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YIELD OPTIMIZATION FOR ARBITRARY STATISTICAL DISTRIBUTIONS

PART I: THEORY

H.L. Abdel-Malek and J.W. Bandler

Group on Simulation, Optimization and Control
and Department of Electrical Engineering
McMaster University, Hamilton, Canada

ABSTRACT

This paper generalizes certain analytical formulas for yield and yield sensitivities so that design centering and yield optimization can be effectively carried out employing given statistical parameter distributions. The tolerance region of possible outcomes is discretized into a set of orthotopic cells. A suitable weight is assigned to each cell in conjunction with an assumed uniform distribution on the cell. Explicit formulas for yield and its sensitivities are derived. To avoid unnecessary evaluations of circuit responses, multidimensional quadratic interpolation is performed. Sparsity is exploited in the determination of these quadratic models leading to reduced computation as well as increased accuracy.

INTRODUCTION

The aim of this paper (see also Part II) is to present some theoretical concepts leading to the most general approach currently available for automatic optimization of production yield which avoids the use of the Monte Carlo method. Thus, the design centering and/or optimal tolerance assignment which is to be performed takes explicitly into account statistical distributions and possible parameter correlations.

The approach is based on the work of Bandler, Liu and Tromp [1] and represents a generalization of the work of Bandler and Abdel-Malek [2,3]. The presentation is directed to a nonlinear programming method of solution, and can be associated with original ideas suggested by a number of other researchers [4,5]. Part II of this paper applies this material to the optimization of yield for a current switch emitter follower.

FUNDAMENTAL CONCEPTS AND DEFINITIONS

A _design_ can be described by a _nominal parameter vector_ $\underset{\sim}{\phi}^0$ of the k designable parameters and a corresponding _tolerance vector_ $\underset{\sim}{\varepsilon}$. The tolerance vector $\underset{\sim}{\varepsilon}$ may be used to define the extremes of the tolerance region or the standard deviation, etc. It is assumed that the parameters can be varied continuously.

An _outcome_ $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}\}$ of a design $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}\}$ implies a point in the parameter space given by

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\, \underset{\sim}{\mu}, \tag{1}$$

where $\underset{\sim}{E}$ is a diagonal matrix with elements set to ε_i, i = 1, 2, .., k, and $\underset{\sim}{\mu}$ is a _random vector_ distributed according to the _joint probability distribution function_ (PDF). The PDF might extend as far as (-∞, ∞), however, for all practical cases it is possible to consider a _tolerance region_ R_ε such that

$$\int_{R_\varepsilon} F(\underset{\sim}{\phi})\, d\phi_1\, d\phi_2 \ldots d\phi_k \simeq 1 , \tag{2}$$

where $F(\underset{\sim}{\phi})$ is the PDF.

For the sake of simplicity as well as the implications of independent design parameters, there is no loss of generality to consider R_ε to be an _orthotope_ defined by

$$R_\varepsilon \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\, \underset{\sim}{\mu} ,\ \underset{\sim}{\mu} \in R_\mu\} , \tag{3}$$

where

$$R_\mu \triangleq \{\underset{\sim}{\mu} \mid -1 \le \mu_i \le 1,\ i = 1, 2, \ldots, k\} . \tag{4}$$

This orthotope is centered at $\underset{\sim}{\phi}^0$ and has edges of length $2\varepsilon_i$, i = 1, 2, ..., k. The extreme points of R_ε are called _vertices_ and the set of vertices is defined by [1]

$$R_V \triangleq \{\underset{\sim}{\phi} \mid \phi_i = \phi_i^0 + \varepsilon_i \mu_i,\ \mu_i \in \{-1,1\},\ i=1,2,\ldots,k\}. \tag{5}$$

The number of these vertices is 2^k and for

$$\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\, \underset{\sim}{\mu}^r,\ \mu_i^r \in \{-1,1\} \tag{6}$$

we have

$$r = 1 + \sum_{i=1}^{k} \left(\frac{\mu_i^r + 1}{2}\right) 2^{i-1}. \tag{7}$$

The _constraint region_ (or _feasible region_)

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itself is given by

$$R_c \stackrel{\Delta}{=} \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) \geq 0,\ i = 1, 2, \ldots, m_c\}\ , \quad (8)$$

where m_c is the number of constraints g_i. The production or manufacturing yield is simply

$$Y \stackrel{\Delta}{=} N/M\ , \quad (9)$$

where M is the total number of outcomes and N is the number of outcomes $\underset{\sim}{\phi}$ which satisfy the specifications, i.e., for which $\underset{\sim}{\phi} \in R_c$.

EXPLOITING SPARSITY IN QUADRATIC INTERPOLATION

Interpolation by Quadratic Polynomials

An approximate representation of a constraint $g(\underset{\sim}{\phi})$ by using its values at a finite set of points is possible [6]. These points are called nodes or base points, and denoted by $\underset{\sim}{\phi}^n$, $n = 1, 2, \ldots, N$.

In order to minimize the computational effort to obtain a quadratic polynomial approximation, the number of base points required will be chosen to be equal to the number of unknown coefficients, i.e., interpolation will be adopted. Hence, the number of base points is

$$N = (k+1)(k+2)/2\ . \quad (10)$$

Let R_I be the interpolation region defined by

$$R_I \stackrel{\Delta}{=} \{\underset{\sim}{\phi} \mid \delta_i \geq |\phi_i - \bar{\phi}_i|\ ,\ i = 1, 2, \ldots, k\}, \quad (11)$$

where $\bar{\underset{\sim}{\phi}}$ is the center of the interpolation region and δ_i, $i = 1, 2, \ldots, k$, are parameters defining the size of the interpolation region. The quadratic polynomial approximation can be expressed as

$$P(\underset{\sim}{\phi}) = a_0 + \underset{\sim}{a}^T(\underset{\sim}{\phi}-\bar{\underset{\sim}{\phi}}) + \frac{1}{2}(\underset{\sim}{\phi}-\bar{\underset{\sim}{\phi}})^T \underset{\sim}{H}(\underset{\sim}{\phi}-\bar{\underset{\sim}{\phi}}) \quad (12)$$

or

$$\begin{aligned} P(\underset{\sim}{\phi}) = {} & b_1(\phi_1)^2 + b_2(\phi_2)^2 + \ldots + b_k(\phi_k)^2 + b_{k+1}\,\phi_1\phi_2 \\ & + b_{k+2}\,\phi_1\,\phi_3 + \cdots + b_{N-k-1}\,\phi_{k-1}\,\phi_k \\ & + b_{N-k}\,\phi_1 + b_{N-k+1}\,\phi_2 + \ldots + b_{N-1}\,\phi_k + b_N, \end{aligned} \quad (13)$$

where $\underset{\sim}{H}$ is the Hessian matrix of the quadratic approximation and is given by

$$\underset{\sim}{H} = \underset{\sim}{\nabla}\,\underset{\sim}{\nabla}^T\, P(\underset{\sim}{\phi})\ , \quad (14)$$

$$\underset{\sim}{\nabla}^T = \left[\frac{\partial}{\partial\phi_1}\ \frac{\partial}{\partial\phi_2}\ \cdots\ \frac{\partial}{\partial\phi_k}\right]\ . \quad (15)$$

The relations between the coefficients in (12) and (13) are given by

$$b_i = h_{ii}/2\ ,\ i = 1, 2, \ldots, k\ , \quad (16)$$

$$b_\ell = h_{ij},\ \ell = j - i + \sum_{p=1}^{i}(k-p+1),\ i < j\ , \quad (17)$$

$$b_{N-k-1+i} = a_i - \sum_{j=1}^{k} h_{ij}\,\bar{\phi}_j\ ,\ i = 1, 2, \ldots, k, \quad (18)$$

$$b_N = a_0 - \sum_{i=1}^{k} a_i\bar{\phi}_i + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} h_{ij}\,\bar{\phi}_i\,\bar{\phi}_j\ , \quad (19)$$

where N is given by (10).

Sparsity and Choice of Base Points

If we have freedom in choosing the base points, we can save computational effort, particularly if the number of variables k is large. The system of linear equations

$$P(\underset{\sim}{\phi}^n) = g(\underset{\sim}{\phi}^n),\ n = 1, 2, \ldots, N, \quad (20)$$

has to be solved for the polynomial coefficients. In general, the resulting matrix is full, however, it is possible to make it sparse by using the following choice of base points. Let

$$[\underset{\sim}{\phi}^1\ \underset{\sim}{\phi}^2\ \ldots\ \underset{\sim}{\phi}^N] = \underset{\sim}{D}\,[\underset{\sim}{1}_k\ -\underset{\sim}{1}_k\ \underset{\sim}{B}\ \underset{\sim}{0}_k] + [\bar{\underset{\sim}{\phi}}\ \bar{\underset{\sim}{\phi}}\ \ldots\ \bar{\underset{\sim}{\phi}}], \quad (21)$$

where $\underset{\sim}{D}$ is a k x k diagonal matrix with diagonal elements δ_i, $\underset{\sim}{1}_k$ is a k-dimensional identity matrix, $\underset{\sim}{0}_k$ is a zero vector of dimension k, $\underset{\sim}{B}$ is a k x L matrix having the structure

$$\underset{\sim}{B} = \left[\begin{array}{c|c|c|c|c} \underset{\sim}{u}_{k-1}^T & \underset{\sim}{0}_{k-2}^T & \underset{\sim}{0}_{k-3}^T & & \\ \hline & \underset{\sim}{u}_{k-2}^T & \underset{\sim}{0}_{k-3}^T & & \underset{\sim}{0}_{k-2} \\ \cline{2-3} \underset{\sim}{T}_{k-1} & & \underset{\sim}{u}_{k-3}^T & & \\ \cline{3-3} & \underset{\sim}{T}_{k-2} & & \cdots & \\ & & \underset{\sim}{T}_{k-3} & & \underset{\sim}{u}_1 \\ & & & & \underset{\sim}{T}_1 \end{array}\right]\ , \quad (22)$$

where $\underset{\sim}{u}_j$ is a column vector of dimension j and having components u_{ij} such that

$$0 < |u_{ij}| \leq 1,\ i = 1, 2, \ldots, j\ , \quad (23)$$

$\underset{\sim}{T}_j$ is a diagonal matrix of dimension j with diagonal elements T_{ij} satisfying

$$0 < |T_{ij}| \leq 1,\ i = 1, 2, \ldots, j\ , \quad (24)$$

and

$$L = k(k-1)/2\ . \quad (25)$$

According to this choice of base points it is clear that

$$a_0 = g(\underset{\sim}{\phi}^N)\ . \quad (26)$$

The system of simultaneous linear equations is now a sparse system and its solution is

$$h_{ii} = [g(\underset{\sim}{\phi}^i) + g(\underset{\sim}{\phi}^{N-k-1+i}) - 2g(\underset{\sim}{\phi}^N)]/\delta_i^2\ , \quad (27)$$

$$a_i = [g(\underset{\sim}{\phi}^i) - g(\underset{\sim}{\phi}^{N-k-1+i})]/2\delta_i\ ,\ i=1,2,\ldots,k, \quad (28)$$

$$h_{ij} = h_{ji} = [g(\phi^{\ell}) - g(\phi^{N}) - (\zeta_i^j)^2 \frac{h_{ii}}{2} - (\zeta_j^i)^2 \frac{h_{jj}}{2}$$

$$- \zeta_i^j a_i - \zeta_j^i a_j]/\zeta_i^j \zeta_j^i , \quad (29)$$

where

$$\ell = j - i + \sum_{p=1}^{i} (k - p + 1) , \; j > i \quad (30)$$

and

$$\zeta_i^j = u_{j-i,k-i} \delta_i, \; \zeta_j^i = T_{j-i,k-i} \delta_j, \; i < j . \quad (31)$$

Subsequently, the number of multiplications/divisions required to obtain the approximation is reduced to $5k^2 - 2k$ instead of $(N^3 + 3N^2 - N)/3$ for Gauss elimination, where N is defined in (10).

Fig. 1 shows the choice of base points in two dimensions and three dimensions [2].

If we are not completely free in choosing the base points, for example, if the function evaluation is expensive and some evaluations, n say, for parameter values inside the interpolation region are known, the linear equations can be ordered such that these n equations come last. In solving the resulting system of simultaneous equations, we proceed with finding the polynomial coefficients using (27), (28) and (29) until we come to the full part of the matrix, i.e., the last n equations. The unknown coefficients beyond this point should be found by solving n simultaneous linear equations, for example, by Gauss elimination.

EVALUATION OF YIELD AND ITS SENSITIVITIES

The Linear Cut [2]

In order to obtain the linear cuts required for yield evaluation [2], consider linearizing the quadratic constraints at a point ϕ^a which may, for example, be the nominal point ϕ^0 or a vertex ϕ^r. Hence, the linear cut based upon the ℓth constraint is given by

$$g_\ell(\phi^a) + (\phi - \phi^a)^T \nabla g_\ell(\phi^a) \geq 0 . \quad (32)$$

Define a reference vertex ϕ^r by

$$\phi^r = \phi^0 + E \mu^r , \quad (33)$$

where

$$\mu_j^r = -\text{sign} \left(\frac{\partial g_\ell(\phi^a)}{\partial \phi_j}\right) , \; j = 1, 2, ..., k . \quad (34)$$

The distance from the reference vertex to the point of intersection with the ℓth cut along the orthotope edge in the jth direction is

$$\alpha_j^\ell = \mu_j^r [g_\ell(\phi^a) + (\phi^r - \phi^a)^T \nabla g_\ell(\phi^a)] / \left(\frac{\partial g_\ell(\phi^a)}{\partial \phi_j}\right) . \quad (35)$$

Accordingly, we have

$$\frac{\partial \alpha_j^\ell}{\partial \phi_i^0} = \mu_j^r \left(\frac{\partial g_\ell(\phi^a)}{\partial \phi_i} + (\phi^r - \phi^a)^T H_i \right) / \left(\frac{\partial g_\ell(\phi^a)}{\partial \phi_j}\right)$$

$$-\mu_j^r \left(g_\ell(\phi^a) + (\phi^r - \phi^a)^T \nabla g_\ell(\phi^a) \right) H_{ji} / \left(\frac{\partial g_\ell(\phi^a)}{\partial \phi_j} \right)^2 , \quad (36)$$

is the Hessian matrix which is a constant matrix for a quadratic function $g_\ell(\phi)$, H_i is the ith column of H and H_{ji} is an element of H. In deriving (36) it is assumed that $(\phi^r - \phi^a)$ is independent of ϕ_i^0, i = 1, 2, ..., k.

The General Distribution

As described earlier, we can assume that all outcomes will lie within the tolerance orthotope R_ϵ. This orthotope is now partitioned into a set of orthocells $R(i_1, i_2, ..., i_k)$ as shown in Fig. 2, where $i_j = 1, 2, ..., n_j$, n_j is the number of intervals in the jth direction and j = 1, 2, ..., k. A weighting factor $W(i_1, i_2, ..., i_k)$ is assigned to each orthocell and is given by

$$W(i) = w(i)/V(R(i)) , \quad (37)$$

where

$$i = (i_1, i_2, ..., i_k) , \quad (38)$$

$$w(i) = \int_{R(i)} F(\phi) \, dv , \quad (39)$$

$$V(R(i)) = \int_{R(i)} dv = \prod_{j=1}^{k} \epsilon_{j,i_j} , \quad (40)$$

$$dv = d\phi_1 \, d\phi_2 \ldots d\phi_k , \quad (41)$$

$\epsilon_{1,i_1}, \epsilon_{2,i_2}, ..., \epsilon_{k,i_k}$ are the dimensions of the orthocell and $F(\phi)$ is the joint probability distribution function (PDF).

The weighting factors $W(i)$ can also be obtained by sampling the parameters or from a histogram if the PDF is not available.

In principle, the problem of finding the yield is now reduced to finding the contribution to the yield given by all of these orthocells. A formula for the weighted nonfeasible hypervolume with respect to the ℓth constraint is constructed and is given by [3]

$$V^\ell = \left(\frac{1}{k!} \prod_{j=1}^{k} \alpha_j^\ell\right) \left(\sum_{i_1=1}^{n_1+1} \sum_{i_2=1}^{n_2+1} \ldots \sum_{i_k=1}^{n_k+1} \Delta W(i)(\delta^\ell(i))^k \right) , \quad (42)$$

where, for indexing with respect to ϕ^r, i.e., numbering starts at this vertex (see Fig. 2), α_j is the distance from the reference vertex to the point of intersection of the ℓth linear cut with the orthotope edge in the jth direction,

$$\delta^{\ell}(\underset{\sim}{i}) = \max\left(0,\ 1-\sum_{j=1}^{k}\frac{1}{\alpha_j^{\ell}}\sum_{p=1}^{i_j}\epsilon_{j,p-1}\right), \quad (43)$$

$$\epsilon_{j,0} = 0\ ,\ j = 1, 2, \ldots, k\ , \quad (44)$$

$$\Delta W(\underset{\sim}{i}) = W(\underset{\sim}{i}) - \sum_{j=1}^{k} W(\underset{\sim}{i}-\underset{\sim}{e}_j) + \sum_{j=1}^{k-1}\sum_{p=j+1}^{k} W(\underset{\sim}{i}-\underset{\sim}{e}_j-\underset{\sim}{e}_p) - \ldots$$

$$+ (-1)^k W(\underset{\sim}{i} - \underset{\sim}{e}_1 - \underset{\sim}{e}_2 - \ldots - \underset{\sim}{e}_k)\ , \quad (45)$$

$$\underset{\sim}{e}_j = (0, 0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0) \quad (46)$$

and where

$$W(\underset{\sim}{i}) = 0 \text{ if } i_j = 0 \text{ or } i_j = n+1 \text{ for any } j. \quad (47)$$

Assuming no overlapping of nonfeasible regions defined by different cuts inside the orthotope R_ϵ, i.e,

$$R_i \underset{i\neq j}{\cap} R_j = \emptyset, \quad (48)$$

where

$$R_\ell = \{\underset{\sim}{\phi} \mid g_\ell(\underset{\sim}{\phi}) < 0\} \cap R_\epsilon\ , \quad (49)$$

the yield can be expressed as

$$Y = 1 - \sum_{\ell=1}^{m} V^{\ell}\ , \quad (50)$$

where m is the number of linear cuts.

Independent Parameters

In the case of independent parameters, (42) can be written as [3]

$$V^{\ell} = \left(\frac{1}{k!}\prod_{j=1}^{k}\alpha_j^{\ell}\right)\left(\sum_{i_1=1}^{n_1+1}\Delta W_1(i_1)\sum_{i_2=1}^{n_2+1}\Delta W_2(i_2)\ \ldots \sum_{i_k=1}^{n_k+1}\Delta W_k(i_k)\ (\delta^{\ell}(\underset{\sim}{i}))^k\right), \quad (51)$$

where $\underset{\sim}{i}$ and $\delta^{\ell}(\underset{\sim}{i})$ are as defined in (38) and (43), respectively, and where

$$\Delta W_j(i_j) = W_j(i_j) - W_j(i_j-1)\ ,\ j = 1,2,\ldots,k, \quad (52)$$

$$W_j(0) = W_j(n_j+1) = 0\ , \qquad j = 1,2,\ldots,k, \quad (53)$$

$$W_j(i_j) = w_j(i_j)/\epsilon_{j,i_j}\ , \qquad i_j = 1,2,\ldots,n_j, \quad (54)$$

$$w_j(i_j) = \int_{R_j(i_j)} f_j(\phi_j)\, d\phi_j\ ,\ i_j = 1,2,\ldots,n_j, \quad (55)$$

$f_j(\phi_j)$ is the PDF of the jth parameter and $R_j(i_j)$ is the ith interval for that parameter. Similarly the yield will be given by (50).

Yield Sensitivities

Formulas for yield sensitivities can be derived assuming that the weighting factors $W(\underset{\sim}{i})$ are independent of $\underset{\sim}{\phi}^0$ as long as the ratios between ϵ_{j,i_j}, $i_j = 1, 2, \ldots, n_j$, are fixed for each parameter $j = 1, 2, \ldots, k$. This is true, for example, if the sizes of the orthocells are fixed.

Let

$$\kappa_{j,i_j} = \epsilon_{j,i_j}/\epsilon_j\ , \quad (56)$$

hence,

$$\sum_{i_j=1}^{n_j}\kappa_{j,i_j} = 2\ ,\ j = 1, 2, \ldots, k\ . \quad (57)$$

The yield sensitivities are now given by

$$\frac{\partial Y}{\partial\phi_i^0} = -\sum_{\ell=1}^{m}\frac{\partial V^{\ell}}{\partial\phi_i^0}\ , \quad (58)$$

$$\frac{\partial Y}{\partial\epsilon_i} = -\sum_{\ell=1}^{m}\frac{\partial V^{\ell}}{\partial\epsilon_i}\ , \quad (59)$$

where

$$\frac{\partial V^{\ell}}{\partial\phi_i^0} = \left(\frac{1}{k!}\sum_{j=1}^{k}\frac{\partial\alpha_j^{\ell}}{\partial\phi_i^0}\prod_{\substack{p=1\\p\neq j}}^{k}\alpha_p^{\ell}\right)B + A\left(k\sum_{i_1=1}^{n_1+1}\sum_{i_2=1}^{n_2+1}\ldots\sum_{i_k=1}^{n_k+1}\Delta W(\underset{\sim}{i})\ (\delta^{\ell}(\underset{\sim}{i}))^{k-1}\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\phi_i^0}\right), \quad (60)$$

$$\frac{\partial V^{\ell}}{\partial\epsilon_i} = \left(\frac{\mu_i^r}{k!}\sum_{j=1}^{k}\frac{\partial\alpha_j^{\ell}}{\partial\phi_i^0}\prod_{\substack{p=1\\p\neq j}}^{k}\alpha_p^{\ell}\right)B + A\left(k\sum_{i_1=1}^{n_1+1}\sum_{i_2=1}^{n_2+1}\ldots\sum_{i_k=1}^{n_k+1}\Delta W(\underset{\sim}{i})\ (\delta^{\ell}(\underset{\sim}{i}))^{k-1}\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\epsilon_i}\right), \quad (61)$$

$$A = \frac{1}{k!}\prod_{j=1}^{k}\alpha_j^{\ell}\ , \quad (62)$$

$$B = \sum_{i_1=1}^{n_1+1}\sum_{i_2=1}^{n_2+1}\ldots\sum_{i_k=1}^{n_k+1}\Delta W(\underset{\sim}{i})\ (\delta^{\ell}(\underset{\sim}{i}))^k \quad (63)$$

and where

$$\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\phi_i^0} = \begin{cases} 0 & \text{if } \delta^{\ell}(\underset{\sim}{i}) = 0, \\ \displaystyle\sum_{j=1}^{k}\frac{1}{(\alpha_j^{\ell})^2}\frac{\partial\alpha_j^{\ell}}{\partial\phi_i^0}\sum_{p=1}^{i_j}\epsilon_{j,p-1} & \text{if } \delta^{\ell}(\underset{\sim}{i}) > 0, \end{cases} \quad (64)$$

$$\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\epsilon_i} = \mu_i^r\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\phi_i^0} - \sum_{j=1}^{k}\frac{1}{\alpha_j^{\ell}}\sum_{p=1}^{i_j}\kappa_{j,p-1}\ . \quad (65)$$

The formulas for $\partial\alpha_j^{\ell}/\partial\phi_i^0$ and for μ_i^r are given by (36) and (34), respectively.

The case of independent parameters is obtained by substituting

$$\Delta W(\underset{\sim}{i}) = \prod_{j=1}^{k} \Delta W_j(i_j) \qquad (66)$$

in (60), (61) and (63).

Example

In order to illustrate the calculation of the weighted hypervolume, consider the two-dimensional example shown in Table I. The reference vertex $\underset{\sim}{\phi}^r$ is given by $\mu^r_1 = -1$, $\mu^r_2 = 1$ while $\alpha_1 = 12$ and $\alpha_2 = 3$. The weighted volume is given by

$$V = \left(\frac{1}{2} \times 12 \times 3\right) \sum_{i_1=1}^{4} \sum_{i_2=1}^{3} \Delta W(i_1, i_2)\ (\delta(i_1, i_2))^2$$
$$= 1813/3600 \ .$$

The same example can be considered as if the parameters are independent as shown in Table II and Table III. The same weighted volume will obviously be obtained.

Assuming that the sizes of the orthocells are fixed, the sensitivities of the weighted hypervolume with respect to the nominal parameter vector $\underset{\sim}{\phi}^0$ can be evaluated. The location of $\underset{\sim}{\phi}^0$ itself is not important. It is the relative location of the constraint with respect to the orthotope that matters. The constraint can be considered as

$$\phi_1/12 - \phi_2/3 \geq 0 \ .$$

According to (36) we have

$$\partial\alpha_1/\partial\phi^0_1 = -1, \ \partial\alpha_1/\partial\phi^0_2 = 4, \ \partial\alpha_2/\partial\phi^0_1 = -1/4$$

and $\partial\alpha_2/\partial\phi^0_2 = 1$.

Using (64), the values of $\partial\delta^\ell(\underset{\sim}{i})/\partial\phi^0_i$ are given in Table IV and Table V. Substituting in (60) we get

$$\frac{\partial V}{\partial\phi^0_1} = -43/720 \ , \quad \frac{\partial V}{\partial\phi^0_2} = 43/180 \ .$$

These sensitivities were verified numerically.

CONCLUSIONS

The yield estimation technique presented provides an inexpensive yield determination without the need for the multitude of circuit simulations required in the Monte Carlo method. The method approximates the integration of the PDF over the feasible region. In addition, the availability of yield sensitivities permits the use of efficient gradient optimization techniques (see Part II). The exploitation of sparsity in choosing the base points reduces the computational effort required for interpolation significantly. A full description of this work is available [7].

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TABLE I
EXAMPLE OF GENERAL HYPERVOLUME CALCULATION

Orthocell		i_1	0	1	2	3	4
dimensions		ε_{1,i_1}	0	3.0	3.0	2.0	-
i_2	ε_{2,i_2}						
0	0	w,W	0	0	0	0	0
		w	0	18/100	12/100	3/10	0
1	2.0	W	0	3/100	1/50	3/40	0
		ΔW	-	3/100	-1/100	11/200	-3/40
		δ	-	1	3/4	1/2	1/3
		w	0	12/100	8/100	2/10	0
2	3.0	W	0	1/75	2/225	1/30	0
		ΔW	-	-1/60	1/180	-11/360	1/24
		δ	-	1/3	1/12	0	0
		w,W	0	0	0	0	0
3	-	ΔW	-	-1/75	1/225	-11/450	1/30
		δ	-	0	0	0	0

TABLE II
LENGTHS AND WEIGHTS OF FIRST PARAMETER INTERVALS

i_1	ε_{1,i_1}	$w(i_1)$	$W(i_1)$	$\Delta W(i_1)$
0	0.0	0	0	-
1	3.0	3/10	1/10	1/10
2	3.0	2/10	1/15	-1/30
3	2.0	5/10	1/4	11/60
4	-	0	0	-1/4

TABLE III
LENGTHS AND WEIGHTS OF SECOND PARAMETER INTERVALS

i_2	ε_{2,i_2}	$w(i_2)$	$W(i_2)$	$\Delta W(i_2)$
0	0.0	0	0	-
1	2.0	6/10	3/10	3/10
2	3.0	4/10	2/15	-1/6
3	-	0	0	-2/15

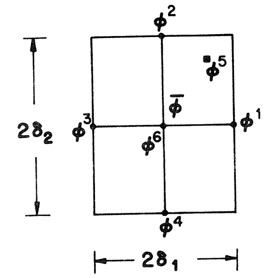
TABLE IV
VALUES OF $\partial\delta^{\ell}(i_1,i_2)/\partial\phi_1^0$

$i_2 \backslash i_1$	1	2	3	4
1	0	- 1/48	-1/24	-1/18
2	-1/18	-11/144	0	0
3	0	0	0	0

TABLE V
VALUES OF $\partial\delta^{\ell}(i_1,i_2)/\partial\phi_2^0$

$i_2 \backslash i_1$	1	2	3	4
1	0	1/12	1/6	2/9
2	2/9	11/36	0	0
3	0	0	0	0

(a)



(b)

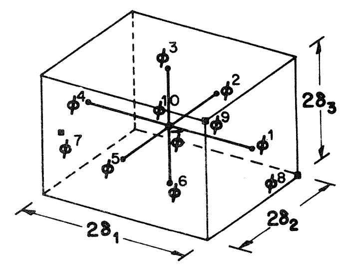


Fig. 1 Arrangement of the base points w.r.t. the centers of interpolation regions (a) two dimensions and (b) three dimensions. For the sparse formulation ϕ^7, ϕ^8 and ϕ^9 are, respectively, placed in the planes containing $\{\bar{\phi}, \phi^1, \phi^2\}$, $\{\bar{\phi}, \phi^1, \phi^3\}$ and $\{\bar{\phi}, \phi^2, \phi^3\}$.

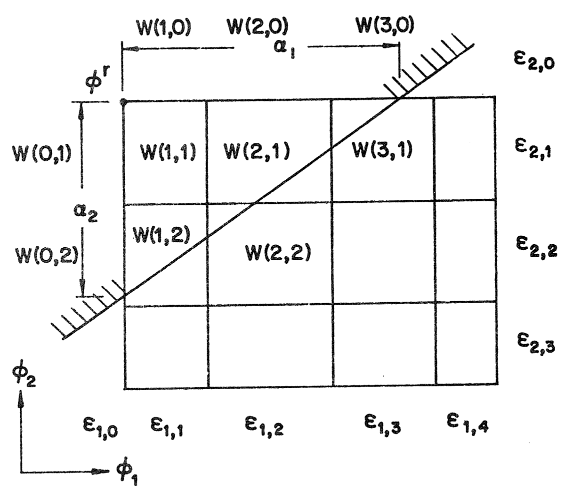


Fig. 2 Two-dimensional illustration of the partitioning of the tolerance region into cells indicating the dimensions and weighting of those cells relevant to the calculation of the weighted nonfeasible hypervolume.