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ALGORITHMS FOR TOLERANCE AND SECOND-ORDER SENSITIVITIES OF CASCADED STRUCTURES

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ABSTRACT

An exact and efficient approach to network analysis for cascaded structures suggested by Bandler et al. is extended to second-order sensitivities and to the evaluation of the response and its first-order sensitivity at the vertices of a tolerance region located in the design parameter space. A substantial saving in computational effort is achieved over the approach of reanalyzing the circuit at every vertex.

INTRODUCTION

A new approach for the chain matrix analysis of cascaded networks has been used efficiently to perform response evaluation as well as simultaneous and arbitrary large-change sensitivity [1]. This paper shows how first- and second-order sensitivities of the response w.r.t. the variable parameters can be obtained using this approach.

In tolerance optimization, the response and its first-order sensitivity at the vertices of the tolerance region [2] are needed by gradient algorithms. This information is useful in a worst-case search algorithm to identify the worst vertex.

THEORETICAL FOUNDATION

Consider the two-port element depicted in Fig. 1. The basic iteration, also summarized by Table

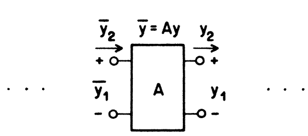

Fig. 1 Notation for an element in the chain, indicating reference directions.

I, is $\bar{y} = A y$, where A is the transmission or chain matrix, y contains the output voltage and current and $\bar{y}$ the corresponding input quantities. Table I presents some of the principal concepts involved in

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TABLE I
PRINCIPAL CONCEPTS INVOLVED IN THE ANALYSES

Concept	Definition	Implication
Basic iteration	$\bar{y} = A y$	$y \Longrightarrow \bar{y}$
Forward operation	$\bar{u}^T A = u^T$	$\bar{u}^T \bar{y} = \bar{u}^T A y = u^T y$
Reverse operation	$\bar{v} = A v$	$y = cv \Longrightarrow \bar{y} = c\bar{v}$

the following analyses. Fig. 2 depicts a cascaded network with appropriate terminations.

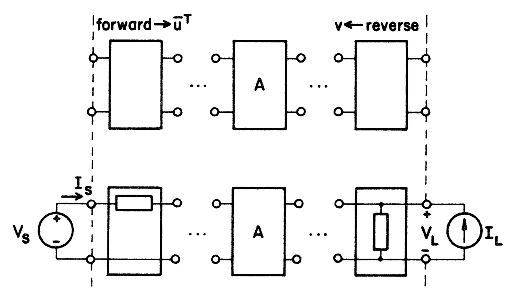

Fig. 2 Cascaded network with terminations.

Forward analysis consists of initializing a $\bar{u}^T$ row vector as either [1 0], [0 1] or a suitable linear combination and successively premultiplying each constant chain matrix by the resulting row vector until an element of interest or a termination is reached.

Reverse analysis, which is similar to conventional analysis of cascaded networks, proceeds by initializing a v column vector as either $[1\ 0]^T$ or $[0\ 1]^T$ or a suitable linear combination and successively postmultiplying each constant matrix by the resulting column vector, again until either an element of interest, or a termination is reached.

For more than one element in the cascade we divide the network into subnetworks by reference planes. These in turn are chosen so that no more than one element is explicitly considered between any pair of reference planes. In Fig. 3 the elements $\underset{\sim}{A}^k$, $\underset{\sim}{A}^i$ and $\underset{\sim}{A}^j$ are considered in the kth, the ith and the jth subnetworks, respectively.

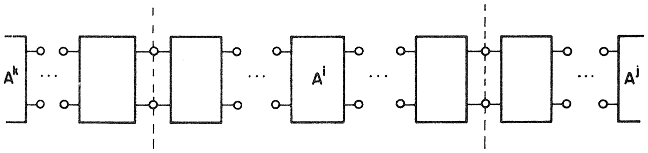

Fig. 3 Subnetwork i cascaded with subnetworks k (at source end) and j (at load end).

Note that the superscripts of $\underset{\sim}{A}$ denote the subnetwork and not the element. Forward and reverse analyses are initiated at the reference planes. A forward iteration of the structure of Fig. 3 is illustrated in Fig. 4, where equivalent (Thevenin) sources are iteratively determined.

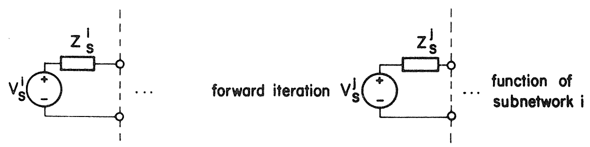

Fig. 4 Forward iteration for Fig. 3, transferring an equivalent source accounting for design variables from subnetwork k from one reference plane to the other.

Equivalent (Norton) sources can also be iteratively determined by reverse iteration [1].

NETWORK FUNCTIONS IN TERMS OF ELEMENTS

Performing forward analysis from the source of the ith subnetwork to the input of $\underset{\sim}{A}^i$ and reverse analysis from the load to the output of $\underset{\sim}{A}^i$ we have

$$V_S^i = (\bar{\underset{\sim}{u}}_1+Z_S^i\bar{\underset{\sim}{u}}_2)^T\,\underset{\sim}{A}^i(V_L^i\underset{\sim}{v}_1+(Y_L^iV_L^i-I_L^i)\underset{\sim}{v}_2) = V_L^k+Z_S^iI_S^i \quad (1)$$

and the current through the voltage source of the ith subnetwork

$$I_S^i = \bar{\underset{\sim}{u}}_2^T\,\underset{\sim}{A}^i(V_L^i\underset{\sim}{v}_1+(Y_L^iV_L^i-I_L^i)\underset{\sim}{v}_2) = V_L^kY_L^k - I_L^k. \quad (2)$$

From (1), letting $I_L^i = 0$ and $Y_L^i = 0$, we have $I_S^j = 0$ and the Thevenin voltage

$$V_S^j = V_L^i = \frac{V_S^i}{(\bar{\underset{\sim}{u}}_1+Z_S^i\bar{\underset{\sim}{u}}_2)^T\underset{\sim}{A}^i\underset{\sim}{v}_1} = \frac{V_S^i}{Q_{11}^i+Z_S^iQ_{21}^i}, \quad (3)$$

where the Q terms are defined in Table II. Letting $V_S^i = 0$ and $Y_L^i = 0$, we have $I_S^j = -I_L^i$ and the output impedance

$$Z_S^j = \frac{V_L^i}{I_L^i} = \frac{(\bar{\underset{\sim}{u}}_1+Z_S^i\bar{\underset{\sim}{u}}_2)^T\underset{\sim}{A}^i\underset{\sim}{v}_2}{(\bar{\underset{\sim}{u}}_1+Z_S^i\bar{\underset{\sim}{u}}_2)^T\underset{\sim}{A}^i\underset{\sim}{v}_1} = \frac{Q_{12}^i+Z_S^iQ_{22}^i}{Q_{11}^i+Z_S^iQ_{21}^i}. \quad (4)$$

These expressions permit equivalent Thevenin sources to be moved in a forward iteration.

Expressions which permit equivalent Norton sources to be moved (if desired) in a reverse iteration are derived analogously [1].

TABLE II
NOTATION AND IMPLIED INITIAL CONDITIONS

Factor		Identification	Initial Conditions Forward	Initial Conditions Reverse
$\bar{\underset{\sim}{u}}_1^T$ (*) $\underset{\sim}{v}_1$	≡	$(\dagger)_{11}$	voltage	voltage
$\bar{\underset{\sim}{u}}_1^T$ (*) $\underset{\sim}{v}_2$	≡	$(\dagger)_{12}$	voltage	current
$\bar{\underset{\sim}{u}}_2^T$ (*) $\underset{\sim}{v}_1$	≡	$(\dagger)_{21}$	current	voltage
$\bar{\underset{\sim}{u}}_2^T$ (*) $\underset{\sim}{v}_2$	≡	$(\dagger)_{22}$	current	current

(*) denotes either $\underset{\sim}{A}$, $\partial\underset{\sim}{A}/\partial\phi$ or $\Delta\underset{\sim}{A}$

(†) denotes Q, Q' or ΔQ

A special case of (3) applicable to Fig. 2 is

$$V_L = V_S/(\bar{\underset{\sim}{u}}_1^T\,\underset{\sim}{A}\,\underset{\sim}{v}_1) = V_S/Q_{11}. \quad (5)$$

Table III gives some useful formulas for variations in a particular element $\underset{\sim}{A}$ involving one network analysis.

TABLE III
FUNCTIONS OF V_L FOR CHANGES IN $\underset{\sim}{A}$ ONLY

Variable	Output
$\underset{\sim}{A}$	$V_L = V_S/Q_{11}$
$\partial\underset{\sim}{A}/\partial\phi$	$\partial V_L/\partial\phi = -(V_L^2/V_S)\,Q_{11}'$
$\Delta\underset{\sim}{A}$	$\Delta V_L = -V_L^2/(V_L+V_S/\Delta Q_{11})$

SECOND-ORDER SENSITIVITIES

The first-order sensitivity of V_L w.r.t. a variable parameter ϕ_1 is given using (5) by

$$\frac{\partial V_L}{\partial\phi_1} = -V_S\,\frac{\partial Q_{11}}{\partial\phi_1}\Big/Q_{11}^2. \quad (6)$$

Differentiating (6) w.r.t. ϕ_2 we get

$$\frac{\partial^2V_L}{\partial\phi_2\partial\phi_1} = -V_L\left[\frac{Q_{11}\dfrac{\partial^2Q_{11}}{\partial\phi_2\partial\phi_1} - 2\dfrac{\partial Q_{11}}{\partial\phi_1}\dfrac{\partial Q_{11}}{\partial\phi_2}}{Q_{11}^2}\right]. \quad (7)$$

The evaluation of $\partial Q_{11}/\partial\phi_1$ and $\partial Q_{11}/\partial\phi_2$ is straightforward (see Table III). For the evaluation of the term $\partial^2Q_{11}/\partial\phi_2\partial\phi_1$, we assume that the variables are numbered consecutively from source to load so that, for example,

$$\frac{\partial^2Q_{11}}{\partial\phi_2\partial\phi_1} = \frac{\partial}{\partial\phi_1}\,(\bar{\underset{\sim}{u}}_1^T)\,\frac{\partial\underset{\sim}{A}}{\partial\phi_2}\,\underset{\sim}{v}_1. \quad (8)$$

Note that $\bar{\underset{\sim}{u}}_1^T$ is a function of a certain chain matrix which contains the variable ϕ_1, $\underset{\sim}{A}$ is the chain matrix containing ϕ_2 and $\underset{\sim}{v}_1$ is evaluated at the reference plane following $\underset{\sim}{A}$.

An algorithm [3] similar to one in [1] can be used to obtain the first- and second-order sensitivities of V_L w.r.t. the design variables.

THE EVALUATION OF V_L AND ITS SENSITIVITIES AT ALL VERTICES OF THE TOLERANCE REGION

Algorithms for finding worst vertices of the tolerance region need the response at the vertices [4] and the sensitivity of this response w.r.t. the design parameters [5]. Each parameter will have a tolerance associated with it so that it has the value $\phi + \varepsilon$ or $\phi - \varepsilon$, where ε is the tolerance [2]. The number of vertices is 2^k, where k is the number of parameters.

Assume we have partitioned the network by reference planes into subnetworks such that each subnetwork contains one chain matrix containing a variable parameter. Each reference plane is chosen to fall immediately after a variable element.

The Thevenin voltage/impedance of the ith subnetwork is considered as the source voltage/ impedance of the (i+1)th subnetwork, given by (3) and (4), respectively, where j = i+1. We note that Q_{11}^i, Q_{21}^i, Q_{12}^i and Q_{22}^i are as in Table II with $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ set to $[1\ 0]^T$ and $[0\ \ 1]^T$, respectively, since the appropriate reference plane immediately follows the element $\underset{\sim}{A}^i$. The number of pairs of terms V_S^{i+1} and Z_S^{i+1} to be evaluated is 2^i, since each subnetwork contains one variable element with two extreme values (assuming that each $\underset{\sim}{A}^i$ contains only one variable parameter).

Differentiating (3) w.r.t. ϕ_h, where ϕ_h does not belong to $\underset{\sim}{A}^i$, but V_S^i and Z_S^i are functions of ϕ_h (i.e., ϕ_h is in a subnetwork h before the ith subnetwork) we get

$$\frac{\partial V_S^{i+1}}{\partial \phi_h} = \frac{(Q_{11}^i + Z_S^i\, Q_{21}^i)\,\frac{\partial V_S^i}{\partial \phi_h} - V_S^i\,\frac{\partial Z_S^i}{\partial \phi_h}\,Q_{21}^i}{(Q_{11}^i + Z_S^i\, Q_{21}^i)^2}\,, \qquad (9)$$

and differentiating (4) w.r.t. ϕ_h, we get

$$\frac{\partial Z_S^{i+1}}{\partial \phi_h} = \frac{\partial Z_S^i}{\partial \phi_h}\,\frac{(Q_{11}^i Q_{22}^i - Q_{12}^i Q_{21}^i)}{(Q_{11}^i + Z_S^i\, Q_{21}^i)^2}\,. \qquad (10)$$

On the other hand, the derivatives w.r.t. ϕ_i which is contained in $\underset{\sim}{A}^i$ (Z_S^i and V_S^i are not functions of ϕ_i), are

$$\frac{\partial V_S^{i+1}}{\partial \phi_i} = \frac{-V_S^i\,(\frac{\partial Q_{11}^i}{\partial \phi_i} + Z_S^i\,\frac{\partial Q_{21}^i}{\partial \phi_i})}{(Q_{11}^i + Z_S^i\, Q_{21}^i)^2}\,, \qquad (11)$$

and

$$\partial Z_S^{i+1}/\partial \phi_1 = (X - Y)/(Q_{11}^i + Z_S^i\, Q_{21}^i)^2, \qquad (12)$$

where

$$X = (Q_{11}^i + Z_S^i\, Q_{21}^i)(\frac{\partial Q_{21}^i}{\partial \phi_i} + Z_S^i\,\frac{\partial Q_{22}^i}{\partial \phi_i})\,,$$

$$Y = (Q_{12}^i + Z_S^i\, Q_{22}^i)(\frac{\partial Q_{11}^i}{\partial \phi_i} + Z_S^i\,\frac{\partial Q_{21}^i}{\partial \phi_i})\,,$$

where $\partial Q_{11}^i/\partial\phi_i$, $\partial Q_{21}^i/\partial\phi_i$, $\partial Q_{12}^i/\partial\phi_i$ and $\partial Q_{22}^i/\partial\phi_i$ correspond to Table II. This sensitivity information is carried through the analysis for each subnetwork. The number of variables for which sensitivities exist at the (i+1)th subnetwork is i so that $2^i.i$ sensitivity calculations are performed. Having Y_L and I_L as zeros, the expression relating V_L and the last sets of V_S and Z_S, is given by (3), so that 2^k values for V_L and its sensitivities can be obtained from appropriate values of V_S, Z_S and $\underset{\sim}{A}$.

Fig. 5 shows the stages involved in the algorithm to obtain the response and its sensitivities at the vertices (3 variables ==> 8 vertices) of the tolerance region.

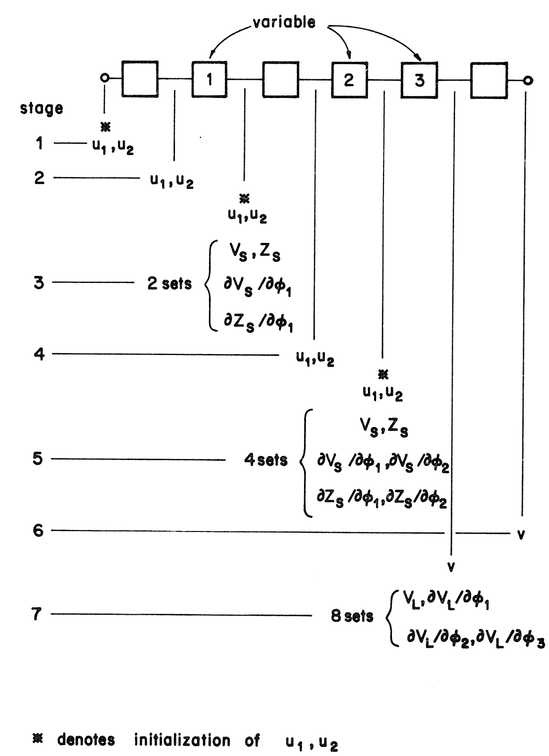

Fig. 5 Illustration of the principal stages of the algorithm.

<u>Algorithm</u>

<u>Step 1</u> Initialize $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$ and $\underset{\sim}{v}$. Set $i \leftarrow 1$, $m \leftarrow 1$, $j \leftarrow n$.

<u>Comment</u> n is the total number of elements in the cascade.

<u>Step 2</u> If $i = \ell_m$ go to Step 6.

<u>Comment</u> The ℓ_m is an element of an index set containing superscripts of k matrices containing k variable parameters and ordered consecutively.

<u>Step 3</u> $\bar{\underset{\sim}{u}}_1^T \leftarrow \bar{\underset{\sim}{u}}_1^T \underset{\sim}{A}^i$. $\bar{\underset{\sim}{u}}_2^T \leftarrow \bar{\underset{\sim}{u}}_2^T \underset{\sim}{A}^i$. Set $i \leftarrow i+1$.

<u>Step 4</u> If $i = \ell_m$ go to Step 5. Go to Step 3.

<u>Step 5</u> If m=k go to Step 7.

TABLE IV

THE RESPONSE V_L AND ITS SENSITIVITIES AT THE VERTICES OF THE TOLERANCE REGION AT NORMALIZED FREQUENCY 0.7

Vertex	V_L	$\partial V_L/\partial Z_1$	$\partial V_L/\partial Z_4$	$\partial V_L/\partial Z_5$	Sign of Tolerance Extreme		
1	0.49135+j0.02351	-0.02450+j0.05953	0.26004-j1.15934	0.02549+j0.32944	-	-	-
2	0.48819+j0.02571	-0.07761+j0.01588	0.28346-j1.05326	0.00954+j0.34878	+	-	-
3	0.49679-j0.04862	0.03751+j0.15916	-0.06631-j0.94430	0.04534+j0.29165	-	+	-
4	0.49677-j0.04046	-0.03384+j0.11417	-0.00426-j0.87724	0.03578+j0.31848	+	+	-
5	0.49209+j0.04341	-0.04367+j0.08072	0.29407-j1.19530	-0.00103+j0.33324	-	-	+
6	0.48786+j0.04670	-0.09378+j0.03123	0.32067-j1.07952	-0.02042+j0.35007	+	-	+
7	0.49889-j0.03101	0.02608+j0.18868	-0.05742-j0.97346	0.02462+j0.29494	-	+	+
8	0.49818-j0.02127	-0.04526+j0.13735	0.01132-j0.90191	0.01113+j0.32057	+	+	+

<u>Step 6</u> Calculate V_S, Z_S, $\partial V_S/\partial \phi_1$, ..., $\partial V_S/\partial \phi_m$, $\partial Z_S/\partial \phi_1$, ..., $\partial Z_S/\partial \phi_m$, 2^m sets all together. Set $m \leftarrow m+1$. $i \leftarrow i+1$. Initialize $\tilde{u}_1$ and $\tilde{u}_2$ and go to Step 4.

<u>Step 7</u> If $n = \ell_k$ go to Step 10.

<u>Step 8</u> $\tilde{v} = \tilde{A}^j \tilde{v}$. Set $j \leftarrow j-1$.

<u>Step 9</u> If $j = \ell_k$ go to Step 10. Go to Step 8.

<u>Step 10</u> Calculate Q, $\partial Q/\partial \phi_1$, ..., $\partial Q/\partial \phi_k$ 2^k times. Stop.

EXAMPLE

The filter shown in Fig. 6 [6] was considered.

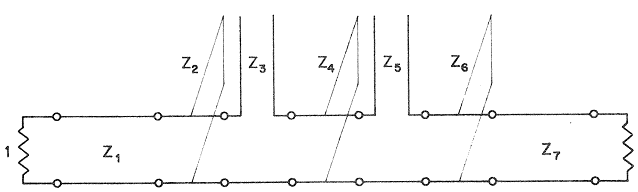

Fig. 6 Seven-section filter containing unit elements and stubs [6]. All sections are quarter-wave at 2.175 GHz.

The optimal minimax characteristic impedances [7] are the nominal values. They are

$$Z_1 = Z_7 = 0.606595,\ Z_2 = Z_6 = 0.303547,$$
$$Z_3 = Z_5 = 0.722287,\ Z_4 = 0.235183.$$

A tolerance of ±0.03 on Z_1, Z_4 and Z_5 was chosen. V_L, $\partial V_L/\partial Z_1$, $\partial V_L/\partial Z_4$ and $\partial V_L/\partial Z_5$ were evaluated at the eight vertices of the tolerance region. The results are in Table IV. They were checked individually by reanalysis.

DISCUSSION AND CONCLUSIONS

The calculation of the first- and second-order sensitivities of a circuit response involves one additional analysis of the adjoint network and k(k+1)/2 analyses to find second-order sensitivities calculated by finite differences. A more efficient approach is to calculate these second-order sensitivities using the adjoint network concept by performing only k analyses. Using our approach for cascaded structures, however, less than k analyses are performed and no additional memory is required.

The seven-section filter example was run with tolerances on the characteristic impedances of the stubs and transmission lines (all seven). It took 0.269 s CPU time to evaluate the response (only) at the 128 vertices. Using the conventional method of reanalysis would take 0.074 x 128 = 9.472 s CPU, where one analysis is performed in approximately 0.074 s. It took 0.118 s CPU time compared with 8 x 0.074 = 0.592 s for 8 analyses to evaluate the response and its sensitivities at vertices.

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CENTERING, TOLERANCING, TUNING AND MINIMAX DESIGN EMPLOYING BIQUADRATIC MODELS

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ABSTRACT

This paper exploits the biquadratic behaviour w.r.t. a variable exhibited in the frequency domain by certain lumped, linear circuits. Boundary points of the constraint region of acceptable designs are explicitly calculated w.r.t. any such variable at any sample point in the frequency domain. An algorithm to exactly determine the constraint region itself for the general nonconvex case has been developed. A minimax algorithm has also been developed and tested to optimize the frequency response w.r.t. any circuit parameter.

INTRODUCTION

A number of researchers have considered properties of response or constraint functions w.r.t. one designable variable at a time in the contexts of sensitivity evaluation of linear circuits [1-3] and the prediction of worst cases in design centering and tolerance assignment [4-7].

We exploit the resulting biquadratic function obtained from the modulus squared of the bilinear function to produce new results. In particular, at any frequency point we can explicitly calculate boundary points of the constraint region of acceptable designs to exactly determine the constraint region itself for the general nonconvex case. This leads to explicit determination of circuit tunability and design centering and tolerance assignment w.r.t. each parameter at a time is facilitated.

We present ideas for predicting worst cases. A globally convergent and extremely efficient minimax algorithm is derived and stated. Examples employing a realistic tunable active filter demonstrate the optimization of the frequency response w.r.t. a circuit parameter.

THEORY

For certain lumped, linear circuits, we can express the response as a bilinear function in a variable parameter ϕ (see, for example, Fidler [1])

$$f(\phi) = (u + a\,\phi)/(1 + b\,\phi), \qquad (1)$$

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where f is the circuit response at a particular frequency s, while u, a and b are complex constants in general. The variable ϕ does not necessarily have the value of the parameter, but it may take the value of the parameter p referred to a reference value p^0. Hence, $\phi = p - p^0$. Note that b is never zero for practical problems. Three analyses to obtain the complex constants in (1) can be efficiently carried out [8].

Since $|f|$ or functions of this magnitude are often of interest, we may write

$$|f(\phi)|^2 = \frac{|u|^2+2\,R(u^*a)\phi+|a|^2\phi^2}{1+2\,R(b)\phi+|b|^2\phi^2}, \qquad (2)$$

where u^* is the complex conjugate of u and $R(\cdot)$ denotes the real part of $(\cdot)$.

For simplicity, we write (2) as

$$F = (A+2B\phi+C\phi^2)/(1+2D\phi+E\phi^2). \qquad (3)$$

Hence,

$$\lim_{\phi\to\pm\infty} F = \frac{C}{E}, \quad E \neq 0. \qquad (4)$$

To find values of ϕ at which F = S, a specification, we replace F by S in (3). Then

$$(SE-C)\phi^2 + 2(SD-B)\phi + S - A = 0. \qquad (5)$$

When $S \neq C/E$, (5) has two finite roots

$$r_{1,2} = -\beta \pm \sqrt{\beta^2-(S-A)/(SE-C)}, \qquad (6)$$

where

$$\beta = (SD-B)/(SE-C). \qquad (7)$$

Consider real roots $r_1 \leq r_2$. F satisfies

$$F \gtrless S \quad \text{for all } \phi \in [r_1,r_2] \quad \text{if } S \gtrless C/E. \qquad (8)$$

If S = C/E, $E \neq 0$, a single root is obtained as

$$r = -0.5(C-AE)/(CD-BE). \qquad (9)$$

We can also derive

$$F \gtrless S \quad \text{for all } \phi \in [r, \infty] \quad \text{if } BE \gtrless CD, \qquad (10)$$

$$F \gtrless S \quad \text{for all } \phi \in [-\infty, r] \quad \text{if } BE \lessgtr CD. \qquad (11)$$

For imaginary roots

$$F \lessgtr S \text{ for all } \phi \in (-\infty,\infty) \text{ if } S \gtrless C/E. \qquad (12)$$

VALID PARAMETER INTERVALS

Consider the set of specifications

$$e_i \overset{\Delta}{=} w_i(F_i - S_i) \leq 0 \;,\; i = 1, 2, \ldots, m, \qquad (13)$$

where $w_i = -1(1)$ for lower(upper) specification S_i and m may be the number of frequency points.

It is possible to define a unique <u>continuous</u> interval I_i so that if the specification is satisfied on I_i then it is violated for all $\phi \notin I_i$ and vice versa. The logical variable t_i is defined by

$$t_i = \text{True} \quad \text{if} \quad I_i \equiv \{\phi | e_i \leq 0\}, \qquad (14)$$

or

$$t_i = \text{False} \quad \text{if} \quad I_i \equiv \{\phi | e_i > 0\}. \qquad (15)$$

A check to investigate meeting the m specifications of (13) simultaneously by adjusting ϕ only can be carried out by finding the feasible region R_S of ϕ given by

$$R_S = \bigcap_{t_i=\text{True}} I_i - \bigcup_{t_i=\text{False}} I_i. \qquad (16)$$

R_S is not necessarily a continuous interval. In general,

$$R_S = \bigcup_{\ell=1}^{k} [\check{\phi}_\ell, \hat{\phi}_\ell], \qquad (17)$$

where k is the number of the closed intervals. A flow diagram is has been developed [8] which provides k and the intervals $[\check{\phi}_\ell, \hat{\phi}_\ell]$, $\ell = 1, 2, \ldots, k$, as well as the indices of the functions F_i which actually define the extreme points of each interval. These indices are denoted $\check{i}_\ell$ and $\hat{i}_\ell$ for the lower and upper extremes, respectively.

Having obtained R_S we center ϕ at

$$\phi^0 = (\hat{\phi}_j + \check{\phi}_j)/2,$$

where

$$(\hat{\phi}_j - \check{\phi}_j) \geq (\hat{\phi}_\ell - \check{\phi}_\ell), \; \ell = 1, 2, \ldots, k.$$

The corresponding tolerance will be

$$\varepsilon = (\hat{\phi}_j - \check{\phi}_j)/2.$$

For several parameters this process may be successively carried out for each parameter independently [9]).

An outcome will be tunable if

$$[\check{\phi}_t, \hat{\phi}_t] \bigcap R_S \neq \emptyset, \qquad (18)$$

where $[\check{\phi}_t, \hat{\phi}_t]$ is the tuning range of ϕ.

EXTREMES OF A BIQUADRATIC FUNCTION

The stationary points of F, see (3), are given by

$$\frac{dF}{d\phi} = 2\,\frac{(B-AD)+(C-AE)\phi+(CD-BE)\phi^2}{(1+2D\phi+E\phi^2)^2} = 0. \qquad (19)$$

For finite stationary points, we solve

$$(CD-BE)\phi^2 + (C-AE)\phi + (B-AD) = 0. \qquad (20)$$

In general, there are two stationary points [5], but if CD - BE = 0, there is only one stationary point given by $\phi = -(B-AD)/(C-AE)$.

For a stationary point we can show that

$$\frac{d^2F}{d\phi^2} = 2\,\frac{C-EF}{1+2D\phi+E\phi^2}. \qquad (21)$$

If it is an inflection point, i.e., if $d^2F/d\phi^2 = 0$, then (21) leads to

$$F = C/E. \qquad (22)$$

The finite point at which F = C/E is obtained by replacing F by C/E in (3) to get

$$\phi = -0.5(C-AE)/(CD-BE). \qquad (23)$$

A stationary point satisfies

$$F = (B+C\phi)/(D+E\phi). \qquad (24)$$

Hence, for a finite stationary point to be an inflection point (22) and (24) have to be satisfied simultaneously for a finite value of ϕ. This is true if

$$BE = CD, \qquad (25)$$

which indicates that ϕ is infinite unless

$$C - AE = 0. \qquad (26)$$

Substituting for C from (26) into (25) for $E \neq 0$

$$B = AD. \qquad (27)$$

But, (25) to (27) make $dF/d\phi = 0$ everywhere. This special case of a constant function F=A is of no interest.

To summarize, the stationary points of a biquadratic function which has no real poles are extreme points.

IMPLICATIONS OF A POLE

A pole of $F = |f|^2$ of order two w.r.t. ϕ at $\phi = -1/b$ is possible only if b is real, otherwise the zeros of the denominator of (2) are complex. Similarly, the numerator of (2) indicates that a real zero of order two w.r.t. ϕ exists if (u^*a) is real at $\phi = -(u^*a)/|u|^2|a|^2$.

Note that

$$\frac{dF}{d\phi} = 2\,\frac{\left[\begin{array}{c}(b\phi+1)^2(\,R(u^*a)+|a|^2\phi) \\ -b(b\phi+1)(|u|^2+2\,R(u^*a)\phi+|a|^2\phi^2)\end{array}\right]}{(b\phi+1)^4}. \qquad (28)$$

Thus, one of the zeros of the numerator will be $\phi = -1/b$, which is a point of infinite gradient and the stationary point is

$$\phi = \frac{b|u|^2 - R(u^*a)}{|a|^2 - b\,R(u^*a)} = \frac{AD-B}{C-DB}. \qquad (29)$$

If $C-DB \neq 0$, this point is a minimum since

$$\frac{d^2F}{d\phi^2} = \frac{2}{(1+b\phi)^4}\,|ub-a|^2 > 0. \qquad (30)$$

THE ONE-DIMENSIONAL MINIMAX ALGORITHM

A minimax algorithm guaranteed to converge [10] to the global optimum (Fig. 1) follows.

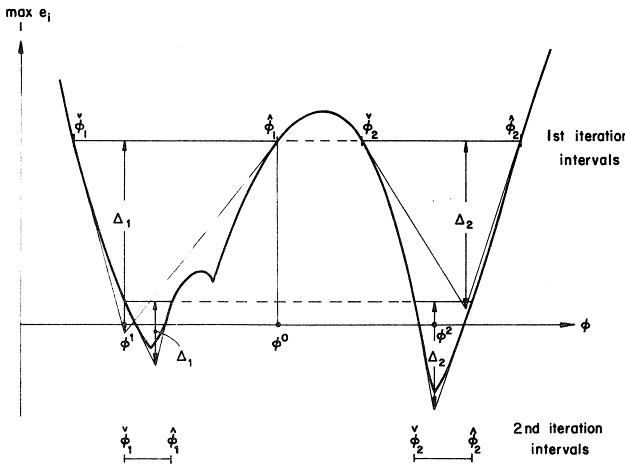

Fig. 1 Illustration of the behaviour of the one-dimensional minimax algorithm. Note that the algorithm switches from interval 1 to interval 2, based on predictions of the decrease in the maximum.

<u>Step 1</u> Find u_i, a_i and b_i, i = 1, 2, ..., m.

<u>Step 2</u> Initialize ϕ.

<u>Step 3</u> Find $\delta = \max_i e_i(\phi)$.

<u>Step 4</u> Find $[\check{\phi}_\ell, \hat{\phi}_\ell]$ and $\check{i}_\ell$, $\hat{i}_\ell$, ℓ = 1, 2, ..., k, using the specifications $e_i \leq \delta$, i = 1, 2, ..., m.

<u>Comment</u> This is carried out using the flow diagram developed [8]. If all functions are convex, k will always be one.

<u>Step 5</u> Find $\check{g}_\ell$ and $\hat{g}_\ell$, ℓ = 1, 2, ..., k, given by

$$\check{g}_\ell = de_{\check{i}_\ell}/d\phi(\check{\phi}_\ell),$$

$$\hat{g}_\ell = de_{\hat{i}_\ell}/d\phi(\hat{\phi}_\ell).$$

<u>Step 6</u> If k = 1, set j ← 1 and go to Step 8.

<u>Step 7</u> Find j such that

$$\Delta_j \geq \Delta_\ell, \quad \ell = 1, 2, \ldots, k,$$

where

$$\Delta_\ell = \hat{g}_\ell \check{g}_\ell (\hat{\phi}_\ell - \check{\phi}_\ell)/(\check{g}_\ell - \hat{g}_\ell).$$

<u>Comment</u> We select the jth interval which appears most promising in terms of expected improvement in the minimax optimum based on linearization. Δ_ℓ should be positive.

<u>Step 8</u> Set $\phi \leftarrow (\check{g}_j\check{\phi}_j - \hat{g}_j\hat{\phi}_j)/(\check{g}_j - \hat{g}_j)$ if $\check{i}_j \neq \hat{i}_j$.

<u>Comment</u> The new value ϕ is the intersection of the linear approximation to the two functions.

<u>Step 9</u> Set ϕ to the minimum of $e_{\hat{i}_j}$ if $\check{i}_j = \hat{i}_j$.

<u>Step 10</u> Set $\phi \leftarrow (\check{\phi}_j + \hat{\phi}_j)/2$ if $\phi \notin (\check{\phi}_j, \hat{\phi}_j)$.

<u>Comment</u> This default value obviates numerical problems arising, say, if $\hat{g}_j = \check{g}_j = 0$.

<u>Step 11</u> Stop if k = 1 and if $(\hat{\phi}_1 - \check{\phi}_1)$ is sufficiently small.

<u>Step 12</u> Go to Step 3.

EXAMPLE

A tunable active filter [8,11] has been chosen to implement the theory and algorithms. The specifications on $F = |V_2/V_g|^2$ are

$F \leq 0.5$ for $f/f_0 \leq 1-10/f_0$,

$F \leq 1.21$ for $1-10/f_0 \leq f/f_0 \leq 1+10/f_0$,

$F \leq 0.5$ for $f/f_0 \geq 1+10/f_0$,

$F \geq 0.5$ for $1-8/f_0 \leq f/f_0 \leq 1+8/f_0$,

$F \geq 1$ for $f = f_0$ Hz,

where f_0 is the center frequency. We use the one pole roll-off model for the operational amplifiers, given by $A(s) = A_0\omega_a/(s+\omega_a)$, where s is the complex frequency, A_0 is the d.c. gain and ω_a the 3 dB radian bandwidth. Refer to [11] for exact details.

A biquadratic model in tuning resistor R_4 was obtained at each frequency, normalized as 1 and $1 \pm 10/f_0$ for the upper specifications, 1 and $1 \pm 8/f_0$ for the lower specifications. The range of R_4 for which the specifications are satisfied (see Fig. 2)

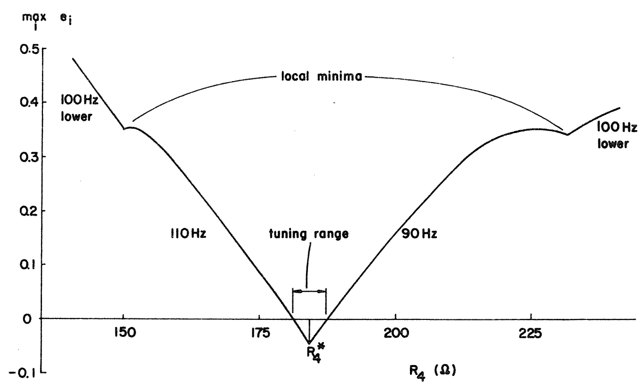

Fig. 2 Max e_i versus the tuning resistor R_4 for specifications defined around f_0 = 100 Hz indicating the active functions (and hence active frequency points).

is that for which $e_i \leq 0$, i = 1, 2, ..., 6. A single run of a computer program indicated that the filter is tunable for the specifications defined at a center frequency of 100 Hz. It meets these specifications if $R_4 \in [181.126, 187.166]$ and with other circuit parameters fixed at R_g = 50 Ω, C_1 = 0.728556 μF, R_1 = 12.446 kΩ, C_2 = 0.728556 μF, R_2 = 26.5 kΩ, $A_0 = 2 \times 10^5$, R_3 = 75 Ω, $\omega_a = 12\pi$ rad/s. It is also tunable around a center frequency of 700 Hz (see Fig. 3) and meets the specifications if $R_4 \in [3.4881, 3.5012]$.

Observe the local minima in Fig. 2. Convergence of other algorithms to the global minimum depends upon the starting point. For our algorithm the results are shown in Table I for

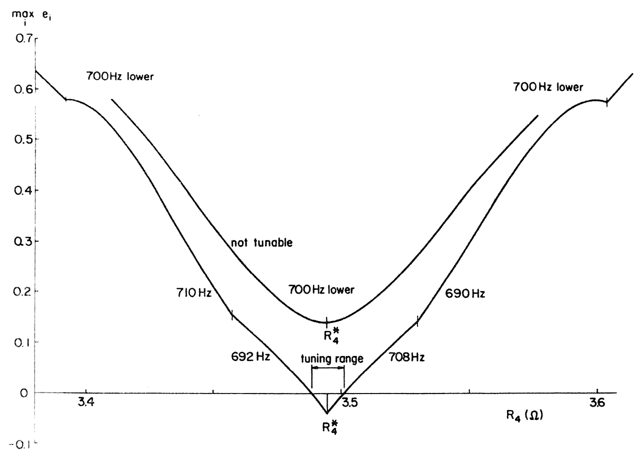

Fig. 3 Max e_i versus R_4 for specifications defined around f_0 = 700 Hz for two cases (a) R_1 = 12.446 kΩ, (b) R_1 = 14 kΩ.

different starting points and at different center frequencies. Note the small number of iterations required.

TABLE I
MINIMAX OPTIMUM OF TUNING RESISTOR R_4

Center Frequency (Hz)	R_4(Ω) Starting	Optimum	Optimum δ	N.O.I.*	CDC Time (s)
100	100.0	184.3998	-0.0458	3	0.14
	300.0	184.3998	-0.0458	3	0.14
	∞	184.3998	-0.0458	3	0.14
700	10.0	3.4946	-0.0403	3	0.14
	200.0	3.4946	-0.0403	3	0.14
	200.0**	3.4940	0.1434	2	0.14

* N.O.I. = number of iterations
** R_1 was altered to 14.0 kΩ and the filter is not tunable since δ>0.

CONCLUSIONS

The explicit determination of the points defining the boundary of the feasible region w.r.t. one parameter led to results on centering and tolerance assignment as well as a simple check on tunability. Detection of worst cases within an interval for any circuit parameter, of course, is also facilitated.

Our minimax algorithm is not only extremely efficient but is also globally convergent. It requires few iterations to reach the global minimax optimum from any starting point. There are no difficulties arising out of multiple local minima unlike a one-dimensional version of the minimax algorithm of Madsen et al. [12].

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