# THE IMPACT OF GENERALIZED SYMMETRY ON COMPUTER-AIDED DESIGN OF CASCADED STRUCTURES

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## ABSTRACT

New theoretical and computational tools are presented which deal with generalized symmetry concepts related to the computer-aided design of cascaded structures. The presentation includes the computational implications of networks consisting of symmetrically located reverse adjoint subnetworks with and without scaling, as well as antisymmetry. Formulas presented are designed to be used in simulation, sensitivity and tolerance analyses as well as in optimal design.

## Introduction

Symmetry or antisymmetry pervades microwave circuits [1]. To date, there has hardly been any systematic exploration or utilization of such features in computer-aided design.

Bandler et al. [2,3] recently provided theoretical tools for handling cascaded structures. Their framework permits the special constraints imposed by symmetry (interpreted generally) to be embodied directly into an overall computational scheme. Advantages include reducing the sizes of design problems with the attendant reduction in computation cost.

This paper presents and interprets a new definition of generalized symmetry described by the term symmetrically located reverse adjoint subnetworks with scaling. Antisymmetrical structures are studied analogously.

## Theoretical Background

Consider a two-port element with transmission or chain matrix $\underset{\sim}{A}$, output vector $\underset{\sim}{y}$ and input vector $\bar{\underset{\sim}{y}}$, containing in the first and second rows, respectively, the voltage and current. Then $\bar{\underset{\sim}{y}} = \underset{\sim}{A}\,\underset{\sim}{y}$ is termed the basic iteration [2,3]. Analysis is expressed by

$$\text{forward: } \bar{\underset{\sim}{u}}^T \underset{\sim}{A} = \underset{\sim}{u}^T, \quad \text{reverse: } \bar{\underset{\sim}{v}} = \underset{\sim}{A}\,\underset{\sim}{v}. \qquad (1)$$

Consider a cascade of n two-ports. Let $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ be unit vectors. Taking

$$\underset{\sim}{v}_j^n = \underset{\sim}{e}_j, \quad j \in \{1,2\} \qquad (2)$$

we have

$$\underset{\sim}{v}_j^i = \underset{\sim}{A}^{i+1}\,\underset{\sim}{A}^{i+2} \cdots \underset{\sim}{A}^n\,\underset{\sim}{e}_j \qquad (3)$$

and, letting

$$\underset{\sim}{u}_j^0 = \bar{\underset{\sim}{u}}_j^{-1} = \underset{\sim}{e}_j, \quad j \in \{1,2\} \qquad (4)$$

we have

$$\underset{\sim}{e}_j^T\,\underset{\sim}{A}^1\,\underset{\sim}{A}^2 \cdots \underset{\sim}{A}^{i-1} = \bar{\underset{\sim}{u}}_j^{i\,T}. \qquad (5)$$

Consider a rotation matrix given by

$$\underset{\sim}{R} \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad (6)$$

Premultiplying a vector by $\underset{\sim}{R}$ interchanges its rows.

Consider a similarity transformation we will refer to as antitransposition given by

$$\underset{\sim}{A}^R = \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underset{\sim}{R}\,\underset{\sim}{A}^T\,\underset{\sim}{R}. \qquad (7)$$

Important properties of this transformation are

$$(\underset{\sim}{A}^1\,\underset{\sim}{A}^2)^R = (\underset{\sim}{A}^2)^R\,(\underset{\sim}{A}^1)^R, \quad (\underset{\sim}{A}^R)^R = \underset{\sim}{A}. \qquad (8)$$

Consider a scaling matrix given by

$$\underset{\sim}{S} \triangleq \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}. \qquad (9)$$

This transformation allows us to define

$$\underset{\sim}{A}_\alpha \triangleq \begin{bmatrix} a_{11} & \alpha a_{12} \\ a_{21}/\alpha & a_{22} \end{bmatrix} = \underset{\sim}{S}\,\underset{\sim}{A}\,\underset{\sim}{S}^{-1}. \qquad (10)$$

Important implications of this are

$$(\underset{\sim}{A}^1\,\underset{\sim}{A}^2)_\alpha = (\underset{\sim}{A}^1)_\alpha\,(\underset{\sim}{A}^2)_\alpha, \quad (\underset{\sim}{A}^T)_\alpha = (\underset{\sim}{A}_{1/\alpha})^T,$$
$$(\underset{\sim}{A}_\alpha)_{1/\alpha} = \underset{\sim}{A}, \quad (\underset{\sim}{A}^R)_\alpha = (\underset{\sim}{A}_\alpha)^R = \underset{\sim}{A}_\alpha^R. \qquad (11)$$

## Generalized Symmetry

Consider a network with a pair of elements placed symmetrically within certain subnetworks as in Fig. 1.

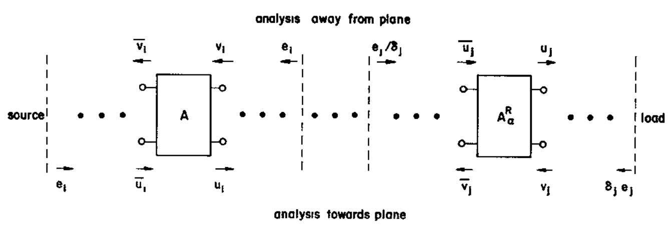


Fig. 1 A network with a pair of elements under consideration. Directions indicate those of analysis and subscripts correspond to unit initialization with i ≠ j.

Consider also that the left subnetwork is being analyzed in the forward direction and the right subnetwork is being analyzed in the reverse direction. This means that we are simultaneously considering for i,j ∈ {1,2} the iteration pair

$$\text{forward: } \bar{\underset{\sim}{u}}_i^T\,\underset{\sim}{A} = \underset{\sim}{u}_i^T, \quad \text{reverse: } \bar{\underset{\sim}{v}}_j = \underset{\sim}{A}_\alpha^R\,\underset{\sim}{v}_j. \qquad (12)$$

We can show in this case that

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$$\mathbf{v}_j = \mathbf{S}\,\mathbf{R}\,\bar{\mathbf{u}}_i \iff \bar{\mathbf{v}}_j = \mathbf{S}\,\mathbf{R}\,\mathbf{u}_i, \quad i \neq j. \tag{13}$$

This means that the condition assumed at the beginning of the iteration is preserved for the next iteration.

Similarly, it is easy to show that

$$\mathbf{R}\,\mathbf{S}\,\bar{\mathbf{u}}_j = \mathbf{v}_i \iff \mathbf{R}\,\mathbf{S}\,\mathbf{u}_j = \bar{\mathbf{v}}_i, \quad i \neq j. \tag{14}$$

The broad implications of the foregoing presentation is that if $\mathbf{A}$ and $\mathbf{A}^R_\alpha$ represent elements in a network then a forward/reverse iteration for $\mathbf{A}$ supplies the results of a reverse/forward iteration for $\mathbf{A}^R_\alpha$ and vice versa. The $\mathbf{RS}$ transformation must hold for the previous iteration.

Initializations at the source and load of a network, for example, may be related by

$$[\delta_j\ \mathbf{e}_j] = \mathbf{S}\,\mathbf{R}\,\mathbf{e}_i, \quad i \neq j, \tag{15}$$

where $\mathbf{e}_j$ refers to the load end and $\mathbf{e}_i$ to the source end. For the opposite case

$$\mathbf{R}\,\mathbf{S}\begin{bmatrix} \mathbf{e}_j \\ \delta_j \end{bmatrix} = \mathbf{e}_i, \quad i \neq j, \tag{16}$$

where $\delta_j = \alpha$ if $j = 1$ and 1 if $j = 2$.

## Antisymmetry with Scaling

Following a very similar approach we replace (12) by

$$\text{forward: } \bar{\mathbf{u}}_i^T \mathbf{A} = \mathbf{u}_i^T, \text{ reverse: } \bar{\mathbf{v}}_i = (\mathbf{A}^T)_\alpha\, \mathbf{v}_i, \tag{17}$$

and, in the opposite direction,

$$\text{reverse: } \bar{\mathbf{v}}_i = \mathbf{A}\,\mathbf{v}_i, \text{ forward: } \bar{\mathbf{u}}_i^T (\mathbf{A}^T)_\alpha = \mathbf{u}_i^T, \tag{18}$$

Then

$$\mathbf{v}_i = \mathbf{S}\,\bar{\mathbf{u}}_i \iff \bar{\mathbf{v}}_i = \mathbf{S}\,\mathbf{u}_i \tag{19}$$

for analyses towards the plane of antisymmetry and

$$\mathbf{S}\,\bar{\mathbf{u}}_i = \mathbf{v}_i \iff \mathbf{S}\,\mathbf{u}_i = \bar{\mathbf{v}}_i \tag{20}$$

for analyses away from the plane of antisymmetry.

Initializations corresponding to (15) and (16) are, respectively, given by

$$[\delta_i\ \mathbf{e}_i] = \mathbf{S}\,\mathbf{e}_i \tag{21}$$

and

$$\mathbf{S}\begin{bmatrix} \mathbf{e}_i \\ \delta_i \end{bmatrix} = \mathbf{e}_i. \tag{22}$$

## Reverse Adjoint Subnetworks

Suppose we have a cascade of two-ports. Let

$$\mathbf{A}^{n-j} = (\mathbf{A}^{j+1})^R_\alpha, \quad j = 0, 1, \ldots, n/2 - 1. \tag{23}$$

Then

$$\mathbf{A}^{n-j}\,\mathbf{A}^{n-j+1} \ldots \mathbf{A}^n = (\mathbf{A}^1\,\mathbf{A}^2 \ldots \mathbf{A}^{j+1})^R_\alpha. \tag{24}$$

This situation is described by the phrase symmetrically located reverse adjoint subnetworks with scaling. The following is an interpretation.

A given element is described by

$$\mathbf{A} \triangleq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{25}$$

Its adjoint is given by [4]

$$\hat{\mathbf{A}} \triangleq \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix} = \frac{1}{a_{11}\,a_{22} - a_{12}\,a_{21}} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{26}$$

Interchanging the ports of the adjoint, i.e., reversing the element, we obtain

$$\hat{\mathbf{A}}^{rev} = \frac{1}{\hat{a}_{11}\,\hat{a}_{22} - \hat{a}_{12}\,\hat{a}_{21}} \begin{bmatrix} \hat{a}_{22} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{11} \end{bmatrix}$$

$$= \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} = \mathbf{A}^R \tag{27}$$

by definition (7). Finally, applying the scaling transformation (10) we obtain the final result.

Fig. 2 shows an equivalent network representation

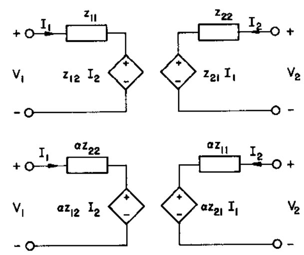


Fig. 2 Impedance matrix representation of a pair of reverse adjoint subnetworks with scaling.

of a pair of symmetrically located reverse adjoint subnetworks or elements with scaling. Notice that, unlike the adjoint element which requires interchanging of controlling and controlled branches of nonreciprocal elements and unlike the reversed element which would, for example, reverse the direction of amplification of amplifiers, the symmetrical element preserves the directions of any nonreciprocal propagation caused by its companion.

Consider a network consisting of symmetrically located identical pairs of elements, each element of which is itself symmetrical:

$$\text{identical pairs: } \mathbf{A}^{n-j} = \mathbf{A}^{j+1}, \ j = 0,1,\ldots,n-1, \tag{28}$$

$$\text{symmetrical elements: } \mathbf{A}^i = (\mathbf{A}^i)^R, \ i = 1,2,\ldots,n. \tag{29}$$

Notice that there is no reciprocity assumption.

Another case is for reciprocal networks: a symmetrically decomposed symmetrical network implied by

$$\text{symmetrical pairs: } \mathbf{A}^{n-j} = (\mathbf{A}^{j+1})^R, \ j=0,1,\ldots,n-1, \tag{30}$$

$$\text{sym. decomposition: } (\mathbf{A}^i)^R = (\mathbf{A}^i)^{rev}, \ i=1,2,\ldots,n, \tag{31}$$

where the designation rev implies a reversed element.

## Antisymmetrical Networks

For antisymmetrical networks

$$\mathbf{A}^{n-j} = ((\mathbf{A}^{j+1})^T)_\alpha, \ j = 0, 1, \ldots, n-1. \tag{32}$$

In this case

$$\mathbf{A}^{n-j}\,\mathbf{A}^{n-j+1} \ldots \mathbf{A}^n = ((\mathbf{A}^1\,\mathbf{A}^2 \ldots \mathbf{A}^{j+1})^T)_\alpha. \tag{33}$$

Ladder network and stepped impedance transmission-line networks are well-known examples.

For a transmission line, for example,

$$\mathbf{A} = \begin{bmatrix} \cos\theta & jZ\sin\theta \\ j\dfrac{\sin\theta}{Z} & \cos\theta \end{bmatrix} \implies (\mathbf{A}^T)_\alpha = \begin{bmatrix} \cos\theta & j\dfrac{\alpha\sin\theta}{Z} \\ j\dfrac{Z\sin\theta}{\alpha} & \cos\theta \end{bmatrix}.$$

This implies that the product of the corresponding characteristic impedances is $\alpha$.

Network Simulation and Design Exploiting Symmetry

Bandler et al. [2] have shown that

$$Q_{k\ell} = \bar{u}_k^T A v_\ell, \quad k, \ell \in \{1,2\}, \tag{34}$$

embodies all the information required in the calculation of responses, first- and higher-order sensitivities as well as large-change sensitivities for that subnetwork (or network) containing $A$, where the $\bar{u}_k$ and $v_\ell$ variables are associated with forward and reverse analyses, respectively, in that subnetwork initiated by appropriate unit vectors.

The corresponding term for a subnetwork containing $A_\alpha^R$ is given by

$$P_{k\ell} = \delta_k [(R S)^{-1} v_m]^T A_\alpha^R [S R \bar{u}_n]/\delta_\ell, \quad k, \ell, m, n \in \{1,2\}, \quad k \neq m, \ell \neq n. \tag{35}$$

Notice that the forward vector must be multiplied by $\delta_k$ and the reverse vector divided by $\delta_\ell$ so that the formula is valid for unit initializations on the symmetrical subnetwork (Fig. 1). Notice also that (35) already incorporates the constraints (13) and (14). Upon substituting for $A_\alpha^R$ we have

$$P_{k\ell} = \delta_k v_m^T A^T \bar{u}_n/\delta_\ell = \delta_k \bar{u}_n^T A v_m/\delta_\ell. \tag{36}$$

From (34) and (35) we have

$$P_{11} = Q_{22}, \quad P_{12} = \alpha Q_{12}, \quad P_{21} = Q_{21}/\alpha, \quad P_{22} = Q_{11}, \tag{37}$$

which can be used directly in <u>any</u> of the recursive formulas of Bandler et al. [2,3].

Network Simulation and Design Exploiting Antisymmetry

By analogy the term corresponding to (35) for a subnetwork containing $(A^T)_\alpha$ is given by

$$P_{k\ell} = \delta_k [S^{-1} v_k]^T (A^T)_\alpha [S \bar{u}_\ell]/\delta_\ell. \tag{38}$$

Substituting for $(A^T)_\alpha$ we have

$$P_{k\ell} = \delta_k v_k^T A^T \bar{u}_\ell/\delta_\ell = \delta_k \bar{u}_\ell^T A v_k/\delta_\ell, \tag{39}$$

hence

$$P_{11} = Q_{11}, \quad P_{12} = \alpha Q_{21}, \quad P_{21} = Q_{12}/\alpha, \quad P_{22} = Q_{22}, \tag{40}$$

which can be utilized to calculate responses, sensitivities and tolerances to reduce effort.

Numerical Example

We consider as a numerical example the optimization over 100 percent relative bandwidth of the response of a quarter-wave transformer having $n = 6$ and terminated resistively in 1 Ω at the source and 100 Ω at the load [1]. See Table 1. Obviously, $\alpha = 100$. Optimization w.r.t. all variables: 6 lengths and 6 characteristic impedances was carried out in two ways. The first follows Bandler et al. [2] in which the 12 variables were treated independently. The second uses the results of the present paper, in which the appropriate constraint is imposed before the analysis and only half the network is analyzed. Only 6 variables are optimized in this case. Least pth optimization by a new package called FLOPT5 was used [5]. The sequence of p was 2, 20, 1,000, 50,000 and 1,000,000. Twenty-one sample points were used during optimization. At each least pth optimum quadratic interpolation at 101 points identified maxima in the reflection coefficient and appropriate sample points in the set of 21 were replaced. The length variable is normalized to the quarter wave length at center frequency. The incredible consistency of the solutions (only one digit differs in the two approaches) was achieved by FLOPT5 on a CDC 6400 in 266 response evaluations (40 s CPU time) and 140 response evaluations (8 s CPU time).

Table 1
The Results of Optimization

| Section | Parameter | Value | |
|---|---|---|---|
| | | Start | Optimum |
| 1 | Z | 1.2 | 1.2960244 |
| | ℓ | 0.8 | 1.0000000 |
| 2 | Z | 2.4 | 2.3894713 |
| | ℓ | 1.1 | 1.0000000 |
| 3 | Z | 6.1 | 5.9778006 |
| | ℓ | 1.5 | 1.0000000 |
| 4 | Z | 100/6.1 | 16.728561 |
| | ℓ | 1.5 | 1.0000000 |
| 5 | Z | 100/2.4 | 41.850262(3) |
| | ℓ | 1.1 | 1.0000000 |
| 6 | Z | 100/1.2 | 77.159040 |
| | ℓ | 0.8 | 1.0000000 |

Conclusions

This paper deals with the exploitation of symmetry and antisymmetry in cascaded networks. The principal goal is to save computational effort in computer-aided design and optimization of such structures. Theoretical descriptions of symmetry and antisymmetry were given for the general case of nonreciprocal as well as scaled networks. Our generalization permits us to consider active and scaled networks with similar computational savings as expected for classical symmetrical and antisymmetrical networks.

References

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