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ADVANCES IN THE MATHEMATICAL PROGRAMMING APPROACH TO DESIGN CENTERING, TOLERANCING AND TUNING

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ABSTRACT

The nonlinear programming approach to the optimal worst-case assignment of parameter tolerances along with design centering taken by Bandler and extended to include tuning by Bandler, Liu and Tromp is reviewed. This work was directed at worst-case design in which, after tuning if necessary, all design outcomes must not fail to meet the specifications. A logical further extension by the present authors which relaxes the requirement of 100% yield is also outlined. Exact descriptions of the boundary of the constraint region via a generalized function of the least pth type are discussed. Consideration of such a function leads to new results applicable to postproduction tuning. Here, a tolerance problem equivalent to the tolerance and tuning problem of Bandler and Liu is presented. Based on this equivalence a mathematical definition of postproduction yield is developed and interpreted.

INTRODUCTION

The nonlinear programming approach to the optimal worst-case assignment of parameter tolerances along with design centering taken by Bandler [1,2] and extended to include tuning by Bandler, Liu and Tromp [3,4] is reviewed. This work was directed at worst-case design in which, after tuning if necessary, all design outcomes must not fail to meet the specifications.

A logical further extension by the present authors which relaxes the requirement of 100% yield is also outlined [5-11]. We developed analytical formulas for the calculation of yield and its sensitivities with respect to design parameters based upon certain approximations and assumptions. Optimization of yield taking into account realistic parameter distributions has been carried out.

The nonlinear programming formulations associated with centering, tolerancing, tuning and production yield can be extremely large. Approaches to reduce the size by exploiting various properties of the systems being designed have been discussed in the literature [12-16]. Methods of obtaining a relatively small number of candidates for constraint violation have been suggested [12-18]. Detection of critical regions, namely candidates for active vertices, has been considered by Bandler, Liu and Chen [12] and by Karafin [17] using sensitivity information. Bandler, Liu and Tromp [14] presented a scheme for vertex selection. Vertices are considered in the optimization process if they are candidates for satisfying the Kuhn-Tucker optimality conditions. Tromp [15,16] developed this work further and described an algorithm based on the Kuhn-Tucker conditions as well as the directions of the derivatives w.r.t. design parameters at the vertices of tolerance regions. Interval arithmetic was suggested by Madsen and Schjaer-Jacobsen [18] for detecting worst cases.

Exact descriptions of the boundary of the constraint region via a generalized function of the least pth type are discussed in this paper. Consideration of such a function leads to new results applicable to postproduction tuning. Here, a tolerance problem equivalent to the tolerance and tuning problem of Bandler and Liu [3] is presented and verified by circuit examples. Based on this equivalence a mathematical definition of postproduction yield is developed and interpreted.

The relevant work of some of the aforementioned and other important researchers [19-21] is

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contrasted with ours. This includes the early work by Karafin [17], who also forced discrete solutions by the branch and bound method, the nonlinear programming approach of Pinel and Roberts [19] employing truncated Taylor expansions of the constraint functions, the Monte Carlo approach of Elias [20] and the simplicial approximation method of Director and Hachtel [21] which employs linear searches in conjunction with linear programming.

FUNDAMENTAL CONCEPTS AND DEFINITIONS

A design is described by a nominal parameter vector $\underset{\sim}{\phi}^0$, a tolerance vector $\underset{\sim}{\varepsilon}$ and a tuning vector $\underset{\sim}{t}$, where

$$\underset{\sim}{\phi}^0 \triangleq \begin{bmatrix} \phi_1^0 \\ \phi_2^0 \\ \cdot \\ \cdot \\ \cdot \\ \phi_k^0 \end{bmatrix}, \qquad \underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_k \end{bmatrix}, \qquad \underset{\sim}{t} \triangleq \begin{bmatrix} t_1 \\ t_2 \\ \cdot \\ \cdot \\ \cdot \\ t_k \end{bmatrix} \tag{1}$$

and k is the number of designable parameters. The tolerance vector ε may be used to define the extremes of the tolerance region or the standard deviation, etc. The tuning vector $\underset{\sim}{t}$, defines the size of the tuning range. See Bandler, Liu and Tromp [4]. It is assumed that the parameters can be varied continuously. Some of these vector elements may be set to zero or held constant.

An outcome $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}\}$ of a design $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t}\}$ implies a point in the parameter space given by

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}, \tag{2}$$

where

$$\underset{\sim}{E} \triangleq \begin{bmatrix} \varepsilon_1 & & & & \\ & \varepsilon_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \varepsilon_k \end{bmatrix}, \qquad \underset{\sim}{\mu} \triangleq \begin{bmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \cdot \\ \mu_k \end{bmatrix} \tag{3}$$

and where $\underset{\sim}{\mu}$ is a random vector distributed according to a joint probability distribution function (PDF). The PDF might extend as far as $(-\infty, \infty)$, however, for all practical cases it is possible to consider a tolerance region R_ε such that

$$\int_{R_\varepsilon} F(\underset{\sim}{\phi})\, d\phi_1\, d\phi_2 \ldots d\phi_k \simeq 1 , \tag{4}$$

where $F(\underset{\sim}{\phi})$ is the PDF.

For the sake of simplicity as well as the implications of independent design parameters, there is no loss of generality in considering R_ε to be an orthotope defined by

$$R_\varepsilon \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu},\ \underset{\sim}{\mu} \in R_\mu\} , \tag{5}$$

where

$$R_\mu \triangleq \{\underset{\sim}{\mu} \mid -1 \leq \mu_i \leq 1,\ i = 1, 2, \ldots, k\} . \tag{6}$$

This orthotope is centered at ϕ^0 and has edges of length $2\varepsilon_i$, i = 1, 2, ..., k. The extreme points of R_ε are called vertices and the set of vertices is defined by [1-4]

$$R_v \triangleq \{\underset{\sim}{\phi} \mid \phi_i = \phi_i^0 + \varepsilon_i \mu_i,\ \mu_i \in \{-1,1\},\ i = 1, 2, \ldots, k\} . \tag{7}$$

The number of these vertices is 2^k and the following enumeration scheme introduced by Bandler

[1,2] will be considered. For a vertex

$$\underset{\sim}{\phi}^{r} = \underset{\sim}{\phi}^{0} + \underset{\sim}{E}\,\underset{\sim}{\mu}^{r}, \quad \mu_i^r \in \{-1,1\} , \tag{8}$$

we have

$$r = 1 + \sum_{i=1}^{k} \left(\frac{\mu_i^r+1}{2}\right) 2^{i-1} . \tag{9}$$

The <u>tuning region</u> is defined by [4]

$$R_t(\underset{\sim}{\mu}) \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^{0} + \underset{\sim}{E}\,\underset{\sim}{\mu} + \underset{\sim}{T}\,\underset{\sim}{\rho},\ \underset{\sim}{\rho} \in R_\rho\} , \tag{10}$$

where

$$\underset{\sim}{T} \triangleq \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_k \end{bmatrix}, \qquad \underset{\sim}{\rho} \triangleq \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{bmatrix} , \tag{11}$$

and R_ρ may be defined, for example, by

$$R_\rho \triangleq \{\underset{\sim}{\rho} \mid -1 \leq \rho_i \leq 1,\ i = 1, 2, \ldots, k\} \tag{12}$$

or in the case of <u>one-way tuning</u> or <u>irreversible trimming</u>,

$$R_\rho \triangleq \{\underset{\sim}{\rho} \mid -1 \leq \rho_i \leq 0,\ i = 1, 2, \ldots, k\}$$

or

$$R_\rho \triangleq \{\underset{\sim}{\rho} \mid 0 \leq \rho_i \leq 1,\ i = 1, 2, \ldots, k\} .$$

The <u>constraint region</u> (or <u>feasible region</u>) itself is given by

$$R_c \triangleq \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) \geq 0,\ i = 1, 2, \ldots, m_c\} , \tag{13}$$

where m_c is the number of constraints g_i. The tolerance, tuning and constraint regions are illustrated in Figure 1.

Production Yield

The production or manufacturing yield is simply defined by

$$Y \triangleq N/M , \tag{14}$$

where M is the total number of outcomes and N is the number of outcomes which satisfy the specifications. Similarly we define the <u>potential yield</u> by

$$Y_p \triangleq N_p/M , \tag{15}$$

where N_p is the number of outcomes which meet the specifications, after tuning if necessary. Hence, the relative frequency of outcomes which require tuning is

$$Y_t = Y_p - Y . \tag{16}$$

One-Dimensional Convexity

A region R is said to be <u>one-dimensionally convex</u> [1,2] if for any direction defined by the unit vector $\underset{\sim}{e}_j$, j = 1, 2, ..., k, and for any two points $\underset{\sim}{\phi}^a$, $\underset{\sim}{\phi}^b \in R$, where

$$\phi^b = \phi^a + c\, e_j\ ,\quad c \text{ is a scalar}\ , \tag{17}$$

then

$$\phi = \phi^a + \lambda(\phi^b - \phi^a) \in R \text{ for all } 0 \leq \lambda \leq 1. \tag{18}$$

One-dimensional convexity is illustrated in Figure 2.

If all vertices of the tolerance orthotope are within a one-dimensionally convex constraint region, then the whole tolerance orthotope lies inside the constraint region. For a proof see Bandler [1,2].

REVIEW OF CENTERING, TOLERANCING, TUNING AND YIELD OPTIMIZATION

Centering via Simplicial Approximation

The simplicial approximation approach of Director and Hachtel [21] involves linear programming as well as one-dimensional search techniques. Their approach is to inscribe a hypersphere inside the constraint region. During the process of enlarging this hypersphere a polytope which approximates the boundary of the constraint region is constructed.

The algorithm initially searches for points on the constraint boundary in both positive and negative directions for each parameter from a feasible point (a point within the constraint region). The convex hull described by these boundary points provides the initial polytope approximating the boundary of the constraint region. This polytope will be an interior approximation only if the constraint region is convex. Using linear programming a hypersphere is to be inflated inside this polytope in a k-dimensional space. The tangent hyperplanes are determined. These hyperplanes, faces of the polytope, are simplices in a space of k-1 dimensions. The largest simplex, i.e., the one which contains the largest hypersphere, is to be broken and replaced by k simplices. This is performed by adding a new vertex to the polytope obtained by searching for a boundary point along the normal direction to the largest simplex from the center of the corresponding hypersphere. The computational effort per iteration can be expressed as [7]

$$CE = LP_k + (k+1)\, LP_{k-1} + LS\ ,$$

where LP_j is the computational effort to solve a j-dimensional linear program and LS is the computational effort in a one-dimensional search.

It is to be noted that the number of constraints for the linear programming problem increases with the number of faces of the polytope. For the k-dimensional linear program and at the nth iteration we have $2^k + (n-1)k$ constraints, while for the k-1 dimensional linear program the number of constraints is fixed and is equal to k. The sequence of approximations is regarded to have converged when

$$r_{n+1} - r_n \leq \delta_1\, r_n + \delta_2\ ,$$

where r_n is the radius of the hypersphere obtained in the nth iteration, δ_1 and δ_2 are given relative and absolute convergence parameters.

The Nonlinear Programming Approach

The method described before does not explicitly optimize values for parameter tolerances, in other words there is no optimal tolerance assignment.

Pinel and Roberts [19] used nonlinear programming to assign parameter tolerances. The nominal parameter values are fixed and the constraints are approximated by truncated Taylor series expansions. Bandler et al. [1-4] treated centering and tolerancing simultaneously with the goal of increased tolerances by permitting the nominal point to move.

A nonlinear programming formulation of the optimal centering, tolerancing and tuning problem is

$$\underset{\phi^0,\ \epsilon,\ t \geq 0}{\text{minimize}}\quad C(\phi^0, \epsilon, \mu, t)\ , \tag{19}$$

subject, for example, to a constraint on yield

$$Y(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}, \underset{\sim}{t}) \geq Y_L , \tag{20}$$

where C is a suitable cost function, sometimes called objective function, and Y_L is a lower yield specification.

The objective function C should reflect a realistic cost-tolerance and tuning relation. Reasonable properties of the objective function are [4]

$$\begin{aligned}
&C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}, \underset{\sim}{t}) \rightarrow \text{constant} && \text{as } \underset{\sim}{\varepsilon} \rightarrow \underset{\sim}{\infty} \\
&C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}, \underset{\sim}{t}) \rightarrow \infty && \text{for any } \varepsilon_i \rightarrow 0 \\
&C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}, \underset{\sim}{t}) \rightarrow C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}) && \text{as } \underset{\sim}{t} \rightarrow \underset{\sim}{0} \\
&C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}, \underset{\sim}{t}) \rightarrow \infty && \text{for any } t_i \rightarrow \infty .
\end{aligned} \tag{21}$$

An orthotope describing the tolerance region is to be inflated by minimizing the cost function. The center of the orthotope provides the nominal parameter values and the lengths of the orthotope edges are twice the absolute tolerances.

Elias [20] presented an approach which applies the Monte Carlo analysis directly to the nonlinear constraints.

Karafin [17] presented an approach using truncated Taylor series approximations to the constraints. The constraint function values are assumed to be normally distributed for all tolerance choices. The parameters are assumed to be statistically independent and each parameter is symmetrically distributed about its nominal value. According to these assumptions, Karafin was able to reduce the k-fold integration of the k-variate probability distribution function to at most 3-fold integration. The yield estimate is based upon the resulting distributions of the values of the constraints. In minimizing the cost, the branch and bound method was used to force the parameters to discrete values.

Quadratic Modeling and Dynamic Linear Cuts

A nonlinear programming approach but employing approximations to the design constraints has been described in detail by the present authors [5-11]. An interpolation region centered at the initial guess to the nominal design is chosen. The simulation program is used to provide the value of the response functions (constraints) at a certain set of base points. The base points are points within the interpolation region and defined in terms of values of the designable parameters. Based upon the corresponding values of the resulting responses, multidimensional quadratic polynomials are constructed. These quadratic polynomials have the general form

$$P(\underset{\sim}{\phi}) = a_0 + \underset{\sim}{a}^T(\underset{\sim}{\phi} - \underset{\sim}{\bar{\phi}}) + \frac{1}{2}(\underset{\sim}{\phi} - \underset{\sim}{\bar{\phi}})^T \underset{\sim}{H}(\underset{\sim}{\phi} - \underset{\sim}{\bar{\phi}}) , \tag{22}$$

where a_0 and $\underset{\sim}{a}$ are, respectively, a constant scalar and a constant vector, $\underset{\sim}{H}$ is a constant symmetric Hessian matrix of the quadratic and $\underset{\sim}{\bar{\phi}}$ is the center of the chosen interpolation region.

The base points are simply those points where the approximated response function and the quadratic polynomial coincide. A system of simultaneous linear equations has to be solved to obtain the polynomial. The number of base points (exactly equal to the number of simulations required) is the minimum necessary to fully describe the responses and is given by

$$N = (k+1)(k+2)/2 , \tag{23}$$

where k is the number of designable parameters. The number N is the number of the unknown coefficients.

The authors have suggested ways of reducing the computational effort in solving the resulting system of N simultaneous linear equations ($N^3/3 + N^2 - N/3$ multiplications or divisions for Gauss elimination). Sparsity was forced in the system matrix [7,9] by a special choice of base points.

The quadratic approximations are updated as the optimization process or accuracy may require [5]. If the optimization indicates an optimum far away from the interpolation region the approximations are updated. Also, if higher accuracy is required the size of the interpolation region is reduced and hence the approximations are updated. Since vertices of the tolerance region are considered to be critical, approximations should be updated to cover potentially active vertices.

Since the approximations embody information about the sensitivities of the approximated constraints w.r.t. the designable parameters, they can be utilized for investigating the effects of slightly perturbing some constraints without requiring any additional simulations.

It is inexpensive to conduct a Monte Carlo analysis in conjunction with the approximation, however, the resulting yield will not be a continuous function of the design parameters due to the finite number of Monte Carlo analyses. Also, yield sensitivities are not available from the Monte Carlo analysis. Both continuity of yield and the availability of its sensitivites are of particular relevance if optimization (of yield or cost) is used. The authors have, therefore, directed their efforts to developing a method incorporating these features. This method can also be used by itself for yield analysis only [6].

The basic idea is to use weighted hypervolumes for evaluating the yield [6]. Evaluating hypervolumes, in general, is expensive because it involves a multidimensional integration. For the special case of cutting an orthotope by a linear constraint, a simple formula can be found [5,6]. A suitable formula was originally stated by Tromp [22], following a concept suggested by Karafin [17].

Our method does not assume that these linear cuts are fixed in the parameter space. It is possible [5,9] that these linear cuts be continuously updated to follow the generally nonlinear constraints. This facilitates a good approximation to the boundary of the constraint region as the tolerance region is allowed to move in the parameter space during, for example, an optimization process. Methods for continuously updating the linear cuts have been given [5,9].

Having an analytical formula for the yield facilities inexpensive yield and yield sensitivity evaluations. The procedure is suitable for optimization, in particular due to the relatively large number of yield analyses required.

THE EQUIVALENT TOLERANCE PROBLEM

A tolerance problem which is equivalent to the tolerance and tuning problem of Bandler and Liu [3] is presented. The <u>generalized least pth function</u> [23] required for constructing the equivalent problem, is given. Based on this equivalence, a mathematical definition of yield is developed.

The Generalized Least pth Function

Given a set of functions $f_j(\underset{\sim}{\phi})$, $j \in J$, we define

$$U(f_j(\underset{\sim}{\phi}), J, p, \lambda) \triangleq \begin{cases} 0 & \text{if } M = 0 \;, \\ \lambda M \left[\sum_{j \in K} \left[\dfrac{\lambda f_j(\underset{\sim}{\phi})}{M} \right]^q \right]^{1/q} & \text{if } M \neq 0 \;, \end{cases} \tag{24}$$

where

$$M = \max_{j \in J} (\lambda f_j) \;, \quad q = p \,\text{sign}\, M \;, \tag{25}$$

$$K = \begin{cases} J \text{ for } M < 0 \;, \\ \{ j \mid j \in J,\ \lambda f_j(\underset{\sim}{\phi}) > 0 \} \text{ for } M > 0 \end{cases} \tag{26}$$

$$\lambda = \begin{cases} 1 \text{ if } U \text{ approximates } \max_{j \in J} f_j(\underset{\sim}{\phi}) , \\ \\ -1 \text{ if } U \text{ approximates } \min_{j \in J} f_j(\underset{\sim}{\phi}) \end{cases} \tag{27}$$

and where p is a scalar greater than one.

Theorem

An outcome, given by (2) as

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\, \underset{\sim}{\mu} ,$$

can be tuned to satisfy the constraints, i.e., there exists $\underset{\sim}{\rho} \in R_\rho$ such that

$$\underset{\sim}{\phi} + \underset{\sim}{T}\, \underset{\sim}{\rho} \in R_c ,$$

if and only if $\underset{\sim}{\phi} \in R_{ct}$, where R_{ct} is the <u>tunable constraint region</u> defined by

$$R_{ct} \overset{\Delta}{=} \{\underset{\sim}{\phi} \mid \max_{\underset{\sim}{\rho} \in R_\rho} U(g_i(\underset{\sim}{\phi} + \underset{\sim}{T}\underset{\sim}{\rho}) , I, \infty, -1) \geq 0\} , \tag{28}$$

where

$$I \overset{\Delta}{=} \{1, 2, \ldots, m_c\}. \tag{29}$$

Proof

Assume that there exists $\underset{\sim}{\rho}^* \in R_\rho$ such that

$$\underset{\sim}{\phi} + \underset{\sim}{T}\, \underset{\sim}{\rho}^* \in R_c .$$

Hence,

$$g_i(\underset{\sim}{\phi} + \underset{\sim}{T}\, \underset{\sim}{\rho}^*) \geq 0 \text{ for all } i \in I .$$

Also,

$$\min_{i \in I} g_i(\underset{\sim}{\phi} + \underset{\sim}{T}\, \underset{\sim}{\rho}^*) \geq 0 .$$

But since

$$\max_{\underset{\sim}{\rho} \in R_\rho} U(g_i(\underset{\sim}{\phi}+\underset{\sim}{T}\underset{\sim}{\rho}), I, \infty, -1) \geq U(g_i(\underset{\sim}{\phi}+\underset{\sim}{T}\underset{\sim}{\rho}^*), I, \infty, -1)$$

and

$$U(g_i(\underset{\sim}{\phi}+\underset{\sim}{T}\underset{\sim}{\rho}^*), I, \infty, -1) = \min_{i \in I} g_i(\underset{\sim}{\phi}+\underset{\sim}{T}\underset{\sim}{\rho}^*) \geq 0 \tag{30}$$

then

$$\underset{\sim}{\phi} \in R_{ct} .$$

Now, $\underset{\sim}{\phi} \in R_{ct}$ implies that there exists $\underset{\sim}{\rho}^* \in R_\rho$ such that (30) is satisfied. Consequently,

$$\underset{\sim}{\phi} + \underset{\sim}{T}\, \underset{\sim}{\rho}^* \in R_c .$$

Example

To illustrate this idea, consider a two-dimensional example in which the constraint region is defined by the two constraints [7]

$$g_1(\underset{\sim}{\phi}) = \phi_2 - \phi_1 \geq 0 ,$$

$$g_2(\underset{\sim}{\phi}) = 5\phi_1 - (\phi_2-5)^2 - 25 \geq 0 .$$

Let

$$\underset{\sim}{\phi}^0 = \begin{bmatrix} 4.5 \\ 8.0 \end{bmatrix} , \quad \underset{\sim}{\varepsilon} = \begin{bmatrix} 2.0 \\ 2.5 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{t} = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix} .$$

Figure 3 shows the constraint region R_c and the tunable constraint region R_{ct}. In the figure R_ε and $R_t(\underset{\sim}{\mu})$ are defined according to (5) and (10), respectively, where R_ρ is assumed as in (12).

Mathematical Definition of Yield

We are now ready to give a mathematical definition of production yield. An outcome $\underset{\sim}{\phi}$ is said to meet the design specifications either if $\underset{\sim}{\phi} \in R_c$ or there exists $\underset{\sim}{\rho} \in R_\rho$ such that $\underset{\sim}{\phi} + \underset{\sim}{T}\,\underset{\sim}{\rho} \in R_c$, i.e., this outcome is tunable. In other words $\underset{\sim}{\phi} \in R_{ct}$.

In an abstract manner, the expected potential yield, i.e., the expected yield after tuning is given by

$$Y_p = \int_{R_{ct}} F(\underset{\sim}{\phi})\, d\phi_1\, d\phi_2 \ldots d\phi_k , \tag{31}$$

where $F(\underset{\sim}{\phi})$ is the joint probability distribution function of the outcomes. The expected yield before tuning is

$$Y = \int_{R_c} F(\underset{\sim}{\phi})\, d\phi_1\, d\phi_2 \ldots d\phi_k . \tag{32}$$

If the outcomes are uniformly distributed between the tolerance extremes, i.e., inside the orthotope R_ε, the expected potential yield and the expected yield can be expressed as

$$Y_p = V(R_\varepsilon \cap R_{ct})/V(R_\varepsilon) \tag{33}$$

and

$$Y = V(R_\varepsilon \cap R_c)/V(R_\varepsilon) , \tag{34}$$

where $V(R)$ denotes the hypervolume of the region R. The expectation of having outcomes which require tuning as a post-manufacturing process is also given by (16).

Worst-Case Design

The worst-case design problem arises when the worst outcome is supposed to meet the specifications. This implies a lower potential yield specification Y_L = 100%. Thus, for the nonlinear program, the constraint (20) reduces to

$$R_\varepsilon \subset R_{ct} . \tag{35}$$

For a one-dimensionally convex region R_{ct}, (35) can be replaced by

$$R_v \subset R_{ct} , \tag{36}$$

where R_v is the set of vertices defined by (7).

At the worst-case optimum, the set of <u>active constraints</u> at a vertex $\underset{\sim}{\phi}^r \in R_v$ is defined by

$$I_{ac}^r = \{i \mid g_i(\underset{\sim}{\phi}^r + \underset{\sim}{T}\,\underset{\sim}{\rho}^{r*}) = 0 , \; i \in I\} , \tag{37}$$

where ρ^{r*} is the optimum setting for the tuning variable for the vertex

$$\phi^r = \phi^{0*} + E^* \phi^r , \tag{38}$$

E^* and ϕ^{0*} are the worst-case optimums of E and ϕ^0, respectively. The set of <u>active vertices</u> is consequently defined by

$$R_{av} = \{\phi^r \mid \phi^r \in R_v, I^r_{ac} \neq \emptyset\} . \tag{39}$$

The set of all active constraints is

$$I_{ac} = \bigcup_{r=1}^{2^k} I^r_{ac} . \tag{40}$$

An alternative approach is to define the set of active vertices for each constraint g_i, $i \in I$ given by

$$R^i_{av} = \{\phi^r \mid g_i(\phi^r + T \rho^{r*}) = 0, \phi^r \in R_v\} , \tag{41}$$

where ϕ^r is given by (38) and ρ^{r*} is the setting of the tuning variable for this vertex at the optimum. Thus, the set of active constraints is defined by

$$I_{ac} = \{i \mid i \in I, R^i_{av} \neq \emptyset\} . \tag{42}$$

The set of all active vertices is

$$R_{av} = \bigcup_{i=1}^{m_c} R^i_{av} . \tag{43}$$

Worst-Case Centering

Worst-case centering is a minimax problem in which the tolerance vector ϵ is fixed either absolutely or relatively w.r.t. the nominal vector ϕ^0 while ϕ^0 and the tuning vector t are variables. The problem can be expressed as

$$\underset{\phi^0 \geq 0,\ 0 \leq t \leq t_{max}}{\text{minimize}} \quad U(-g_i(\phi^0 + E\,\mu + T\,\rho), I, \infty, 1) , \tag{44}$$

where t_{max} is an upper bound on the tuning range, U is the least pth function defined by (24) and μ is chosen to give the worst outcome.

EXAMPLES

LC Filter

Consider the LC filter shown in Figure 4. Taking L_1 as a tunable parameter with 5% tuning range, a worst-case design is obtained by minimizing the cost function [4]

$$L^0_2/\epsilon_2 + C^0/\epsilon_C ,$$

where L^0_2 and C^0 are the nominal values of L_2 and C, respectively, ϵ_2 and ϵ_C are the corresponding tolerances. We considered five variables, namely, L^0_1, L^0_2, C^0, ϵ_2 and ϵ_C. The same problem has been solved by Bandler, Liu and Tromp [4].

A quadratic approximation to the boundary of the tunable constraint region R_{ct} was employed. The constraint region is defined by the specifications given in Table I and the quadratic approximation is obtained by interpolation at the set of 10 points defined by the columns of

$$\begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -.9 & .6 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -.7 & 0 & .5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & .5 & -.6 & 0 \end{bmatrix} + \begin{bmatrix} \bar{L}_1 & \bar{L}_1 & & \bar{L}_1 \\ \bar{L}_2 & \bar{L}_2 & \cdots & \bar{L}_2 \\ \bar{C} & \bar{C} & & \bar{C} \end{bmatrix} ,$$

where δ is a parameter defining the size of the interpolation region and $\{\bar{L}_1, \bar{L}_2, \bar{C}\}$ the center of the region. The approximation was conducted at the critical regions [7] for each constraint function. These critical regions are ultimately centered at the vertices of the tolerance region under the one-dimensional convexity assumption. The results are shown in Table I. The approximation was updated once using smaller δ in order to improve accuracy. The centers of interpolation for the different constraints are obtained by detecting the worst vertices and perturbing the parameters by either the tolerance value or to one of the extremes of the tuning range. The results are in good agreement with published values [4].

Tunable Active Filter

Another example is the tunable active filter [8] illustrated in Figure 5. The specifications w.r.t. frequency f on the transfer function $F = |V_2/V_g|$ are

$$F \le 1/\sqrt{2} \quad \text{for } f/f_0 \le 1 - 10/f_0 ,$$

$$F \le 1.1 \quad \text{for } 1-10/f_0 \le f/f_0 \le 1+10/f_0 ,$$

$$F \le 1/\sqrt{2} \quad \text{for } f/f_0 \ge 1+10/f_0 ,$$

$$F \ge 1/\sqrt{2} \quad \text{for } 1-8/f_0 \le f/f_0 \le 1+8/f_0 ,$$

$$F \ge 1 \quad \text{for } f = f_0 \text{ Hz},$$

where f_0 is the center frequency. Two sample center frequencies, namely f_0 = 100 Hz and f_0 = 700 Hz, were found to be enough for the tuning frequency range 100 $\le f_0 \le$ 700 Hz. The normalized sample frequencies considered are 1 and 1 $\pm$ 10/f_0 for the relevant upper specifications, 1 and 1 $\pm$ 8/f_0 for the relevant lower specifications.

Again, the boundary of the tunable constraint region is approximated by a quadratic polynomial. Interpolation is performed using the points defined by the columns of

$$\begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & .8 & -.7 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -.6 & 0 & .5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & .9 & -.4 & 0 \end{bmatrix} + \begin{bmatrix} \bar{R}_1 & \bar{R}_1 & & \bar{R}_1 \\ \bar{C}_1 & \bar{C}_1 & \cdots & \bar{C}_1 \\ \bar{C}_2 & \bar{C}_2 & & \bar{C}_2 \end{bmatrix} .$$

At each of these points the optimum value of the tunable resistor R_4 is obtained [24] and the corresponding value of a constraint is used for interpolation. The assumed objective cost function is

$$R_1^0/\varepsilon_{R1} + C^0/\varepsilon_{C1} + C^0/\varepsilon_{C2} .$$

The design variables are R_1^0, C^0, ε_{R1}, ε_{C1} and ε_{C2}, where $C^0 = C_1^0 = C_2^0$. Table II shows the starting point and the optimum obtained after each modeling step. It is to be noted that each optimum was used as a starting point for the succeeding optimization as well as providing the new centers of interpolation. The vertices are numbered according to (9).

Computational Methods Used

In both examples subroutines MODEL4 and QPE [25] are used for carrying out the approximations and evaluating them along with their gradients, respectively. Nonlinear programming problems are solved using the Bandler-Charalambous [26] nonlinear programming to minimax transformation with the subroutine FLOPT4 [27].

CONCLUSIONS

Having a tolerance problem which is equivalent to a tolerance-tuning problem allows us to deal solely with tolerance assignment. It permits the evaluation of yield to be based upon hypervolume computation as has been dealt with by Bandler and Abdel-Malek [5-7]. Subsequently or alternatively Monte Carlo analysis using the approximations to the tunable constraint region can be conducted to verify or find the production yield. Our approach presented here is significantly more efficient than the one used by Bandler et al. [8]. Almost two minutes of CPU time on a CDC 6400 were previously required to solve a somewhat simpler problem involving only one center frequency, whereas about 20s were required for the example of Table II.

TABLE I

OPTIMIZATION OF LC FILTER WITH 5% TUNING OF L_1

Situation	L_1^0	L_2^0	C^0	ε_2 (%)	ε_C (%)	Frequency point (rad/s)
Starting	1.999	1.999	.906	10.0	10.0	
Interpolation centers (δ = .16)	1.9 1.9 2.1	2.2 2.2 1.8	.906 .906 .906			0.55 1.00 2.50
Optimum 1	2.14	1.84	.906	15.7	12.6	
Interpolation centers (δ = .04)	2.03 2.03 2.24	2.12 2.12 1.55	.792 1.021 .792			0.55 1.00 2.50
Optimum 2	2.19	1.79	.907	16.1	12.6	

Insertion loss should be less than 1.5 dB at 0.55 and 1.0 rad/s and greater than 25 dB at 2.5 rad/s

TABLE II

OPTIMIZATION OF TUNABLE ACTIVE FILTER WITH UNRESTRICTED $R_4 > 0$

Situation	R_1^0 (10^4 Ω)	C^0 (10^{-7}F)	ε_1 (%)	ε_{C1} (%)	ε_{C2} (%)	CDC Time(s) M	CDC Time(s) O	Model information
Starting	1.200	7.200	1.00	1.00	1.00	11.2	-	Conducted at all vertices with $\delta_1 = .06$, $\delta_2 = \delta_3 = .36$
Optimum 1	1.256	7.016	1.58	2.56	1.83	4.1	1.9	Detected active vertices are $\underset{\sim}{\phi}^3$ at 100 Hz and $\underset{\sim}{\phi}^6$ at 100 and 700 Hz. $\underset{\sim}{\phi}^3$ and $\underset{\sim}{\phi}^6$ are subsequently used as interpolation centers with $\delta_1 = .06$, $\delta_2 = \delta_3 = .36$
Optimum 2	1.248	7.224	1.55	2.24	2.09	2.7	0.5	$\underset{\sim}{\phi}^3$ at 100 Hz and $\underset{\sim}{\phi}^6$ at 700 Hz and $\underset{\sim}{\delta}$ reduced so that $\delta_1 = .015$, $\delta_2 = \delta_3 = .09$
Optimum 3	1.245	7.233	1.65	2.51	2.25	-	0.3	

Vector of designable parameters $\underset{\sim}{\phi} = [R_1 \;\; C_1 \;\; C_2]^T$.

M = modeling

O = optimization

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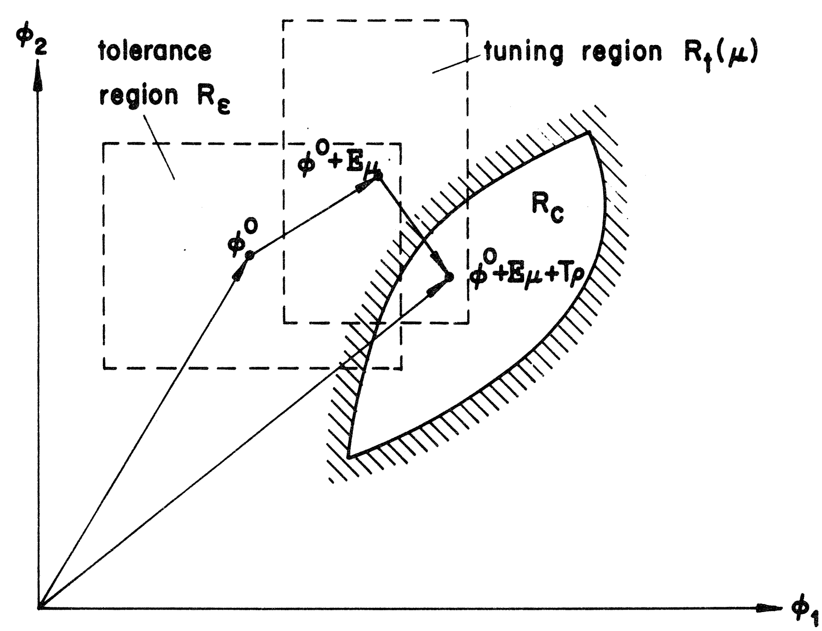

Figure 1 Illustration of regions R_c, R_ε and R_t. (From Bandler and Liu [3].)

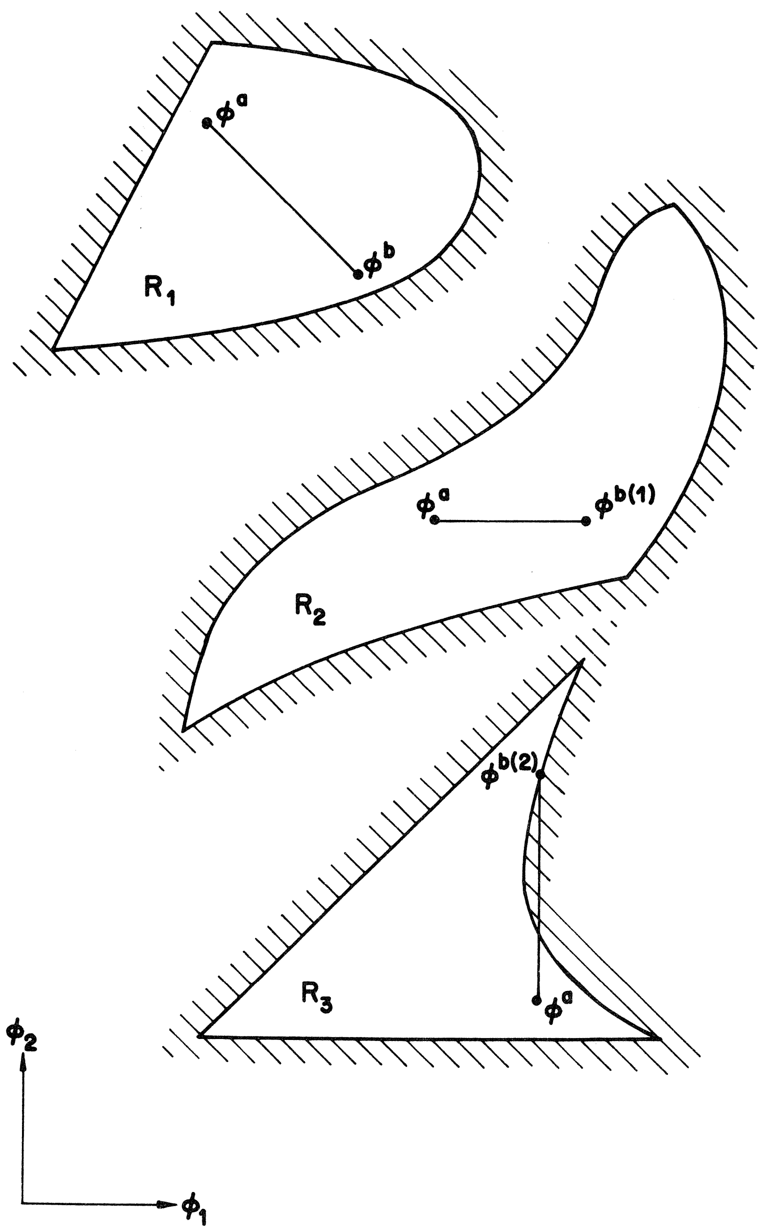

Figure 2 Illustrations of convex, one-dimensionally convex and nonconvex regions.

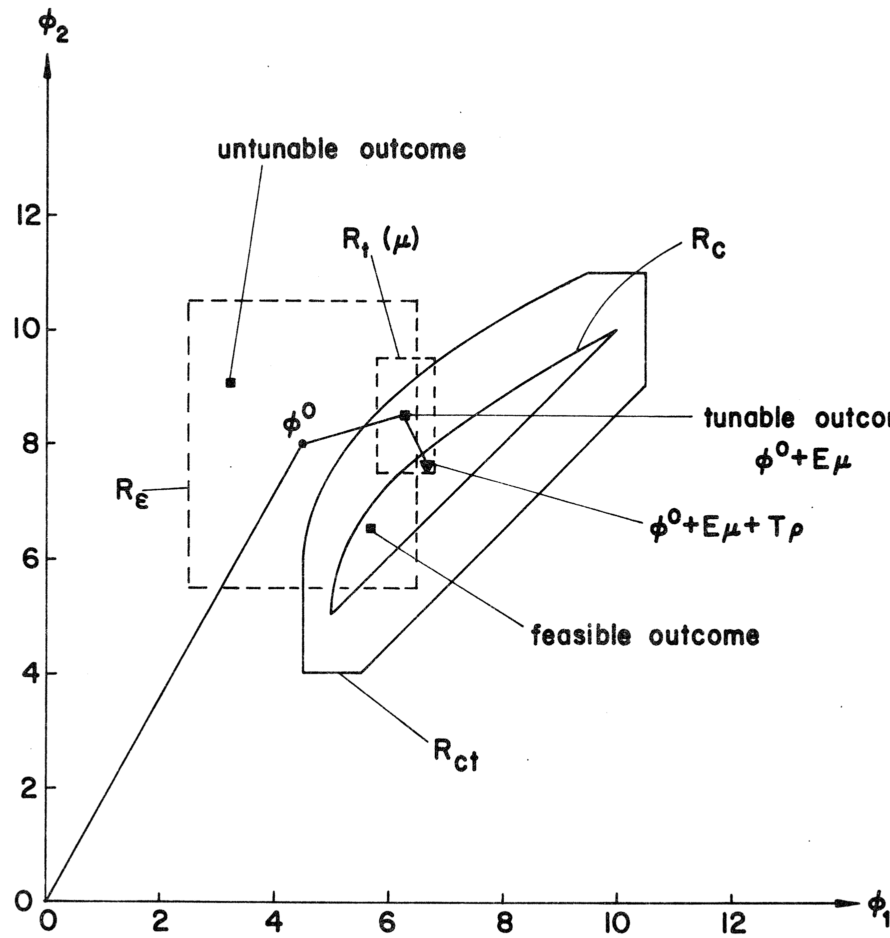

Figure 3 Geometric interpretation of the tolerance problem equivalent to the tolerance-tuning problem.

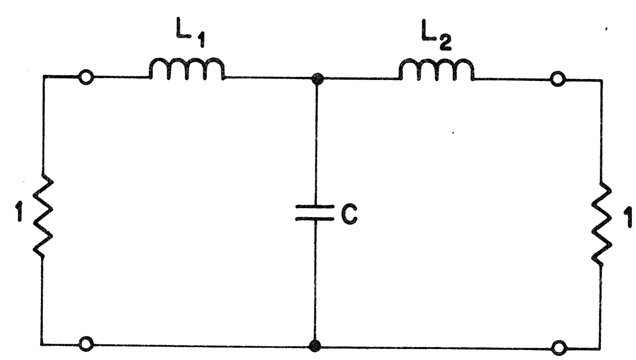

Figure 4 The LC filter example.

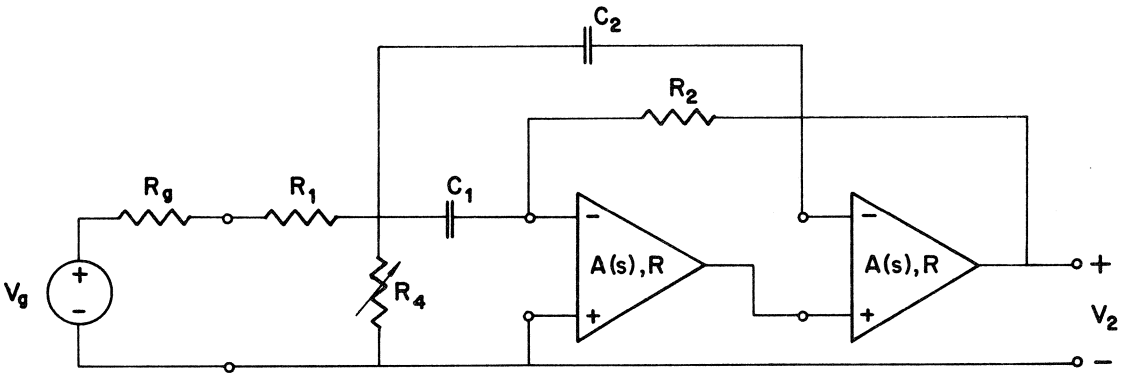

Figure 5 Tunable active filter with R_g = 50 Ω, R_2 = 26.5 kΩ, R = 75 Ω and one-pole roll-off for $A(s) = A_0\omega_a/(s+\omega_a)$, where $A_0 = 2x10^5$ and ω_a = 12 π rad/s.