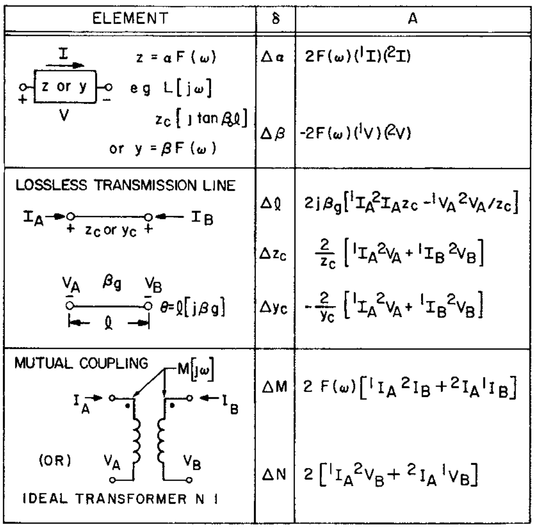

ELEMENT	δ	A
$z = \alpha F(\omega)$ e g $L[j\omega]$ $z_c[j \tan \beta l]$ or $y = \beta F(\omega)$	$\Delta\alpha$	$2F(\omega)(^1I)(^2I)$
	$\Delta\beta$	$-2F(\omega)(^1V)(^2V)$
LOSSLESS TRANSMISSION LINE	Δl	$2j\beta_g[^1I_A{}^2I_A z_c - {}^1V_A{}^2V_A/z_c]$
	Δz_c	$\frac{2}{z_c}[^1I_A{}^2V_A + {}^1I_B{}^2V_B]$
	Δy_c	$-\frac{2}{y_c}[^1I_A{}^2V_A + {}^1I_B{}^2V_B]$
MUTUAL COUPLING $M[j\omega]$ (OR) IDEAL TRANSFORMER N 1	ΔM	$2F(\omega)[^1I_A{}^2I_B + {}^2I_A{}^1I_B]$
	ΔN	$2[^1I_A{}^2V_B + {}^2I_A{}^1V_B]$

Fig. 14. Tabulation of expressions for sensitivity coefficients for circuit elements used in filter equivalent circuits. All values normalized to $Z_0 = 1\ \Omega$ and $^1V_0$ or $^2V_0 = 1$ V.

of the effects of the element changes, it generally is necessary to iterate the procedure a few times. Fig. 14 provides a tabulation of S_{12} sensitivity coefficients for various element types. All currents, voltages, and impedances are assumed to be normalized to the case where $Z_0 = 1\ \Omega$ and $^1V_0$ or $^2V_0 = 1$ V. The values for S_{11} or S_{22} may be obtained as special cases by letting all superscripts be 1 or 2, respectively.

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REFERENCES

[1] A. E. Atia and A. E. Williams, "General TE_{011} mode waveguide bandpass filters," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 640–648, Oct. 1976.

[2] G. L. Matthaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance-Matching Networks, and Coupling Structures*. New York: McGraw-Hill, 1964, pp. 231–235, pp. 511–513.

[3] S. B. Cohn, "Determination of aperture parameters by electrolytic tank measurements," *Proc. IRE*, vol. 39, pp. 1416–1421, Nov. 1951.

[4] A. E. Williams, "A four-cavity elliptic waveguide filter," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 1109–1114, Dec. 1970.

[5] R. D. Wanselow, "Prototype characteristics for a class of dual-mode filters," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 708–711, Aug. 1975.

[6] H. L. Thal, "Loss mechanisms in coupled cavity microwave filters," in *IEEE Int. Microwave Symp. Digest*, June 1977, pp. 415–418.

New Results in Network Simulation, Sensitivity, and Tolerance Analysis for Cascaded Structures

JOHN W. BANDLER, FELLOW, IEEE, MOHAMED R. M. RIZK, STUDENT MEMBER, IEEE, AND
HANY L. ABDEL-MALEK, STUDENT MEMBER, IEEE

***Abstract*—An attractive, exact, and efficient approach to network analysis for cascaded structures is presented. It is useful for sensitivity and tolerance analyses, in particular, for a multiple of simultaneous large changes in design parameter values. It also facilitates the exploitation of symmetry to reduce computational effort for the analysis. Responses at different loads in branched networks, which may be connected in series or in parallel with the main cascade, can be obtained analytically in terms of the variable elements. Sensitivity and large-change effects with respect to these variables can be easily evaluated. The approach is not confined to 2-port elements but can be generalized to 2*p*-port cascaded elements.**

I. INTRODUCTION

THIS PAPER presents a new and comprehensive treatment of computer-oriented cascaded network analysis. The analysis of cascaded networks plays a very important role in the design and optimization of microwave circuits, so that an attractive approach which facilitates efficient analytical and numerical investigations of response, first- and higher, order sensitivities of response, and simultaneous and arbitrary large-change sensitivity evaluation is highly desirable. As is well known, first-

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J. W. Bandler and M. R. M. Rizk are with the Group on Simulation, Optimization, and Control and the Department of Electrical Engineering, McMaster University, Hamilton, Ont., Canada L8S 4L7.

H. L. Abdel-Malek was with the Group on Simulation, Optimization, and Control and the Department of Electrical Engineering, McMaster University, Hamilton, Ont., Canada L8S 4L7. He is now with the Department of Engineering Physics and Mathematics, Faculty of Engineering, Cairo University, Giza, Egypt.

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order sensitivities, for example, are useful in network optimization by gradient methods. Large-change sensitivities are important in tolerance analysis and design centering.

The approach we have developed permits the efficient

1) exact analysis of cascaded networks in any direction,
2) exact evaluation of first-order response sensitivities at any location,
3) exact evaluation of the effects of any number of simultaneous large changes in any elements,
4) exploitation of network structure, i.e., branches, symmetry, reciprocity, etc.,
5) evaluation of the exact effect due to simultaneously growing elements in appropriate locations, and
6) exact response and response sensitivity evaluation for branches connected in series or in parallel with the main cascade.

The conceptual advantages enjoyed by our approach and applicable to 2-port elements are as follows.

1) All calculations are applied directly to the given network; no auxiliary or adjoint network is defined.

2) All calculations involve at most the premultiplication of 2 by 2 matrices by row vectors or postmultiplications by column vectors; no explicit matrix inversion is ever required.

3) Response functions, sensitivities, or large-change effects are represented analytically in terms of the parameters to be investigated; all parts of the network to be kept constant are reduced numerically to a few 2-element vectors appearing as constants in the formulas.

4) Calculations can be carried out easily by hand, if appropriate or are readily programmed.

II. THEORETICAL FOUNDATION

Consider the 2-port element depicted in Fig. 1. The basic iteration, also summarized by Table I, is $\bar{y}=Ay$, where A is the transmission or chain matrix, y contains the output voltage and current, and $\bar{y}$ the corresponding input quantities.

Forward analysis as shown in Fig. 2 and Table I consists of initializing a $\bar{u}^T$ row vector as either $[1\ \ 0]$, $[0\ \ 1]$, or a suitable linear combination and successively premultiplying each constant chain matrix by the resulting row vector until an element of interest or a termination is reached.

Reverse analysis, which is similar to conventional analysis of cascaded networks, proceeds by initializing a v column vector as either $[1\ \ 0]^T$, $[0\ \ 1]^T$, or a suitable linear combination and successively postmultiplying each constant matrix by the resulting column vector, again until either an element of interest or a termination is reached.

In summary, assuming a cascade of n 2-ports we have

$$\bar{y}^1 = y^0 = A^1A^2 \cdots A^i \cdots A^n y^n \tag{1}$$

and, applying forward and reverse analyses up to A^i, this

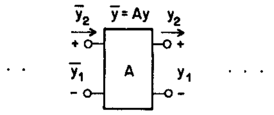

Fig. 1. Notation for an element in the chain, indicating reference directions and voltage and current variables.

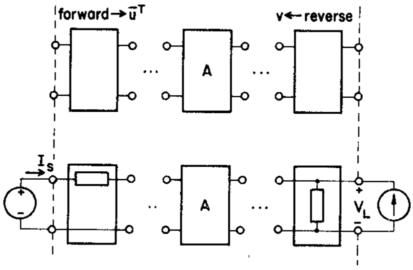

Fig. 2. Forward and reverse analyses of a cascaded network with source and load impedances assumed constant.

TABLE I
PRINCIPAL CONCEPTS INVOLVED IN THE ANALYSES

Concept	Definition	Implication
Basic iteration	$\bar{\underset{\sim}{y}} = \underset{\sim}{A}\,\underset{\sim}{y}$	$\underset{\sim}{y} \Longrightarrow \bar{\underset{\sim}{y}}$
Forward operation	$\bar{\underset{\sim}{u}}^T\underset{\sim}{A} = \underset{\sim}{u}^T$	$\bar{\underset{\sim}{u}}^T\bar{\underset{\sim}{y}} = \bar{\underset{\sim}{u}}^T\underset{\sim}{A}\underset{\sim}{y} = \underset{\sim}{u}^T\underset{\sim}{y}$
Reverse operation	$\bar{\underset{\sim}{v}} = \underset{\sim}{A}\underset{\sim}{v}$	$\underset{\sim}{y} = c\underset{\sim}{v} \Longrightarrow \bar{\underset{\sim}{y}} = c\bar{\underset{\sim}{v}}$
Voltage selector	$\underset{\sim}{e}_1 \overset{\Delta}{=} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\underset{\sim}{e}_1 \Longrightarrow \underset{\sim}{u}_1$ or $\underset{\sim}{v}_1$
Current selector	$\underset{\sim}{e}_2 \overset{\Delta}{=} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\underset{\sim}{e}_2 \Longrightarrow \underset{\sim}{u}_2$ or $\underset{\sim}{v}_2$
Equivalent source	$\underset{\sim}{y} = \begin{bmatrix} V_S - Z_S I_S \\ I_S \end{bmatrix}$	$\underset{\sim}{e}_1^T\underset{\sim}{y} = V_S - Z_S I_S$, $\underset{\sim}{e}_2^T\underset{\sim}{y} = I_S$
Equivalent load	$\underset{\sim}{y} = \begin{bmatrix} V_L \\ Y_L V_L - I_L \end{bmatrix}$	$\underset{\sim}{y} = V_L\underset{\sim}{e}_1 + (Y_L V_L - I_L)\underset{\sim}{e}_2$

reduces to an expression of the form

$$d = \bar{u}^{1^T}\bar{y}^1 = c\bar{u}^{i^T}A^i v^i \tag{2}$$

where

$$y^n = cv^n \tag{3}$$

and c and d relate selected output and input variables of interest explicitly with A^i.

The typical formula, therefore, will contain factors of the form as follows.

Function evaluation:

$$\bar{u}^T A v \Rightarrow Q. \tag{4}$$

First-order sensitivity:

$$\bar{u}^T \delta A v \Rightarrow \delta Q. \tag{5}$$

Partial derivative:

$$\bar{u}^T \frac{\partial A}{\partial \phi} v \Rightarrow Q'. \tag{6}$$

Large-change sensitivity:

$$\bar{u}^T \Delta A v \Rightarrow \Delta Q \tag{7}$$

where the parameter ϕ is contained in A. A full reverse analysis taking

$$[v_1^n \ v_2^n] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yields

$$[v_1^i \ v_2^i] = A^{i+1} A^{i+2} \cdots A^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and a corresponding full forward analysis taking

$$[\bar{u}_1^1 \ \bar{u}_2^1]^T = [u_1^0 \ u_2^0]^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yields

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^1 A^2 \cdots A^{i-1} = [\bar{u}_1^i \ \bar{u}_2^i]^T.$$

A. Symmetrical Networks Consisting of Symmetrical Elements

In many practical cases we encounter symmetrical networks (around a central plane) which consist of reciprocal and symmetrical elements. Series impedances, shunt admittances, transmission lines, and RC lines are examples of such elements. Assume that, for each element,

$$a_{11} = a_{22}$$

and, for the network,

$$A^{n-i+1} = A^i.$$

Let

$$A \triangleq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{8}$$

If we take the transformation

$$A^a = [e_2 \ e_1] A^T [e_2 \ e_1]$$

then

$$[v_1^i \ v_2^i]^a = [\bar{u}_1^{n-i+1} \ \bar{u}_2^{n-i+1}]^T.$$

This equality can be used to reduce computational effort.

B. Reference Planes

In considering more than one element in the cascade, we divide the network into subnetworks by reference planes. These in turn are chosen so that no more than one element is to be explicitly considered between any pair of reference planes. In Fig. 2 the element A is the only element whose effect is to be considered. In Fig. 3 the elements A^k, A^i, and A^j are considered in the kth, the ith, and the jth subnetworks, respectively. Note that the superscripts of A here and from now on denote the subnetwork and not the element. Forward and reverse analyses are initiated at the reference planes. A forward iteration of the structure of Fig. 3 is illustrated in Fig. 4, where equivalent (Thevenin) sources are iteratively determined. Reverse iteration is shown in Fig. 5, where equivalent (Norton) sources are iteratively determined.

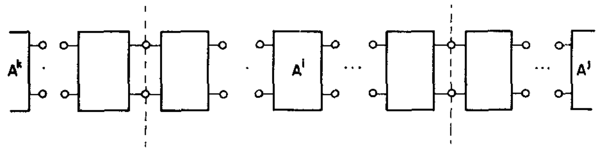

Fig. 3. Subnetwork i cascaded with subnetworks k (at source end) and j (at load end).

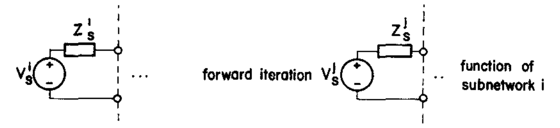

Fig. 4. Forward iteration for Fig. 3, transferring an equivalent source accounting for design variables from subnetwork k from one reference plane to the other.

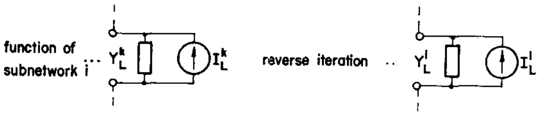

Fig. 5. Reverse iteration for Fig. 3, transferring an equivalent source accounting for design variables from subnetwork j from one reference plane to the other.

III. NETWORK FUNCTIONS IN TERMS OF ELEMENTS UNDER CONSIDERATION

Performing forward analysis from the source of the ith subnetwork to the input of A^i and reverse analysis from the load to the output of A^i we have

$$V_S^i = (\bar{u}_1 + Z_S^i \bar{u}_2)^T A^i \left(V_L^i v_1 + (Y_L^i V_L^i - I_L^i) v_2 \right) = V_L^k + Z_S^i I_S^i \tag{9}$$

and the current through the voltage source of the ith subnetwork

$$I_S^i = \bar{u}_2^T A^i \left(V_L^i v_1 + (Y_L^i V_L^i - I_L^i) v_2 \right) = V_L^k Y_L^k - I_L^k. \tag{10}$$

From (9), letting $I_L^i = 0$ and $Y_L^i = 0$, we have $I_S^j = 0$ and the Thevenin voltage

$$V_S^j = V_L^i = \frac{V_S^i}{(\bar{u}_1 + Z_S^i \bar{u}_2)^T A^i v_1} = \frac{V_S^i}{Q_{11}^i + Z_S^i Q_{21}^i} \tag{11}$$

where the Q terms have been defined in (4). Letting $V_S^i = 0$ and $Y_L^i = 0$, we have $I_S^j = -I_L^i$ and the output impedance

$$Z_S^j = \frac{V_L^i}{I_L^i} = \frac{(\bar{u}_1 + Z_S^i \bar{u}_2)^T A^i v_2}{(\bar{u}_1 + Z_S^i \bar{u}_2)^T A^i v_1} = \frac{Q_{12}^i + Z_S^i Q_{22}^i}{Q_{11}^i + Z_S^i Q_{21}^i} \tag{12}$$

where, again, the Q terms of (4) are used to obtain a compact expression. These expressions for V_S^j and Z_S^j

TABLE II
NOTATION AND IMPLIED INITIAL CONDITIONS

Factor	Identification	Initial Conditions	
		Forward	Reverse
$\bar{\mathbf{u}}_1^T\,(*)\,\mathbf{v}_1$	$(\dagger)_{11}$	voltage	voltage
$\bar{\mathbf{u}}_1^T\,(*)\,\mathbf{v}_2$	$(\dagger)_{12}$	voltage	current
$\bar{\mathbf{u}}_2^T\,(*)\,\mathbf{v}_1$	$(\dagger)_{21}$	current	voltage
$\bar{\mathbf{u}}_2^T\,(*)\,\mathbf{v}_2$	$(\dagger)_{22}$	current	current

(*)Denotes either A, δA, $\partial A/\partial\phi$, or ΔA.
(†)Denotes Q, δQ, Q', or ΔQ, as taken from (4), (5), (6), or (7), respectively.

TABLE III
ANALYSES REQUIRED BY CERTAIN TERMS

Term	Analysis Required
$\mathbf{u}^T\mathbf{v}$	Forward and reverse (conventional) cascade analysis to any corresponding reference plane, whichever is convenient
$\mathbf{u}_1^T\mathbf{v}$, $\mathbf{u}_2^T\mathbf{v}$	Preferably one reverse analysis to source reference plane (avoiding calculation of $\mathbf{u}_1$ and $\mathbf{u}_2$)
$\mathbf{u}^T\mathbf{v}_1$, $\mathbf{u}^T\mathbf{v}_2$	Preferably one forward analysis to load reference plane (avoiding calculation of $\mathbf{v}_1$ and $\mathbf{v}_2$)
$\bar{\mathbf{u}}^T \cdot \mathbf{v}$	One forward analysis to input of $\mathbf{A}$ and one reverse analysis to output of $\mathbf{A}$
$\bar{\mathbf{u}}_1^T \cdot \mathbf{v}$, $\bar{\mathbf{u}}_2^T \cdot \mathbf{v}$	One full forward analysis to input of $\mathbf{A}$ and one reverse analysis to output of $\mathbf{A}$
$\bar{\mathbf{u}}^T \cdot \mathbf{v}_1$, $\bar{\mathbf{u}}^T \cdot \mathbf{v}_2$	One full reverse analysis to output of $\mathbf{A}$ and one forward analysis to input of $\mathbf{A}$
$\bar{\mathbf{u}}_1^T \cdot \mathbf{v}_1$, $\bar{\mathbf{u}}_1^T \cdot \mathbf{v}_2$ $\bar{\mathbf{u}}_2^T \cdot \mathbf{v}_1$, $\bar{\mathbf{u}}_2^T \cdot \mathbf{v}_2$	One full forward analysis to input of $\mathbf{A}$ and one full reverse analysis to output of $\mathbf{A}$

TABLE IV
FUNCTIONS OF INPUT CURRENT I_S AND OUTPUT VOLTAGE V_L FOR CHANGES IN A ONLY

Variable	Input	Output
$\mathbf{A}$	$I_S = V_S \frac{Q_{21}}{Q_{11}}$	$V_L = \frac{V_S}{Q_{11}}$
$\delta\mathbf{A}$	$\delta I_S = \frac{V_S\delta Q_{21} - I_S\delta Q_{11}}{Q_{11}}$	$\delta V_L = -\frac{V_L^2}{V_S}\,\delta Q_{11}$
$\frac{\partial\mathbf{A}}{\partial\phi}$	$\frac{\partial I_S}{\partial\phi} = \frac{V_S Q_{21}' - I_S Q_{11}'}{Q_{11}}$	$\frac{\partial V_L}{\partial\phi} = -\frac{V_L^2}{V_S}\,Q_{11}'$
$\Delta\mathbf{A}$	$\Delta I_S = \frac{V_S\Delta Q_{21} - I_S\Delta Q_{11}}{Q_{11}+\Delta Q_{11}}$	$\Delta V_L = -\frac{V_L^2}{V_L + V_S/\Delta Q_{11}}$

permit equivalent Thevenin sources to be moved in a forward iteration.

From (9) and (10), letting $I_L^i = 0$ and $Z_S^i = 0$, we have $I_L^k = 0$ and the input admittance

$$Y_L^k = \frac{I_S^i}{V_S^i} = \frac{\bar{\boldsymbol{u}}_2^T A^i(\boldsymbol{v}_1 + Y_L^i \boldsymbol{v}_2)}{\bar{\boldsymbol{u}}_1^T A^i(\boldsymbol{v}_1 + Y_L^i \boldsymbol{v}_2)} = \frac{Q_{21}^i + Y_L^i Q_{22}^i}{Q_{11}^i + Y_L^i Q_{12}^i}. \qquad (13)$$

Letting $V_S^i = 0$ and $Z_S^i = 0$, we have $V_L^k = 0$ and the Norton current

$$I_L^k = -I_S^i = -I_L^i(Y_L^k \bar{\boldsymbol{u}}_1 - \bar{\boldsymbol{u}}_2)^T A^i \boldsymbol{v}_2 = -I_L^i(Y_L^k Q_{12}^i - Q_{22}^i). \qquad (14)$$

These expressions for I_L^k and Y_L^k permit equivalent Norton sources to be moved (if desired) in a reverse iteration.

The input current I_S^i for $I_L^i = 0$ is obtained via (13) as

$$I_S^i = V_S^i \Bigg/ \left[Z_S^i + \frac{\bar{\boldsymbol{u}}_1^T A^i(\boldsymbol{v}_1 + Y_L^i \boldsymbol{v}_2)}{\bar{\boldsymbol{u}}_2^T A^i(\boldsymbol{v}_1 + Y_L^i \boldsymbol{v}_2)} \right]$$

$$= \frac{V_S^i \bar{\boldsymbol{u}}_2^T A^i(\boldsymbol{v}_1 + Y_L^i \boldsymbol{v}_2)}{(\bar{\boldsymbol{u}}_1 + Z_S^i \bar{\boldsymbol{u}}_2)^T A^i(\boldsymbol{v}_1 + Y_L^i \boldsymbol{v}_2)}$$

$$= \frac{V_S^i(Q_{21}^i + Y_L^i Q_{22}^i)}{Q_{11}^i + Y_L^i Q_{12}^i + Z_S^i Q_{21}^i + Z_S^i Y_L^i Q_{22}^i}. \qquad (15)$$

Tables II and III summarize the procedures and the effort required in evaluating the different factors in the derived equations.

Useful special cases of these formulas for I_S and V_L in Fig. 2 are, from (15) and (11), respectively,

$$I_S = V_S \frac{\bar{\boldsymbol{u}}_2^T A \boldsymbol{v}_1}{\bar{\boldsymbol{u}}_1^T A \boldsymbol{v}_1} = V_S \frac{Q_{21}}{Q_{11}} \qquad (16)$$

and

$$V_L = \frac{V_S}{\bar{\boldsymbol{u}}_1^T A \boldsymbol{v}_1} = \frac{V_S}{Q_{11}}. \qquad (17)$$

Table IV gives some useful formulas which can be obtained for variations in a particular element A. We note, for example, that, since A is arbitrary and at most only one full analysis yields all Q_{11}, δQ_{11}, Q_{11}', and ΔQ_{11}, the corresponding V_L, δV_L, $\partial V_L/\partial\phi$, and ΔV_L with respect to all possible parameters anywhere in the cascade can be evaluated exactly for one network analysis. This particular special case is equivalent to the results of previous researchers [1], [2].

IV. NUMERICAL EXAMPLE

The cascaded seven-section bandpass filter shown in Fig. 6 [3], [4] serves as a numerical example. All sections are quarter-wave at 2.175 GHz. The normalized minimax characteristic impedances are [4]

$$Z_1^0 = Z_7^0 = 0.606\,463$$

$$Z_2^0 = Z_6^0 = 0.303\,051$$

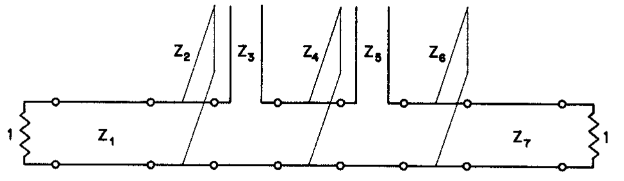

Fig. 6. Seven-section filter containing unit elements and stubs [3]. All sections are quarter-wave at 2.175 GHz.

$$Z_3^0 = Z_5^0 = 0.722\,061$$
$$Z_4^0 = 0.235\,593.$$

The output voltage V_L at a normalized frequency of 0.7 is $0.497\,407\,90 - j3.901\,159\,4 \times 10^{-3}$, verified twice using (11), once associating A^i with Z_3, and once with Z_4. Furthermore, one analysis yielded

$$V_L(Z_4^0 + 0.03) = 0.498\,389\,50 - j0.034\,901\,610$$
$$V_L(Z_4^0 - 0.03) = 0.490\,629\,12 + j0.034\,959\,186.$$

The open-circuit voltage at the load end was calculated using (11) as

$$V_{\mathrm{OC}} = 0.986\,245\,07 + j0.092\,266\,904$$

and the Thevenin impedance using (12) is

$$Z_{\mathrm{TH}} = 0.981\,192\,53 + j0.201\,033\,91$$

which further verified V_L.

One analysis taking $\epsilon_2 = 0.021$ and $\epsilon_5 = 0.024$ yielded

$$V_L(Z_2^0 - \epsilon_2, Z_5^0 - \epsilon_5) = 0.497\,197\,16 + j2.219\,136\,0 \times 10^{-3}$$
$$V_L(Z_2^0 + \epsilon_2, Z_5^0 - \epsilon_5) = 0.495\,835\,38 - j2.363\,631\,4 \times 10^{-2}$$
$$V_L(Z_2^0 - \epsilon_2, Z_5^0 + \epsilon_5) = 0.497\,324\,62 + j1.790\,991\,2 \times 10^{-2}$$
$$V_L(Z_2^0 + \epsilon_2, Z_5^0 + \epsilon_5) = 0.497\,514\,27 - j8.372\,647\,0 \times 10^{-3}.$$

A multidimensional quadratic approximation was carried out for V_L following the approach of Bandler and Abdel-Malek [5]. The variables for the approximation were the characteristic impedances as well as the normalized frequency. The circuit responses at 45 base points (which is equal to $(k+1)(k+2)/2$, where k is 8) were needed to evaluate the coefficients of the quadratic polynomial approximating the response function [5]. A base point is a point where the approximation and the actual function coincide. The center base point, which is the center of the interpolation region in which the approximation is assumed to be valid, had the characteristic impedances given before and a normalized frequency of 0.7. Sixteen base points were determined by varying one parameter at a time by $\pm\delta$ with respect to its value at the center of interpolation. For the characteristic impedances, δ was chosen to be 0.03, and for the normalized frequency, it was 0.01. At the remaining 28 base points only two parameters were perturbed at a time from their values at the center of interpolation by a percentage of their δ.

The symmetry of the structure was taken into consideration in choosing these base points. Letting $\bar{\phi}$ be the center of the interpolation region, the base points can be expressed by [6]

$$[\phi^1\ \phi^2 \cdots \phi^N] = \boldsymbol{D}[\mathbf{1}_k\ \ -\mathbf{1}_k\ \ \boldsymbol{B}\ \ \mathbf{0}_k] + [\bar{\phi}\ \bar{\phi} \cdots \bar{\phi}] \tag{18}$$

where N is equal to 45 in our case, $\mathbf{1}_k$ is a k-dimensional identity matrix, $\mathbf{0}_k$ is a zero vector of dimension k,

$$\boldsymbol{D} = \begin{bmatrix} 0.03 & & & & \\ & 0.03 & & & \\ & & \ddots & & \\ & & & 0.03 & \\ & & & & 0.01 \end{bmatrix}$$

and $\boldsymbol{B}$ is a $k \times [k(k-1)/2]$ matrix shown as follows:

$$\boldsymbol{B} = \left[\begin{array}{cccccccccccccccccccccccccccc}
0.5 & -0.2 & 0.9 & -0.2 & 0.5 & 0.3 & 0.7 & & & & & & & & & & & & & & & & & & & & & \\
0.7 & & & & & & & 0.7 & 0.5 & 0.7 & 0.9 & 0.7 & -0.3 & & & & & & & & & & & & & & & \\
 & 0.8 & & & & & & 0.8 & & & & & & 0.7 & -0.4 & 0.8 & 0.8 & 0.8 & & & & & & & & & & \\
 & & -0.5 & & & & & & 0.9 & & & & & -0.9 & & & & & -0.9 & 0.9 & -0.5 & 0.8 & & & & & & \\
 & & & 0.8 & & & & & & 0.8 & & & & & 0.8 & & & & 0.7 & & & & 0.8 & 0.8 & 0.8 & & & \\
 & & & & 0.7 & & & & & & -0.2 & & & & & 0.7 & & & & 0.5 & & & 0.7 & & & 0.7 & -0.3 & \\
 & & & & & -0.9 & & & & & & 0.5 & & & & & -0.2 & & & & 0.9 & & & -0.2 & & 0.5 & & 0.7 \\
 & & & & & & 0.6 & & & & & & 0.4 & & & & & 0.9 & & & & 0.4 & & & 0.9 & & 0.4 & 0.6
\end{array}\right].$$

Examining this $\boldsymbol{B}$ matrix we note that the entries for perturbing two parameters at a time are the same as for their corresponding symmetrical parameters. The choice of base points given by (18) preserves symmetry in the appropriate coefficients of the multidimensional polynomials.

Taking the optimal minimax characteristic impedances [4]

$$Z_1 = Z_7 = 0.606\,595$$
$$Z_2 = Z_6 = 0.303\,547$$
$$Z_3 = Z_5 = 0.722\,287$$
$$Z_4 = 0.235\,183$$

and calculating the group delay using the derivative of V_L with respect to ω obtained from the quadratic approximation yielded

$$T_G = 0.893 \text{ ns}$$

while the exact group delay is [7]

$$T_{G_{\text{exact}}} = 0.895 \text{ ns}.$$

V. Two Algorithms for Evaluation of Large Changes

The two following algorithms were used to obtain responses at the base points for the interpolation performed in the previous section. The first was used when one parameter at a time was perturbed, and the second was used when pairs of parameters were perturbed simultaneously. Note that when the normalized frequency was perturbed a whole new analysis had to be performed.

A. Algorithm 1—Multiple One-at-a-Time Changes

```
Step 1:   Initialize $\bar{u}$ and $v$.
          Set $i \leftarrow 1$, $m \leftarrow 1$, $j \leftarrow n$.
Comment:  $n$ is the total number of elements in the cascade, and $m$ is a counter for the variable elements.
Step 2:   If $i = l_m$, go to Step 5.
Comment:  $l_m$ is an element of $L$, an index set containing superscripts of the $k$ matrices containing the $k$ variable parameters and ordered consecutively. It is assumed that each matrix contains only one variable.
Step 3:   $\bar{u}^T \leftarrow \bar{u}^T A^i$.
          $i \leftarrow i + 1$.
Step 4:   If $i = l_m$, go to Step 5.
          Go to Step 3.
Step 5:   Let $x^m \leftarrow \bar{u}$.
          If $i = l_k$, go to Step 7.
Comment:  $x^1, x^2, \cdots, x^k$ are working arrays to store the $\bar{u}$ vectors required in the evaluation of the large changes taking place.
Step 6:   $m \leftarrow m + 1$. Go to Step 3.
Step 7:   If $n = l_k$, go to Step 10.
Step 8:   $v = A^j v$.
          $j \leftarrow j - 1$.
Step 9:   If $j = l_m$, go to Step 10.
          Go to Step 8.
Step 10:  Evaluate $Q$ using the stored $x^m$, $v$, and the perturbed $A^j$.
          If $j = l_1$, stop.
Comment:  Positive and negative extremes of the variable in $A^j$ are considered simultaneously.
Step 11:  $m \leftarrow m - 1$. Go to Step 8.
```

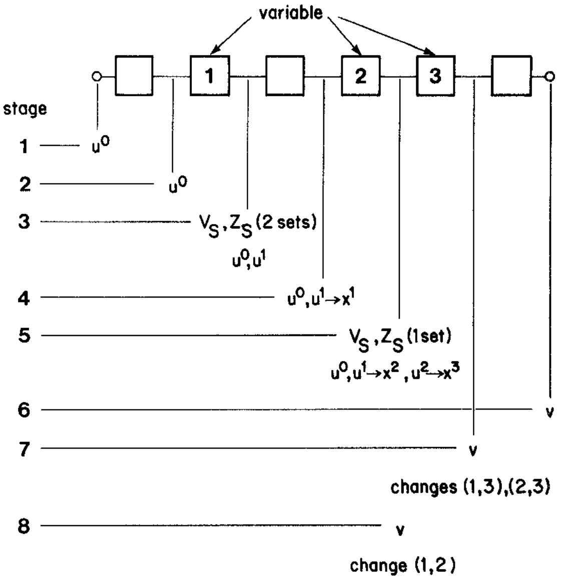

Fig. 7. Illustration for a cascade of six 2-ports of the principal stages in the calculations involved in the multiple-pairwise-changes algorithm. Three variable elements are considered; hence three sets of simultaneous analyses are effectively performed.

B. Algorithm 2—Multiple Pairwise Changes

This algorithm is for evaluating the response at the $k(k-1)/2$ base points where two parameters are perturbed at a time. At the first $k-1$ points following those considered in Algorithm 1, the parameters indicated by the subscripts $1,2 \;\; 1,3 \cdots 1,k$ are changed; at the next $k-2$ points the parameters indicated by the subscripts $2,3 \;\; 2,4 \cdots 2,k$ are changed, and so on, until the final point at which parameters $k-1$ and k are perturbed. Fig. 7 serves to illustrate the analyses involved.

```
Step 1:   Initialize $\mathbf{u}_1^0$, $\mathbf{u}_2^0$, $\mathbf{u}_1^1$, and $\mathbf{u}_2^1$.
          Set $i \leftarrow 1$, $m \leftarrow 1$, $q \leftarrow 0$, $r \leftarrow 1$, and $s \leftarrow k - 1$.
Comment:  $\mathbf{u}_1^1$ and $\mathbf{u}_2^1$ are vectors to be initialized as $\mathbf{u}_1^0$ and $\mathbf{u}_2^0$, respectively. They have the same role as $\mathbf{u}_1^0$ and $\mathbf{u}_2^0$ in the forward analysis initiated at a reference plane immediately following the first variable element.
Step 2:   If $i = l_m$, go to Step 4.
Comment:  $l_m$ is an element of $L$, an index set containing superscripts of the $k$ matrices containing the $k$ variable parameters as indicated in Algorithm 1.
Step 3:   $\mathbf{u}_1^{0T} \leftarrow \mathbf{u}_1^{0T} A^i$.
          $\mathbf{u}_2^{0T} \leftarrow \mathbf{u}_2^{0T} A^i$.
Step 4:   If $m = 1$, go to Step 5.
          $\mathbf{u}_1^{1T} \leftarrow \mathbf{u}_1^{1T} A^i$.
```

```
         $\mathbf{u}_2^{1T} \leftarrow \mathbf{u}_2^{1T} \mathbf{A}^i$.
           .
           .
           .
         $\mathbf{u}_1^{qT} \leftarrow \mathbf{u}_1^{qT} \mathbf{A}^i$.
         $\mathbf{u}_2^{qT} \leftarrow \mathbf{u}_2^{qT} \mathbf{A}^i$.
Comment: This step is not performed until we reach a variable element, since the analyses involving the $\mathbf{u}^j$ do not begin until the $j$th variable element has been considered.
Step 5:  Set $i \leftarrow i+1$.
Step 6:  If $i = l_m$, go to Step 7.
         Go to Step 3.
Step 7:  If $m = k$, go to Step 9.
         Calculate the Thevenin impedances and voltages:
         $Z_S(m,1), \cdots, Z_S(m,s)$
         $V_S(m,1), \cdots, V_S(m,s)$.
         $s \leftarrow s-1$.
Comment: For the first variable element $k-1$ sets of $Z_S$ and $V_S$ have to be evaluated since changes in this element will be coupled one at a time with changes in the next $k-1$ variable elements. For the second variable element $k-2$ sets of $Z_S$ and $V_S$ are calculated, and so on. See Fig. 7.
Step 8:  If $m = 1$, go to Step 13.
Step 9:  Set $p \leftarrow 1$.
Comment: $p$ is an internal counter.
Step 10: $\mathbf{x}^r \leftarrow \mathbf{u}^p$.
         If $p = q$, go to Step 12.
Comment: When the analysis has reached a reference plane immediately preceding an element containing a variable whose change is to be associated with any previously encountered variable, a snapshot of the appropriate $\mathbf{u}$ vectors is taken and stored in the $\mathbf{x}$ arrays. See Fig. 7.
Step 11: Set $r \leftarrow r+1$.
         $p \leftarrow p+1$.
         Go to Step 10.
Step 12: Set $r \leftarrow r+1$.
Step 13: If $m = k$, go to Step 16.
         $\mathbf{u}_1^{0T} \leftarrow \mathbf{u}_1^{0T} \mathbf{A}^i$.
         $\mathbf{u}_2^{0T} \leftarrow \mathbf{u}_2^{0T} \mathbf{A}^i$.
Step 14: If $m = 1$, go to Step 15.
         $\mathbf{u}_1^{1T} \leftarrow \mathbf{u}_1^{1T} \mathbf{A}^i$.
         $\mathbf{u}_2^{1T} \leftarrow \mathbf{u}_2^{1T} \mathbf{A}^i$.
           .
           .
           .
         $\mathbf{u}_1^{qT} \leftarrow \mathbf{u}_1^{qT} \mathbf{A}^i$.
         $\mathbf{u}_2^{qT} \leftarrow \mathbf{u}_2^{qT} \mathbf{A}^i$.
Comment: In Step 7 we calculated sets of $Z_S$ and $V_S$ accounting for variations in $\mathbf{A}^i$. In Steps 13 and 14, however, we carry forward the analyses for which $\mathbf{A}^i$ is considered fixed.
Step 15: Set $i \leftarrow i+1$.
         $m \leftarrow m+1$.
         $q \leftarrow q+1$.
         Initialize $\mathbf{u}_1^q$ and $\mathbf{u}_2^q$ and go to Step 6.
Comment: $\mathbf{u}_1^q$ and $\mathbf{u}_2^q$ are initialized to start a forward analysis at a reference plane immediately following a variable element $\mathbf{A}^i$.
Step 16: Set $r \leftarrow r-1$.
         $m \leftarrow m-1$.
         Initialize $\mathbf{v}_1$ and $\mathbf{v}_2$.
Comment: At this step we start the analysis from the load end.
Step 17: If $n = l_k$, go to Step 20.
         Set $j \leftarrow n$.
Comment: $n$ is the total number of elements in the cascade.
Step 18: $\mathbf{v}_1 \leftarrow \mathbf{A}^j \mathbf{v}_1$.
         $\mathbf{v}_2 \leftarrow \mathbf{A}^j \mathbf{v}_2$.
         $j \leftarrow j-1$.
Step 19: If $j = l_m$, go to Step 20.
         Go to Step 18.
Step 20: $p \leftarrow 1$.
Step 21: Calculate $Q$ using $V_S$, $Z_S$, $\mathbf{A}^j$, and $\mathbf{v}$, and the appropriate $\mathbf{x}$.
Comment: When we reach the $k$th variable element we calculate $k-1$ values of $Q$, and when the variable element $k-1$ is reached we calculate $k-2$ values of $Q$, and so on, as illustrated in Fig. 7.
Step 22: If $p = q$, go to Step 23.
         Set $r \leftarrow r-1$.
         $p \leftarrow p+1$.
         Go to Step 21.
Step 23: If $m = 1$, stop.
         Set $q \leftarrow q-1$.
         $m \leftarrow m-1$.
         Go to Step 18.
```

VI. BRANCHED CIRCUITS

Consider, as an example, the cascaded circuit shown in Fig. 8 which has two branches, one connected in series and one in parallel. In the series and parallel branches, we highlight, for example, the elements $\mathbf{B}$ and $\mathbf{C}$, respectively. The series branch can be thought of equivalently as an element consisting of a series impedance connected in cascade with the main circuit as shown in Fig. 8. This impedance Z may be taken as the inverse of the input admittance derived in (13) and is given by

$$Z = \frac{\bar{\mathbf{u}}_{1B}^T \mathbf{B} \mathbf{v}_{1B}}{\bar{\mathbf{u}}_{2B}^T \mathbf{B} \mathbf{v}_{1B}} \tag{19}$$

where the subscript B distinguishes terms associated with the branch from that of the cascaded main circuit. The forward analysis is initiated at reference plane d, and the reverse analysis is initiated at reference plane b, as shown in Fig. 8.

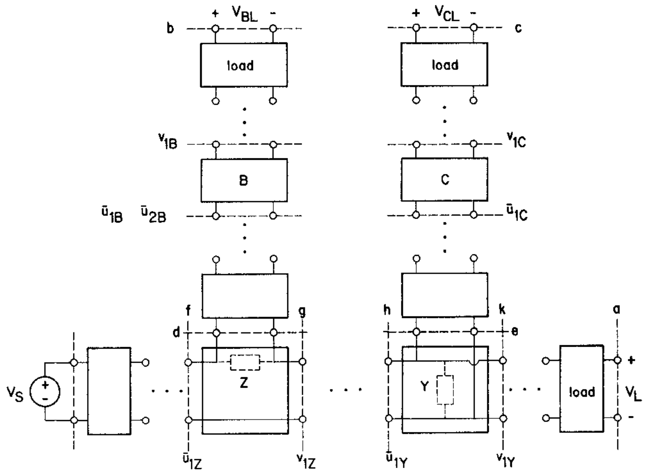

Fig. 8. An example of a cascaded circuit with a branch connected in series and a branch connected in parallel. Branches are represented in the cascade by their equivalents. Reference planes where different analyses are initiated are labeled.

Similarly, the parallel branch can be thought of equivalently as an admittance Y connected in shunt in the cascade. The admittance Y (as in (13)) is given by

$$Y=\frac{\bar{\boldsymbol{u}}_{2C}^T \boldsymbol{C} \boldsymbol{v}_{1C}}{\bar{\boldsymbol{u}}_{1C}^T \boldsymbol{C} \boldsymbol{v}_{1C}} \tag{20}$$

where the forward analysis is initiated at reference plane e, and the reverse analysis is initiated at reference plane c.

Different formulas relating the load voltages of the branches to the variables can be derived. The load voltage of the series branch can be derived, as shown in Appendix I, as a function of $\boldsymbol{B}$ as

$$V_{BL}(\boldsymbol{B})=\frac{e_2^T \boldsymbol{v}_{1Z} V_S}{\bar{\boldsymbol{u}}_{2B}^T \boldsymbol{B} \boldsymbol{v}_{1B} \bar{\boldsymbol{u}}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \boldsymbol{v}_{1Z}} \tag{21}$$

where $\bar{\boldsymbol{u}}_{1Z}^T$ is the result at reference plane f of a forward analysis initiated at the source, and $\boldsymbol{v}_{1Z}$ is the result at reference plane g of a reverse analysis initiated at the load reference plane a.

It can also be obtained, as shown in Appendix II, as a function of $\boldsymbol{C}$ as

$$V_{BL}(\boldsymbol{C})=\frac{\left[\bar{\boldsymbol{u}}_{1Yf}^T-\bar{\boldsymbol{u}}_{1Yg}^T\right]\begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \boldsymbol{v}_{1Y} V_S}{\bar{\boldsymbol{u}}_{1B}^T \boldsymbol{B} \boldsymbol{v}_{1B} \bar{\boldsymbol{u}}_{1Y}^T \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \boldsymbol{v}_{1Y}} \tag{22}$$

where $\bar{\boldsymbol{u}}_{1Y}^T$ is the result at reference plane h of a forward analysis, $\boldsymbol{v}_{1Y}$ is the result at reference plane k of a reverse analysis, $\bar{\boldsymbol{u}}_{1Yf}^T$ is the result at reference plane h of a forward analysis initiated at reference plane f, and $\bar{\boldsymbol{u}}_{1Yg}^T$ is the result at reference plane h of a forward analysis initiated at reference plane g.

The load voltage of the parallel branch can also be derived, as shown in Appendix III, as a function of $\boldsymbol{C}$ as

$$V_{CL}(\boldsymbol{C})=\frac{e_1^T \boldsymbol{v}_{1Y} V_S}{\bar{\boldsymbol{u}}_{1C}^T \boldsymbol{C} \boldsymbol{v}_{1C} \bar{\boldsymbol{u}}_{1Y}^T \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \boldsymbol{v}_{1Y}} \tag{23}$$

and, as shown in Appendix IV, as a function of $\boldsymbol{B}$ as

$$V_{CL}(\boldsymbol{B})=\frac{e_1^T \boldsymbol{v}_{1Y} V_S}{\bar{\boldsymbol{u}}_{1C}^T \boldsymbol{C} \boldsymbol{v}_{1C} \bar{\boldsymbol{u}}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \boldsymbol{v}_{1Z}}. \tag{24}$$

VII. Cascaded Networks of $2p$-Port Elements

The approach we have developed can also be utilized in the analysis and design of cascaded networks consisting of $2p$-port elements. Consider the $2p$-port element shown in Fig. 9, possessing p input ports and p output ports. Its transmission matrix is given by

$$\boldsymbol{A} \triangleq \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}$$

where $\boldsymbol{A}_{11}$, $\boldsymbol{A}_{12}$, $\boldsymbol{A}_{21}$, and $\boldsymbol{A}_{22}$ are $p\times p$ matrices. The input quantities in this case are

$$\bar{\boldsymbol{y}}=\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \cdot \\ \cdot \\ \bar{y}_p \\ \bar{y}_{p+1} \\ \bar{y}_{p+2} \\ \cdot \\ \cdot \\ \bar{y}_{2p} \end{bmatrix}$$

and the output quantities are

$$\boldsymbol{y}=\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_p \\ y_{p+1} \\ y_{p+2} \\ \cdot \\ \cdot \\ y_{2p} \end{bmatrix}$$

where the element with subscripts 1 to p denote voltages and from $p+1$ to $2p$ denote currents.

For the forward and reverse analyses the matrices $\bar{\boldsymbol{U}}_1$, $\bar{\boldsymbol{U}}_2$, $\boldsymbol{V}_1$, and $\boldsymbol{V}_2$ are initialized such that

$$\boldsymbol{E}_1 \Rightarrow \bar{\boldsymbol{U}}_1 \text{ or } \boldsymbol{V}_1$$

$$\boldsymbol{E}_2 \Rightarrow \bar{\boldsymbol{U}}_2 \text{ or } \boldsymbol{V}_2$$

where

$$\boldsymbol{E}_1 \triangleq \begin{bmatrix} \boldsymbol{1}_p \\ \boldsymbol{0}_p \end{bmatrix}$$

and

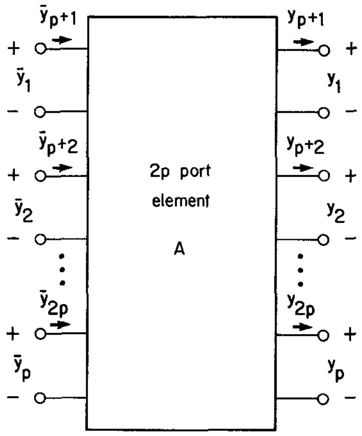

Fig. 9. A 2p-port element: a generalization of Fig. 1.

$$E_2 \triangleq \begin{bmatrix} \mathbf{0}_p \\ \mathbf{1}_p \end{bmatrix}$$

and where $\mathbf{1}_p$ is the unit matrix of order p, and $\mathbf{0}_p$ is the null matrix of order p.

We can now derive in an analogous manner to the derivation of (9)

$$V_S = \left(\bar{U}_1^T + Z_S \bar{U}_2^T\right) A \left(V_1 V_L + V_2(Y_L V_L - I_L)\right) \quad (25)$$

where

$\bar{U}_1$, $\bar{U}_2$, V_1, and V_2	matrices obtained from forward and reverse analyses,
V_S	vector containing the p source voltages,
V_L	vector of load voltages,
I_L	vector of current sources at the loads (if any),
Z_S	diagonal matrix containing the impedances of the sources,
Y_L	diagonal matrix containing the load admittances.

To evaluate the unknowns V_L, having obtained numerical values for (25), a system of p linear equations is solved. When A is perturbed or when derivatives are required, only $6p^3$ additional multiplications and the solution of a p-system of linear equations are needed and not a whole reanalysis of the entire cascaded circuit.

VIII. CONCLUSIONS

An important claim we make in this paper is that (9)–(15) can be used to generate in a straightforward manner, following differencing or differentiating (as appropriate), any desired exact formulas for multiple network analyses, sensitivity, and tolerance analysis with simultaneous large changes. All calculations are carried forward simultaneously, and redundant calculations are obviated as demonstrated by the examples and algorithms presented.

The symmetry of the networks analyzed can be exploited leading to the saving of computational effort. Branched circuits can be handled readily. Formulas, similar to (21)–(24), can be derived for other branched structures using the same concepts so as to render the sensitivity analysis and design of these circuits as simple as possible. The approach should prove to be very suitable for the computer-aided design of cascaded microwave circuits and systems consisting of 2-ports. It appears to be readily extendable to 2p-port networks.

APPENDIX I
TO OBTAIN V_{BL} AS A FUNCTION OF V_S AND $\boldsymbol{B}$

The voltage across the impedance Z representing the branched circuit in terms of V_{BL} is given by

$$V_Z = \bar{\boldsymbol{u}}_{1B}^T \boldsymbol{B} \boldsymbol{v}_{1B} V_{BL} \quad \text{(A1)}$$

and it can be expressed in terms of voltages in the main cascaded circuit as

$$V_Z = \boldsymbol{e}_1^T \left[\bar{\boldsymbol{v}}_{1Z} - \boldsymbol{v}_{1Z}\right] V_L \quad \text{(A2)}$$

where $\bar{\boldsymbol{v}}_{1Z}$ is the result of the reverse analysis at reference plane f. So (A2) can be written, substituting for the chain matrix of the element representing the branch, as

$$V_Z = \boldsymbol{e}_1^T \left[\begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \boldsymbol{v}_{1Z} - \boldsymbol{v}_{1Z} \right] V_L \quad \text{(A3)}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \boldsymbol{v}_{1Z} V_L \quad \text{(A4)}$$

$$= \boldsymbol{e}_2^T \boldsymbol{v}_{1Z} Z V_L. \quad \text{(A5)}$$

The load voltage of the main cascade V_L can be expressed by

$$V_L = \frac{V_S}{\bar{\boldsymbol{u}}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \boldsymbol{v}_{1Z}} \quad \text{(A6)}$$

and (A1) can be rewritten as

$$V_{BL} = \frac{V_Z}{\bar{\boldsymbol{u}}_{1B}^T \boldsymbol{B} \boldsymbol{v}_{1B}}. \quad \text{(A7)}$$

Substituting for V_Z of (A5) we have

$$V_{BL} = \frac{\boldsymbol{e}_2^T \boldsymbol{v}_{1Z} Z V_L}{\bar{\boldsymbol{u}}_{1B}^T \boldsymbol{B} \boldsymbol{v}_{1B}} \quad \text{(A8)}$$

and substituting for V_L from (A6) and Z from (19), we get

$$V_{BL} = \frac{\boldsymbol{e}_2^T \boldsymbol{v}_{1Z} \dfrac{\bar{\boldsymbol{u}}_{1B}^T \boldsymbol{B} \boldsymbol{v}_{1B}}{\bar{\boldsymbol{u}}_{2B}^T \boldsymbol{B} \boldsymbol{v}_{1B}} V_S}{\bar{\boldsymbol{u}}_{1B}^T \boldsymbol{B} \boldsymbol{v}_{1B} \bar{\boldsymbol{u}}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \boldsymbol{v}_{1Z}}. \quad \text{(A9)}$$

Hence,

$$V_{BL}(\boldsymbol{B}) = \frac{\boldsymbol{e}_2^T \boldsymbol{v}_{1Z} V_S}{\bar{\boldsymbol{u}}_{2B}^T \boldsymbol{B} \boldsymbol{v}_{1B} \bar{\boldsymbol{u}}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \boldsymbol{v}_{1Z}}.$$

APPENDIX II

TO OBTAIN V_{BL} AS A FUNCTION OF V_S AND $\boldsymbol{C}$

From (A7) and (A2) we can write V_{BL} as

$$V_{BL}=\frac{\boldsymbol{e}_1^T[\bar{\boldsymbol{v}}_{1Z}-\boldsymbol{v}_{1Z}]V_L}{\bar{\boldsymbol{u}}_{1B}^T\boldsymbol{B}\boldsymbol{v}_{1B}}. \tag{A10}$$

The load voltage V_L can be expressed, (cf. (A6)) by

$$V_L=\frac{V_S}{\bar{\boldsymbol{u}}_{1Y}^T\begin{bmatrix}1 & 0\\ Y & 1\end{bmatrix}\boldsymbol{v}_{1Y}}. \tag{A11}$$

We can write, using the notation defined for (22),

$$\boldsymbol{e}_1^T\bar{\boldsymbol{v}}_{1Z}=\bar{\boldsymbol{u}}_{1Yf}^T\begin{bmatrix}1 & 0\\ Y & 1\end{bmatrix}\boldsymbol{v}_{1Y}. \tag{A12}$$

Similarly,

$$\boldsymbol{e}_1^T\boldsymbol{v}_{1Z}=\bar{\boldsymbol{u}}_{1Yg}^T\begin{bmatrix}1 & 0\\ Y & 1\end{bmatrix}\boldsymbol{v}_{1Y}. \tag{A13}$$

Substituting these terms and V_L of (A11) into (A10) we obtain

$$V_{BL}(\boldsymbol{C})=\frac{\left[\bar{\boldsymbol{u}}_{1Yf}^T-\bar{\boldsymbol{u}}_{1Yg}^T\right]\begin{bmatrix}1 & 0\\ Y & 1\end{bmatrix}\boldsymbol{v}_{1Y}V_S}{\bar{\boldsymbol{u}}_{1B}^T\boldsymbol{B}\boldsymbol{v}_{1B}\bar{\boldsymbol{u}}_{1Y}^T\begin{bmatrix}1 & 0\\ Y & 1\end{bmatrix}\boldsymbol{v}_{1Y}}. \tag{A14}$$

APPENDIX III

TO OBTAIN V_{CL} AS A FUNCTION OF V_S AND $\boldsymbol{C}$

The voltage across Y in terms of V_{CL} is given by

$$V_Y=\bar{\boldsymbol{u}}_{1C}^T\boldsymbol{C}\boldsymbol{v}_{1C}V_{CL} \tag{A15}$$

and in terms of V_L as

$$V_Y=\boldsymbol{e}_1^T\boldsymbol{v}_{1Y}V_L. \tag{A16}$$

But V_L is also given by

$$V_L=\frac{V_S}{\bar{\boldsymbol{u}}_{1Y}^T\begin{bmatrix}1 & 0\\ Y & 1\end{bmatrix}\boldsymbol{v}_{1Y}}. \tag{A17}$$

So, substituting this V_L into (A16) and the resulting V_Y into (A15) we get

$$V_{CL}(\boldsymbol{C})=\frac{\boldsymbol{e}_1^T\boldsymbol{v}_{1Y}V_S}{\bar{\boldsymbol{u}}_{1C}^T\boldsymbol{C}\boldsymbol{v}_{1C}\bar{\boldsymbol{u}}_{1Y}^T\begin{bmatrix}1 & 0\\ Y & 1\end{bmatrix}\boldsymbol{v}_{1Y}}. \tag{A18}$$

APPENDIX IV

TO OBTAIN V_{CL} AS A FUNCTION OF V_S AND $\boldsymbol{B}$

From (A15), (A16), and (A6) we can write V_{CL} as

$$V_{CL}(\boldsymbol{B})=\frac{\boldsymbol{e}_1^T\boldsymbol{v}_{1Y}V_S}{\bar{\boldsymbol{u}}_{1C}^T\boldsymbol{C}\boldsymbol{v}_{1C}\bar{\boldsymbol{u}}_{1Z}^T\begin{bmatrix}1 & Z\\ 0 & 1\end{bmatrix}\boldsymbol{v}_{1Z}}. \tag{A19}$$

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REFERENCES

[1] C. W. Therrien, "Use of the adjoint for computing exact changes in response of cascaded two-port networks," *IEEE Trans. Circuits Syst.*, vol. CAS-21, pp. 217–218, Mar. 1974.

[2] J. W. Bandler and R. E. Seviora, "Current trends in network optimization," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 1159–1170, Dec. 1970.

[3] M. C. Horton and R. J. Wenzel, "General theory and design of optimum quarter-wave TEM filters," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 316–327, May 1965.

[4] J. W. Bandler, C. Charalambous, J. H. K. Chen, and W. Y. Chu, "New results in the least *p*th approach to minimax design," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 116–119, Feb. 1976.

[5] J. W. Bandler and H. L. Abdel-Malek, "Optimal centering, tolerancing and yield determination using multidimensional approximations," *Proc. IEEE Int. Symp. Circuits Syst.* (Phoenix, AZ), 1977, pp. 219–222.

[6] H. L. Abdel-Malek and J. W. Bandler, "Yield optimization for arbitrary statistical distributions, Part I: Theory," *Proc. IEEE Int. Symp. Circuits Syst.* (New York, NY), 1978, pp. 664–669.

[7] J. W. Bandler, M. R. M. Rizk, and H. Tromp, "Efficient calculation of exact group delay sensitivities," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 188–194, Apr. 1976.