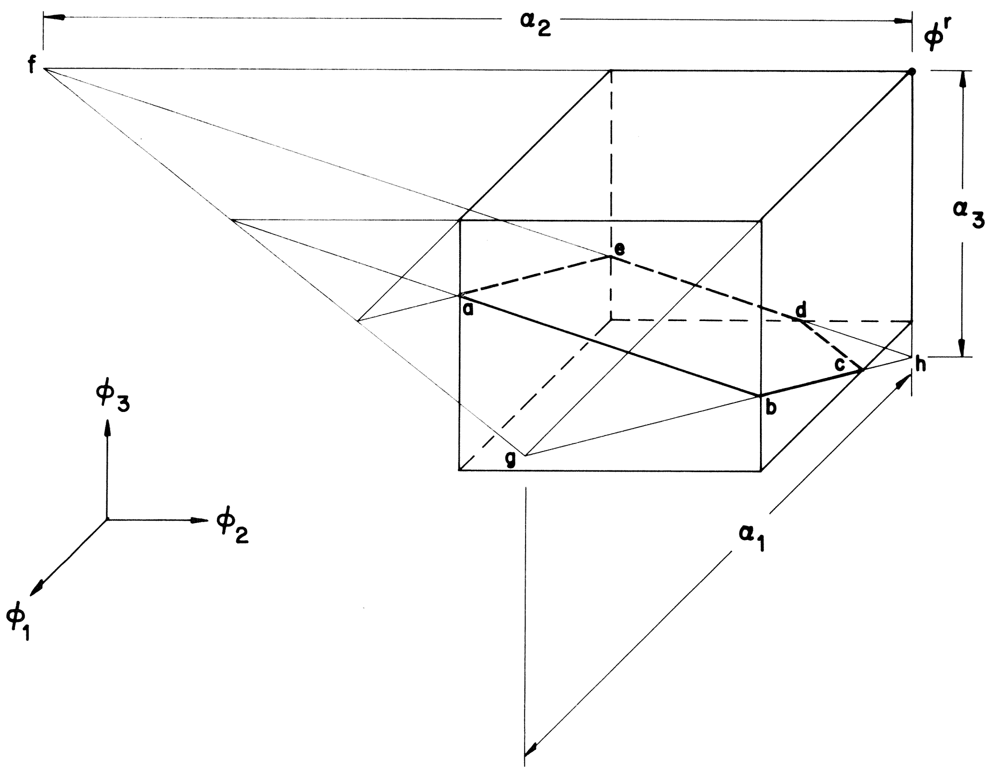

Figure 2. Three-dimensional example illustrating the calculation of nonfeasible hypervolumes in the case of a partially feasible tolerance region

otherwise

$$\frac{\partial V^l}{\partial \phi_i^0} = \left\{ \frac{q_i^l}{k!} \sum_{p=1}^{k} \left[\frac{\mu_p^r}{q_p^l} \prod_{\substack{j=1 \\ j \neq p}}^{k} \alpha_j^l \right] \right\} A + B \left\{ k q_i^l \sum_{s \in S_l} (-1)^{\nu s} (\delta_l^s)^{k-1} \left(\sum_{j=1}^{k} \frac{\mu_j^r}{q_j^l} \frac{\varepsilon_j}{(\alpha_j^l)^2} |\mu_j^s - \mu_j^r| \right) \right\} \tag{21}$$

$$\frac{\partial V^l}{\partial \varepsilon_i} = \mu_i^r \frac{\partial V^l}{\partial \phi_i^0} - B \left\{ \frac{k}{\alpha_i^l} \sum_{s \in S_l} |\mu_i^s - \mu_i^r| (-1)^{\nu s} (\delta_l^s)^{k-1} \right\} \tag{22}$$

where

$$A = \sum_{s \in S_l} (-1)^{\nu s} (\delta_l^s)^k \tag{23}$$

$$B = \frac{1}{k!} \prod_{j=1}^{k} \alpha_j^l \tag{24}$$

It is to be noted that the yield sensitivities are discontinuous whenever a vertex $\boldsymbol{\phi}^s$ satisfies the equation $g_l(\boldsymbol{\phi}^s) = 0$ for any $l = 1, 2, \ldots, m$. Also for the case of having $\alpha_j \to \infty$ there exists a limit for the hypervolume formula and its sensitivities.

For an alternative way of calculating $V(R_l)$ we define a complementary vertex

$$\boldsymbol{\phi}^{\bar{r}} = \boldsymbol{\phi}^0 + \mathbf{E}\boldsymbol{\mu}^{\bar{r}} \tag{25}$$

where

$$\mu_i^{\bar{r}} = -\mu_i^r = \text{sign}\,(q_i^l) \qquad i = 1, 2, \ldots, k \tag{26}$$

If $g_l(\boldsymbol{\phi}^{\bar{r}})\leq 0$, then $V(R_l)=V(R_\varepsilon)$. Otherwise we find the distance between the intersection of the hyperplane $g_l(\boldsymbol{\phi})=0$ and the complementary vertex $\boldsymbol{\phi}^{\bar{r}}$ along an edge of R_ε in the ith direction given by

$$\bar{\alpha}_i^l=\mu_i^{\bar{r}}g_l(\boldsymbol{\phi}^{\bar{r}})/q_i^l, \qquad i=1,2,\ldots,k \tag{27}$$

Hence we find the following equations:

$$V^l=V(R_l)=2^k\prod_{j=1}^k \varepsilon_j-\left\{\frac{1}{k!}\prod_{j=1}^k \bar{\alpha}_j^l\right\}\left\{\sum_{s\in\bar{S}_l}(-1)^{\bar{\nu}^s}(\bar{\delta}_l^s)^k\right\} \tag{28}$$

where

$$\bar{\delta}_l^s=1-\sum_{j=1}^k \frac{\varepsilon_j}{\bar{\alpha}_j^l}|\mu_j^s-\mu_j^{\bar{r}}| \tag{29}$$

$$\bar{S}_l\triangleq\{s\,|\,g_l(\boldsymbol{\phi}^s)>0,\quad \boldsymbol{\phi}^s=\boldsymbol{\phi}^0+\mathbf{E}\boldsymbol{\mu}^s,\ \mu_i^s\in\{-1,1\},\ i=1,2,\ldots,k\} \tag{30}$$

$$\bar{\nu}^s=\sum_{i=1}^k |\mu_i^s-\mu_i^{\bar{r}}|/2 \tag{31}$$

Equations (19) and (20) remain as before.

We take

$$\frac{\partial V^l}{\partial \phi_i^0}=0 \quad\text{and}\quad \frac{\partial V^l}{\partial \varepsilon_i}=2^k\prod_{\substack{j=1\\ j\neq i}}^k \varepsilon_j \quad \text{if } g_l(\boldsymbol{\phi}^{\bar{r}})\leq 0$$

otherwise

$$\frac{\partial V^l}{\partial \phi_i^0}=-\left\{\frac{q_i^l}{k!}\sum_{p=1}^k\left[\frac{\mu_p^{\bar{r}}}{q_p^l}\prod_{\substack{j=1\\ j\neq p}}^k \bar{\alpha}_j^l\right]\right\}\bar{A}$$
$$-\bar{B}\left\{kq_i^l\sum_{s\in\bar{S}_l}(-1)^{\bar{\nu}^s}(\bar{\delta}_l^s)^{k-1}\left(\sum_{j=1}^k \frac{\mu_j^{\bar{r}}}{q_j^l}\frac{\varepsilon_j}{(\bar{\alpha}_j^l)^2}|\mu_j^s-\mu_j^{\bar{r}}|\right)\right\} \tag{32}$$

$$\frac{\partial V^l}{\partial \varepsilon_i}=2^k\prod_{\substack{j=1\\ j\neq i}}^k \varepsilon_j+\mu_i^{\bar{r}}\frac{\partial V^l}{\partial \phi_i^0}+\bar{B}\left\{\frac{k}{\bar{\alpha}_i^l}\sum_{s\in\bar{S}_l}|\mu_i^s-\mu_i^{\bar{r}}|(-1)^{\bar{\nu}^s}(\bar{\delta}_l^s)^{k-1}\right\} \tag{33}$$

where

$$\bar{A}=\sum_{s\in\bar{S}_l}(-1)^{\bar{\nu}^s}(\bar{\delta}_l^s)^k \tag{34}$$

$$\bar{B}=\frac{1}{k!}\prod_{j=1}^k \bar{\alpha}_j^l \tag{35}$$

In order to obtain the hypervolume and its sensitivities efficiently we use the following criteria:

(i) If $g_l(\boldsymbol{\phi}^r)\geq 0$, use reference vertex approach.
(ii) If $g_l(\boldsymbol{\phi}^{\bar{r}})\leq 0$, use complementary vertex approach.
(iii) If $g_l(\boldsymbol{\phi}^r)<0$ and $g_l(\boldsymbol{\phi}^{\bar{r}})>0$, then
if $|g_l(\boldsymbol{\phi}^r)|\leq|g_l(\boldsymbol{\phi}^{\bar{r}})|$, use reference vertex approach,
if $|g_l(\boldsymbol{\phi}^r)|>|g_l(\boldsymbol{\phi}^{\bar{r}})|$, use complementary vertex approach.

The cases (i) and (ii) are clear since the tolerance orthotope will be either completely feasible or completely nonfeasible, respectively. Case (iii) follows from the theorem in the Appendix.

Example 1

Consider the following four-dimensional example, with a linear constraint

$$g(\boldsymbol{\phi})=\frac{\phi_1}{24}+\frac{\phi_2}{15}+\frac{\phi_3}{60}+\frac{\phi_4}{240}-1\geqslant 0$$

and where

$$\boldsymbol{\phi}^0=\begin{bmatrix}9\\7\\9\\26\end{bmatrix}\qquad \boldsymbol{\varepsilon}=\begin{bmatrix}5\\2\\4\\6\end{bmatrix}$$

Hence

$$\boldsymbol{\phi}^r=\begin{bmatrix}9\\7\\9\\26\end{bmatrix}+\begin{bmatrix}5&0&0&0\\0&2&0&0\\0&0&4&0\\0&0&0&6\end{bmatrix}\begin{bmatrix}-1\\-1\\-1\\-1\end{bmatrix}=\begin{bmatrix}4\\5\\5\\20\end{bmatrix}$$

and

$$V=\left[\frac{1}{4!}8\times5\times20\times80\right]\left[1-\left(1-\frac{4}{5}\right)^4-\left(1-\frac{8}{20}\right)^4-\left(1-\frac{12}{80}\right)^4+\left(1-\frac{4}{5}-\frac{12}{80}\right)^4+\left(1-\frac{8}{20}-\frac{12}{80}\right)^4\right]$$

$$=1034{\cdot}15.$$

Table I shows the nonfeasible vertices. A check for the analytical formulae for the gradients and the numerical gradients obtained by central differences is shown in Table II.

Table I. Nonfeasible vertices for Example 1

Vertex	ϕ_1	ϕ_2	ϕ_3	ϕ_4	μ_1	μ_2	μ_3	μ_4	Nonfeasible vertices
1	4	5	5	20	−1	−1	−1	−1	X
2	14	5	5	20	1	−1	−1	−1	
3	4	9	5	20	−1	1	−1	−1	X
4	14	9	5	20	1	1	−1	−1	
5	4	5	13	20	−1	−1	1	−1	X
6	14	5	13	20	1	−1	1	−1	
7	4	9	13	20	−1	1	1	−1	
8	14	9	13	20	1	1	1	−1	
9	4	5	5	32	−1	−1	−1	1	X
10	14	5	5	32	1	−1	−1	1	
11	4	9	5	32	−1	1	−1	1	X
12	14	9	5	32	1	1	−1	1	
13	4	5	13	32	−1	−1	1	1	X
14	14	5	13	32	1	−1	1	1	
15	4	9	13	32	−1	1	1	1	
16	14	9	13	32	1	1	1	1	

The alternative approach will lead to

$$\boldsymbol{\phi}^{\bar{r}}=\begin{bmatrix}9\\7\\9\\26\end{bmatrix}+\begin{bmatrix}5&0&0\\0&2&0\\0&0&4\\0&0&0\end{bmatrix}\begin{bmatrix}0&1\\0&1\\0&1\\6&1\end{bmatrix}=\begin{bmatrix}14\\9\\13\\32\end{bmatrix}$$

Table II. Hypervolume gradient check for Example 1

Parameters	Analytical gradients	Numerical gradients
ϕ_1^0	−337·50	−337·50
ϕ_2^0	−540·00	−540·00
ϕ_3^0	−135.00	−135·00
ϕ_4^0	−33·75	−33·75
ε_1	337·50	337·50
ε_2	573·60	573·60
ε_3	268·20	268·20
ε_4	173·18	173·18

and

$$V = 2^4 \times 5 \times 2 \times 4 \times 6 - \left[\frac{1}{4!}(8\times 1{\cdot}6)(5\times 1{\cdot}6)(20\times 1{\cdot}6)(80\times 1{\cdot}6)\right]$$
$$\cdot \left[1 - \left(1 - \frac{10}{8\times 1{\cdot}6}\right)^4 - \left(1 - \frac{4}{5\times 1{\cdot}6}\right)^4 - \left(1 - \frac{8}{20\times 1{\cdot}6}\right)^4 + \left(1 - \frac{4}{5\times 1{\cdot}6} - \frac{8}{20\times 1{\cdot}6}\right)^4\right.$$
$$- \left(1 - \frac{12}{80\times 1{\cdot}6}\right)^4 + \left(1 - \frac{10}{8\times 1{\cdot}6} - \frac{12}{80\times 1{\cdot}6}\right)^4 + \left(1 - \frac{4}{5\times 1{\cdot}6} - \frac{12}{80\times 1{\cdot}6}\right)^4$$
$$\left. + \left(1 - \frac{8}{20\times 1{\cdot}6} - \frac{12}{80\times 1{\cdot}6}\right)^4 - \left(1 - \frac{4}{5\times 1{\cdot}6} - \frac{8}{20\times 1{\cdot}6} - \frac{12}{80\times 1{\cdot}6}\right)^4\right]$$
$$= 3840 - 2805{\cdot}85 = 1034{\cdot}15.$$

3. YIELD WITH STATISTICAL DISTRIBUTIONS

The probability distribution function (PDF) might extend as far as $(-\infty, \infty)$; however, for all practical cases we consider a tolerance region R_ε such that

$$\int_{R_\varepsilon} F(\boldsymbol{\phi})\, d\phi_1\, d\phi_2 \ldots d\phi_k \simeq 1 \tag{36}$$

where $F(\boldsymbol{\phi})$ is the PDF.

The orthotope R_ε is now partitioned into a set of orthocells $R(i_1, i_2, \ldots, i_k)$ as in Figure 3, where $i_j = 1, 2, \ldots, n_j$, n_j is the number of intervals in the jth direction and $j = 1, 2, \ldots, k$. A weighting factor $W(i_1, i_2, \ldots, i_k)$ is assigned to each orthocell and is given by

$$W(i_1, i_2, \ldots, i_k) = w(i_1, i_2, \ldots, i_k)/V(R(i_1, i_2, \ldots, i_k)) \tag{37}$$

where

$$w(i_1, i_2, \ldots, i_k) = \int_{R(i_1, i_2, \ldots, i_k)} F(\boldsymbol{\phi})\, dv \tag{38}$$

$$V(R(i_1, i_2, \ldots, i_k)) = \int_{R(i_1, i_2, \ldots, i_k)} dv = \prod_{j=1}^{k} \varepsilon_{j,i_j} \tag{39}$$

$$dv = d\phi_1\, d\phi_2 \ldots d\phi_k \tag{40}$$

and $\varepsilon_{1,i_1}, \varepsilon_{2,i_2}, \ldots, \varepsilon_{k,i_k}$ are the dimensions of the orthocell.

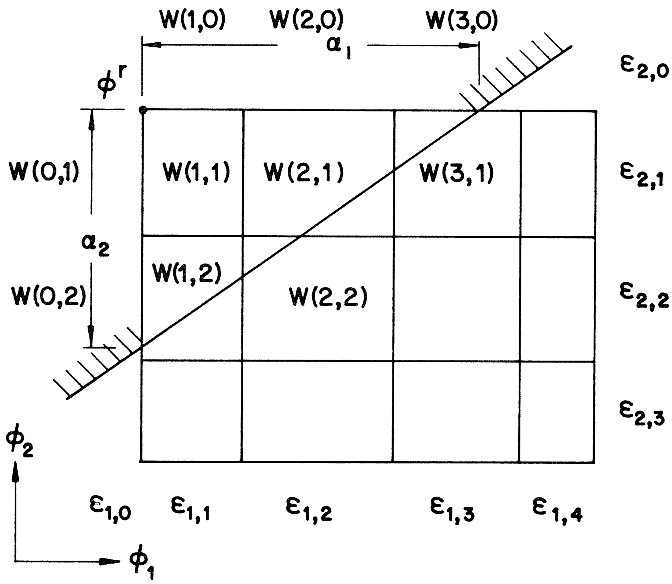

Figure 3. Two-dimensional illustration of the partitioning of the tolerance region into cells indicating the dimensions and weighting of those cells relevant to the calculation of the weighted nonfeasible hypervolume

In principle, the problem of finding the yield is now reduced to finding the contribution to the yield given by any of these orthocells. However, it will be a tedious job to consider

$$\prod_{j=1}^{k} n_j \text{ orthocells.}$$

By exploiting the way (14) is constructed, a formula for the weighted nonfeasible hypervolume with respect to the lth constraint is constructed and is given by

$$V^l = \left[\frac{1}{k!}\prod_{j=1}^{k}\alpha_j^l\right]\left[\sum_{i_1=1}^{n_1+1}\sum_{i_2=1}^{n_2+1}\cdots\sum_{i_k=1}^{n_k+1}\Delta W(i_1, i_2, \ldots, i_k)(\delta_l(i_1, i_2, \ldots, i_k))^k\right] \tag{41}$$

where, for *indexing with respect to* $\boldsymbol{\phi}^r$ (see Figure 3), α_j^l = the distance from the reference vertex to the point of intersection in the jth direction,

$$\delta_l(i_1, i_2, \ldots, i_k) = \max\left[0, \left(1 - \sum_{j=1}^{k}\frac{1}{\alpha_j^l}\sum_{p=1}^{i_j}\varepsilon_{j,p-1}\right)\right] \tag{42}$$

$$\varepsilon_{j,0} = 0, \qquad j = 1, 2, \ldots, k \tag{43}$$

$$\begin{aligned}\Delta W(i_1, i_2, \ldots, i_k) = {} & W(i_1, i_2, \ldots, i_k) - \sum_{j=1}^{k} W(i_1, i_2, \ldots, i_{j-1}, i_j - 1, i_{j+1}, \ldots, i_k)\\ & + \sum_{j=1}^{k-1}\sum_{p=j+1}^{k} W(i_1, i_2, \ldots, i_j - 1, \ldots, i_p - 1, \ldots, i_k) - \cdots\\ & + (-1)^k W(i_1 - 1, i_2 - 1, \ldots, i_k - 1)\end{aligned} \tag{44}$$

$$W(i_1, i_2, \ldots, i_k) = 0 \quad \text{if} \quad i_j = 0 \quad \text{or} \quad i_j = n_j + 1 \text{ for any } j. \tag{45}$$

For the case of independent parameters (41) can be written as

$$V^l = \left[\frac{1}{k!}\prod_{j=1}^{k}\alpha_j^l\right]\left[\sum_{i_1=1}^{n_1+1}\Delta W_1(i_1)\sum_{i_2=1}^{n_2+1}\Delta W_2(i_2)\cdots\sum_{i_k=1}^{n_k+1}\Delta W_k(i_k)(\delta_l(i_1, i_2, \ldots, i_k))^k\right] \tag{46}$$

where

$$\Delta W_j(i_j) = W_j(i_j) - W_j(i_j - 1) \tag{47}$$

$$W_j(0) = W_j(n_j + 1) = 0 \tag{48}$$

$$W_j(i_j) = \int_{R_j(i_j)} f_j(\phi_j)\, d\phi_j / \varepsilon_{j,i_j}, \qquad i_j = 1, 2, \ldots, n_j \tag{49}$$

$f_j(\phi_j)$ is the PDF of the jth parameter and $R_j(i_j)$ is the ith interval for that parameter. Table III illustrates the calculation of weighted hypervolume.

Table III. Example of calculation of weighted hypervolume by the general formula

Orthocell dimensions		i_1 / ε_{1,i_1}	0 / 0	1 / 3·0	2 / 3·0	3 / 2·0	4 / —
i_2	ε_{2,i_2}						
0	0	w, W	0	0	0	0	0
1	2·0	w	0	18/100	12/100	3/10	0
		W	0	3/100	1/50	3/40	0
		ΔW	—	3/100	−1/100	11/200	−3/40
		δ	—	1	3/4	1/2	1/3
2	3·0	w	0	12/100	8/100	2/10	0
		W	0	1/75	2/225	1/30	0
		ΔW	—	−1/60	1/180	−11/360	1/24
		δ	—	1/3	1/12	0	0
3	—	w, W	0	0	0	0	0
		ΔW	—	−1/75	1/225	−11/450	1/30
		δ	—	0	0	0	0

Reference vertex $\boldsymbol{\phi}^r$ given by $\mu_1^r = -1$, $\mu_2^r = 1$.
Intersections of the linear constraint are $\alpha_1 = 12$, $\alpha_2 = 3$.
Weighted volume $V = 1813/3600$.

Again, assuming nonoverlapping, nonfeasible regions defined by different constraints inside the orthotope R_ε, the yield can be expressed as

$$Y = 1 - \sum_{l=1}^{m} V^l \tag{50}$$

In short, the method approximates the integration of the PDF over the feasible region. It allows freedom in discretizing the PDF which is an advantage particularly if a worst-case solution is already known.

Example 2

The bandpass filter,[6,8] shown in Figure 4, was used for verification of the yield formula. The specifications are shown in Table IV. All inductors have the same Q at the nominal value given in Reference 8 as the corresponding inductors in Reference 6. The results given in Reference 8 as indicated by the authors violate the specifications at unconsidered frequency points. The adjoint network technique was used for evaluating the sensitivities and, hence, linearizing the constraints at these frequency points. The linearization was done at the worst violating vertex, i.e. the vertex which gives the most negative value for that particular constraint. The yields obtained by the present approach and applying the Monte Carlo method with the nonlinear constraints for a uniform distribution are shown in Table V. Further, as the tolerances were increased more frequency points were considered. In order to avoid overlapping

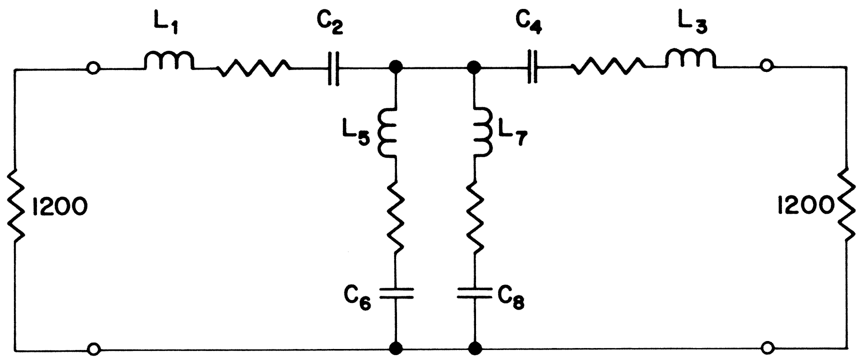

Figure 4. Karafin's bandpass filter. The values of the resistances are related to nominal values of the corresponding inductances by the same ratio used by Karafin (Reference 6, p. 112)

constraints, for each nonfeasible vertex the frequency point corresponding to the worst violated constraint is considered.

In addition, a uniform distribution of outcomes was considered but with the more accurate components removed. This gives $w_i(1)=w_i(3)=0{\cdot}5$ and $w_i(2)=0$. The problem is equivalent to having 2^8 different orthotopes. The results are shown in Table VI.

Consider now the case of a normal distribution which has a probability distribution function[9]

$$F(\boldsymbol{\phi})=\frac{1}{(2\pi)^{k/2}}\frac{1}{\sqrt{|COV|}}\exp\left[-\tfrac{1}{2}(\boldsymbol{\phi}-\boldsymbol{\phi}^0)^T(COV)^{-1}(\boldsymbol{\phi}-\boldsymbol{\phi}^0)\right]$$

where

k is the number of parameters,
$\boldsymbol{\phi}^0$ is the mean value of the parameter vector $\boldsymbol{\phi}$,
COV is the covariance matrix.

In the case of no correlation, COV is a diagonal matrix with variances σ_i^2, $i=1,2,\ldots,k$, along the diagonal. Hence,

$$F(\boldsymbol{\phi})=\frac{1}{(2\pi)^{k/2}}\frac{1}{\prod_{i=1}^{k}\sigma_i}\exp\left[-\sum_{i=1}^{k}\left(\frac{\phi_i-\phi_i^0}{2\sigma_i}\right)^2\right]$$

Table IV. Specifications for the bandpass filter

Frequency range (Hz)	Relative insertion loss (dB)	Type
0–240	35	lower (stopband)
360–490	3	upper (passband)
700–1000	35	lower (stopband)

Reference frequency 420 Hz (fixed, therefore, ripples higher than 3 dB are to be expected in the passband).
Nominal values $L_1^0=3{\cdot}0142$, $C_2^0=4{\cdot}975\times10^{-8}$, $L_3^0=2{\cdot}902$, $C_4^0=5{\cdot}0729\times10^{-8}$, $L_5^0=0{\cdot}82836$, $C_6^0=5{\cdot}5531\times10^{-7}$, $L_7^0=0{\cdot}30319$ and $C_8^0=1{\cdot}6377\times10^{-7}$.

Table V. Comparison with the Monte Carlo analysis for uniform distribution between tolerance extremes

Tolerances (%)								Sample points	Yield (%)		CDC Time (sec)	
ε_1/L_1^0	ε_2/C_2^0	ε_3/L_3^0	ε_4/C_4^0	ε_5/L_5^0	ε_6/C_6^0	ε_7/L_7^0	ε_8/C_8^0	(Hz)	Approx.	M.C.	Approx.*	M.C.†
6·99	6·52	6·97	6·55	4·36	5·69	6·80	5·25	188, 700, 876	100·00	99·75	0·67	24·0
7·00	7·00	7·00	7·00	5·00	6·00	7·00	6·00	188, 700, 876	100·00	99·65	0·66	24·2
8·00	8·00	8·00	8·00	6·00	7·00	8·00	7·00	188, 700, 876	99·99	99·60	0·67	24·4
8·00	8·00	8·00	8·00	6·00	7·00	8·00	7·00	190, 240, 360, 480, 490, 700, 860	99·94	99·35	1·56	52·4
10·00	10·00	10·00	10·00	10·00	10·00	10·00	10·00	190, 240, 360, 480, 490, 700, 860	92·62	93·00	1.67	51·4

CDC time for selecting frequency points = 7·65 sec.
* This time includes the linearization time.
† 2000 points were used in Monte Carlo (M.C.) analyses with the nonlinear constraints.

Table VI. Comparison with the Monte Carlo analysis for accurate components removed

$\frac{\phi_i - \phi_i^0}{\phi_i^0}$ (%)	Yield (%)		CDC Time (sec)	
	Approx.	M.C.	Approx.	M.C.
[−10, −5], [5, 10]	68·9	71·0	4·9	45·6

Frequency points used are 190, 240, 360, 480, 490, 700 and 860 Hz.

Using the described approach and dividing the interval $[\phi_i^0 - 2\sigma_i, \phi_i^0 + 2\sigma_i]$ for each parameter into three different subintervals the weights are obtained in the following manner. Let [10]

$$I_i = \frac{1}{\sqrt{(2\pi)}\sigma_i} \int_{\phi_i^0 - 2\sigma_i}^{\phi_i^0 - 2\sigma_i/3} \exp\left[-\left(\frac{\phi_i - \phi_i^0}{2\sigma_i}\right)^2\right] d\phi_i = 0{\cdot}2298$$

$$I_2 = \frac{1}{\sqrt{(2\pi)}\sigma_i} \int_{\phi_i^0 - 2\sigma_i/3}^{\phi_i^0 + 2\sigma_i/3} \exp\left[-\left(\frac{\phi_1 - \phi_i^0}{2\sigma_i}\right)^2\right] d\phi_i = 0{\cdot}4950$$

$$I_3 = \frac{1}{\sqrt{(2\pi)}\sigma_i} \int_{\phi_i^0 + 2\sigma_i/3}^{\phi_i^0 + 2\sigma_i} \exp\left[-\left(\frac{\phi_i - \phi_i^0}{2\sigma_i}\right)^2\right] d\phi_i = 0{\cdot}2298$$

Considering a probability of unity for finding ϕ_i in the interval $[\phi_i - 2\sigma_i, \phi_i + 2\sigma_i]$, the weights for each interval are given by (see Figure 5)

$$w_1 = w_3 = 0{\cdot}2298/(I_1 + I_2 + I_3)$$

$$w_2 = 0{\cdot}4950/(I_1 + I_2 + I_3)$$

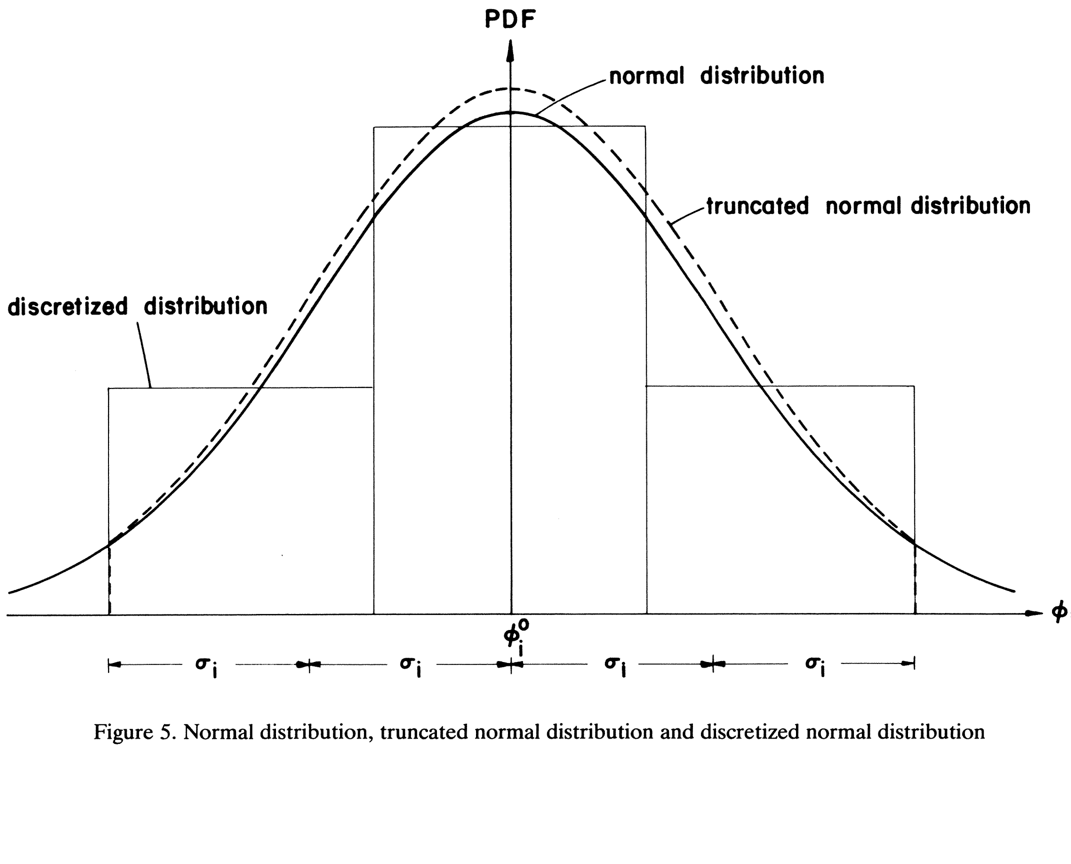

Figure 5. Normal distribution, truncated normal distribution and discretized normal distribution

The results are shown in Table VII for equal standard deviations for all of the eight parameters and for two values, namely, 5 per cent and 6 per cent. Table VIII shows the execution time if Monte Carlo analysis is applied to the linear constraints for the case of normally distributed parameters.

Table VII. Comparison with Monte Carlo analysis for normally distributed components

$\frac{\sigma_i}{\phi_i^0}$(%)	Yield (%)		CDC Time (sec)	
	Approx.	M.C.	Approx.	M.C.
5·0	96·5	95·1	4·9	69·2
6·0	88·4	87·0	7·4	68·0

Table VIII. Effect of number of Monte Carlo analyses on the yield based upon the linearized constraints

$\frac{\sigma_i}{\phi_i^0}$(%)	N.O.M.P.*	Yield (%)	CDC Time (sec)
5·0	2000	94·4	24·6
	500	94·2	7·0
	200	91·5	2·8
6·0	2000	86·6	24·3
	500	85·2	6·9
	200	84·0	2·8

* N.O.M.P. denotes the number of Monte Carlo points used.

CONCLUSIONS

It has been shown how yield may be estimated for arbitrary statistical distributions in an efficient way without recourse to the Monte Carlo method. Examples involving a number of distributions have been presented and the results contrasted with those given by the Monte Carlo method.

For the case of a uniform distribution between tolerance extremes yield sensitivity formulae have been derived with respect to nominal parameter values and tolerances assuming independent variables. These can be useful in optimization.[11,12] Since the uniform distribution is basic to the subsequent consideration of arbitrary distributions, it is felt that the ideas on sensitivity could be carried through to effect design centring with respect to given distributions.

As usual in iterative schemes, the choice of starting point may be important. In the present work it is recommended that a rough solution to a worst-case centring and tolerance assignment problem be used to provide and identify suitable active constraints. This allows only essential constraints to be considered and provides some justification for a worst-case solution even if less than 100 per cent yield is subsequently contemplated.[11,12]

APPENDIX

Theorem

If $g_l(\boldsymbol{\phi}^r)<0$, $g_l(\boldsymbol{\phi}^{\bar{r}})>0$ and $|g_l(\boldsymbol{\phi}^r)|\leq|g_l(\boldsymbol{\phi}^{\bar{r}})|$, then

$$\text{Order }(S_l)\leq\text{Order }(\bar{S}_l).$$

Proof. In the case under consideration the order of a set is simply the number of its elements. Assume that $s \in S_l$, then

$$g_l(\boldsymbol{\phi}^s) = g_l(\boldsymbol{\phi}^r) + (\boldsymbol{\phi}^s - \boldsymbol{\phi}^r)^T \nabla g_l(\boldsymbol{\phi}^r) < 0$$
$$= g_l(\boldsymbol{\phi}^r) + \sum_{i=1}^{k} \varepsilon_i(\mu_i^s - \mu_i^r) q_i^l < 0$$

or

$$-g_l(\boldsymbol{\phi}^r) + \sum_{i=1}^{k} \varepsilon_i(-\mu_i^s + \mu_i^r) q_i^l > 0$$

But, since

$$-g_l(\boldsymbol{\phi}^r) \leq g_l(\boldsymbol{\phi}^{\bar{r}}) \quad \text{and} \quad \mu_i^{\bar{r}} = -\mu_i^r$$

then

$$g_l(\boldsymbol{\phi}^{\bar{r}}) + \sum_{i=1}^{k} \varepsilon_i(-\mu_i^s - \mu_i^{\bar{r}}) q_i^l > 0$$

i.e.

$$g_l(\boldsymbol{\phi}^{\bar{s}}) > 0$$

where

$$\boldsymbol{\phi}^{\bar{s}} = \boldsymbol{\phi}^0 - \mathbf{E}\boldsymbol{\mu}^s$$

Hence

$$\bar{s} \in \bar{S}_l$$

This means that for each vertex $s \in S_l$ there exists a vertex $\bar{s} \in \bar{S}_l$; thus

$$\text{Order}\,(S_l) \leq \text{Order}\,(\bar{S}_l).$$

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