Proc. 1979 IEEE International Symposium on Circuits and Systems, Tokyo, Japan, July 1979

CENTERING, TOLERANCING, TUNING AND MINIMAX DESIGN EMPLOYING BIQUADRATIC MODELS

H.L. Abdel-Malek and J.W. Bandler

Group on Simulation, Optimization and Control, Faculty of Engineering
McMaster University, Hamilton, Canada L8S 4L7

ABSTRACT

This paper exploits the biquadratic behaviour w.r.t. a variable exhibited in the frequency domain by certain lumped, linear circuits. Boundary points of the constraint region of acceptable designs are explicitly calculated w.r.t. any such variable at any sample point in the frequency domain. An algorithm to exactly determine the constraint region itself for the general nonconvex case has been developed. A minimax algorithm has also been developed and tested to optimize the frequency response w.r.t. any circuit parameter.

INTRODUCTION

A number of researchers have considered properties of response or constraint functions w.r.t. one designable variable at a time in the contexts of sensitivity evaluation of linear circuits [1-3] and the prediction of worst cases in design centering and tolerance assignment [4-7].

We exploit the resulting biquadratic function obtained from the modulus squared of the bilinear function to produce new results. In particular, at any frequency point we can explicitly calculate boundary points of the constraint region of acceptable designs to exactly determine the constraint region itself for the general nonconvex case. This leads to explicit determination of circuit tunability and design centering and tolerance assignment w.r.t. each parameter at a time is facilitated.

We present ideas for predicting worst cases. A globally convergent and extremely efficient minimax algorithm is derived and stated. Examples employing a realistic tunable active filter demonstrate the optimization of the frequency response w.r.t. a circuit parameter.

THEORY

For certain lumped, linear circuits, we can express the response as a bilinear function in a variable parameter ϕ (see, for example, Fidler [1])

$$f(\phi) = (u + a\,\phi)/(1 + b\,\phi), \tag{1}$$

This work was supported by the Natural Sciences and Engineering Research Council of Canada under Grant A7239.

where f is the circuit response at a particular frequency s, while u, a and b are complex constants in general. The variable ϕ does not necessarily have the value of the parameter, but it may take the value of the parameter p referred to a reference value p^0. Hence, $\phi = p - p^0$. Note that b is never zero for practical problems. Three analyses to obtain the complex constants in (1) can be efficiently carried out [8].

Since $|f|$ or functions of this magnitude are often of interest, we may write

$$|f(\phi)|^2 = \frac{|u|^2+2\,R(u^*a)\phi+|a|^2\phi^2}{1+2\,R(b)\phi+|b|^2\phi^2}, \tag{2}$$

where u^* is the complex conjugate of u and $R(\cdot)$ denotes the real part of $(\cdot)$.

For simplicity, we write (2) as

$$F = (A+2B\phi+C\phi^2)/(1+2D\phi+E\phi^2). \tag{3}$$

Hence,

$$\lim_{\phi\to\pm\infty} F = \frac{C}{E}, \quad E \neq 0. \tag{4}$$

To find values of ϕ at which F = S, a specification, we replace F by S in (3). Then

$$(SE-C)\phi^2 + 2(SD-B)\phi + S - A = 0. \tag{5}$$

When $S \neq C/E$, (5) has two finite roots

$$r_{1,2} = -\beta \pm \sqrt{\beta^2-(S-A)/(SE-C)}, \tag{6}$$

where

$$\beta = (SD-B)/(SE-C). \tag{7}$$

Consider real roots $r_1 \leq r_2$. F satisfies

$$F \gtrless S \quad \text{for all } \phi \in [r_1,r_2] \quad \text{if } S \gtrless C/E. \tag{8}$$

If S = C/E, $E \neq 0$, a single root is obtained as

$$r = -0.5(C-AE)/(CD-BE). \tag{9}$$

We can also derive

$$F \gtrless S \quad \text{for all } \phi \in [r, \infty] \quad \text{if } BE \gtrless CD, \tag{10}$$

$$F \gtrless S \quad \text{for all } \phi \in [-\infty, r] \quad \text{if } BE \lessgtr CD. \tag{11}$$

For imaginary roots

$$F \lessgtr S \text{ for all } \phi \in (-\infty,\infty) \text{ if } S \gtrless C/E. \tag{12}$$

VALID PARAMETER INTERVALS

Consider the set of specifications

$$e_i \overset{\Delta}{=} w_i(F_i - S_i) \leq 0 \ , \ i = 1, 2, \ldots, m, \qquad (13)$$

where $w_i = -1(1)$ for lower(upper) specification S_i and m may be the number of frequency points.

It is possible to define a unique <u>continuous</u> interval I_i so that if the specification is satisfied on I_i then it is violated for all $\phi \notin I_i$ and vice versa. The logical variable t_i is defined by

$$t_i = \text{True} \quad \text{if} \quad I_i \equiv \{\phi | e_i \leq 0\}, \qquad (14)$$

or

$$t_i = \text{False} \quad \text{if} \quad I_i \equiv \{\phi | e_i > 0\}. \qquad (15)$$

A check to investigate meeting the m specifications of (13) simultaneously by adjusting ϕ only can be carried out by finding the feasible region R_S of ϕ given by

$$R_S = \bigcap_{t_i=\text{True}} I_i - \bigcup_{t_i=\text{False}} I_i. \qquad (16)$$

R_S is not necessarily a continuous interval. In general,

$$R_S = \bigcup_{\ell=1}^{k} [\check{\phi}_\ell, \hat{\phi}_\ell], \qquad (17)$$

where k is the number of the closed intervals. A flow diagram is has been developed [8] which provides k and the intervals $[\check{\phi}_\ell, \hat{\phi}_\ell]$, $\ell = 1, 2, \ldots, k$, as well as the indices of the functions F_i which actually define the extreme points of each interval. These indices are denoted $\check{i}_\ell$ and $\hat{i}_\ell$ for the lower and upper extremes, respectively.

Having obtained R_S we center ϕ at

$$\phi^0 = (\hat{\phi}_j + \check{\phi}_j)/2,$$

where

$$(\hat{\phi}_j - \check{\phi}_j) \geq (\hat{\phi}_\ell - \check{\phi}_\ell), \ \ell = 1, 2, \ldots, k.$$

The corresponding tolerance will be

$$\varepsilon = (\hat{\phi}_j - \check{\phi}_j)/2.$$

For several parameters this process may be successively carried out for each parameter independently [9]).

An outcome will be tunable if

$$[\check{\phi}_t, \hat{\phi}_t] \bigcap R_S \neq \emptyset, \qquad (18)$$

where $[\check{\phi}_t, \hat{\phi}_t]$ is the tuning range of ϕ.

EXTREMES OF A BIQUADRATIC FUNCTION

The stationary points of F, see (3), are given by

$$\frac{dF}{d\phi} = 2\,\frac{(B-AD)+(C-AE)\phi+(CD-BE)\phi^2}{(1+2D\phi+E\phi^2)^2} = 0. \qquad (19)$$

For finite stationary points, we solve

$$(CD-BE)\phi^2 + (C-AE)\phi + (B-AD) = 0. \qquad (20)$$

In general, there are two stationary points [5], but if $CD - BE = 0$, there is only one stationary point given by $\phi = -(B-AD)/(C-AE)$.

For a stationary point we can show that

$$\frac{d^2F}{d\phi^2} = 2\,\frac{C-EF}{1+2D\phi+E\phi^2}. \qquad (21)$$

If it is an inflection point, i.e., if $d^2F/d\phi^2 = 0$, then (21) leads to

$$F = C/E. \qquad (22)$$

The finite point at which $F = C/E$ is obtained by replacing F by C/E in (3) to get

$$\phi = -0.5(C-AE)/(CD-BE). \qquad (23)$$

A stationary point satisfies

$$F = (B+C\phi)/(D+E\phi). \qquad (24)$$

Hence, for a finite stationary point to be an inflection point (22) and (24) have to be satisfied simultaneously for a finite value of ϕ. This is true if

$$BE = CD, \qquad (25)$$

which indicates that ϕ is infinite unless

$$C - AE = 0. \qquad (26)$$

Substituting for C from (26) into (25) for $E \neq 0$

$$B = AD. \qquad (27)$$

But, (25) to (27) make $dF/d\phi = 0$ everywhere. This special case of a constant function F=A is of no interest.

To summarize, the stationary points of a biquadratic function which has no real poles are extreme points.

IMPLICATIONS OF A POLE

A pole of $F = |f|^2$ of order two w.r.t. ϕ at $\phi = -1/b$ is possible only if b is real, otherwise the zeros of the denominator of (2) are complex. Similarly, the numerator of (2) indicates that a real zero of order two w.r.t. ϕ exists if (u^*a) is real at $\phi = -(u^*a)/|u|^2|a|^2$.

Note that

$$\frac{dF}{d\phi} = 2\,\frac{\left[\begin{array}{c}(b\phi+1)^2(\,R(u^*a)+|a|^2\phi) \\ -b(b\phi+1)(|u|^2+2\,R(u^*a)\phi+|a|^2\phi^2)\end{array}\right]}{(b\phi+1)^4}. \qquad (28)$$

Thus, one of the zeros of the numerator will be $\phi = -1/b$, which is a point of infinite gradient and the stationary point is

$$\phi = \frac{b|u|^2 - R(u^*a)}{|a|^2 - b\,R(u^*a)} = \frac{AD-B}{C-DB}. \qquad (29)$$

If $C-DB \neq 0$, this point is a minimum since

$$\frac{d^2F}{d\phi^2} = \frac{2}{(1+b\phi)^4} |ub-a|^2 > 0. \qquad (30)$$

THE ONE-DIMENSIONAL MINIMAX ALGORITHM

A minimax algorithm guaranteed to converge [10] to the global optimum (Fig. 1) follows.

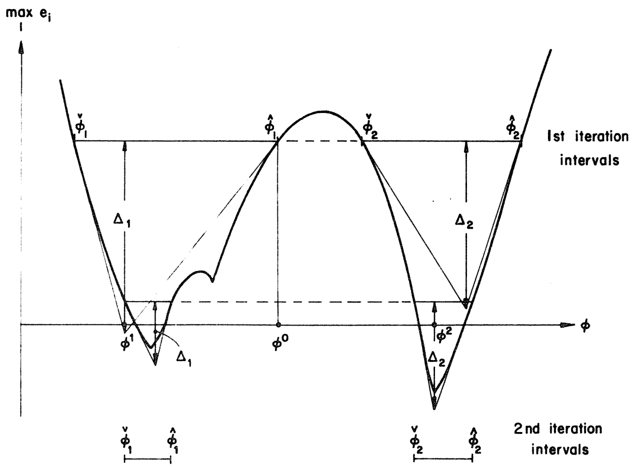

Fig. 1 Illustration of the behaviour of the one-dimensional minimax algorithm. Note that the algorithm switches from interval 1 to interval 2, based on predictions of the decrease in the maximum.

Step 1 Find u_i, a_i and b_i, $i = 1, 2, \ldots, m$.

Step 2 Initialize ϕ.

Step 3 Find $\delta = \max_i e_i(\phi)$.

Step 4 Find $[\check{\phi}_\ell, \hat{\phi}_\ell]$ and $\check{i}_\ell$, $\hat{i}_\ell$, $\ell = 1, 2, \ldots, k$, using the specifications $e_i \leq \delta$, $i = 1, 2, \ldots, m$.

Comment This is carried out using the flow diagram developed [8]. If all functions are convex, k will always be one.

Step 5 Find $\check{g}_\ell$ and $\hat{g}_\ell$, $\ell = 1, 2, \ldots, k$, given by

$$\check{g}_\ell = de_{\check{i}_\ell}/d\phi(\check{\phi}_\ell),$$

$$\hat{g}_\ell = de_{\hat{i}_\ell}/d\phi(\hat{\phi}_\ell).$$

Step 6 If $k = 1$, set $j \leftarrow 1$ and go to Step 8.

Step 7 Find j such that

$$\Delta_j \geq \Delta_\ell, \quad \ell = 1, 2, \ldots, k,$$

where

$$\Delta_\ell = \hat{g}_\ell \check{g}_\ell (\hat{\phi}_\ell - \check{\phi}_\ell)/(\check{g}_\ell - \hat{g}_\ell).$$

Comment We select the jth interval which appears most promising in terms of expected improvement in the minimax optimum based on linearization. Δ_ℓ should be positive.

Step 8 Set $\phi \leftarrow (\check{g}_j\check{\phi}_j - \hat{g}_j\hat{\phi}_j)/(\check{g}_j - \hat{g}_j)$ if $\check{i}_j \neq \hat{i}_j$.

Comment The new value ϕ is the intersection of the linear approximation to the two functions.

Step 9 Set ϕ to the minimum of $e_{\hat{i}_j}$ if $\check{i}_j = \hat{i}_j$.

Step 10 Set $\phi \leftarrow (\check{\phi}_j + \hat{\phi}_j)/2$ if $\phi \notin (\check{\phi}_j, \hat{\phi}_j)$.

Comment This default value obviates numerical problems arising, say, if $\hat{g}_j = 0$.

Step 11 Stop if $k = 1$ and if $(\hat{\phi}_1 - \check{\phi}_1)$ is sufficiently small.

Step 12 Go to Step 3.

EXAMPLE

A tunable active filter [8,11] has been chosen to implement the theory and algorithms. The specifications on $F = |V_2/V_g|^2$ are

$F \leq 0.5$ for $f/f_0 \leq 1-10/f_0$,

$F \leq 1.21$ for $1-10/f_0 \leq f/f_0 \leq 1+10/f_0$,

$F \leq 0.5$ for $f/f_0 \geq 1+10/f_0$,

$F \geq 0.5$ for $1-8/f_0 \leq f/f_0 \leq 1+8/f_0$,

$F \geq 1$ for $f = f_0$ Hz,

where f_0 is the center frequency. We use the one pole roll-off model for the operational amplifiers, given by $A(s) = A_0\omega_a/(s+\omega_a)$, where s is the complex frequency, A_0 is the d.c. gain and ω_a the 3 dB radian bandwidth. Refer to [11] for exact details.

A biquadratic model in tuning resistor R_4 was obtained at each frequency, normalized as 1 and $1 \pm 10/f_0$ for the upper specifications, 1 and $1 \pm 8/f_0$ for the lower specifications. The range of R_4 for which the specifications are satisfied (see Fig. 2)

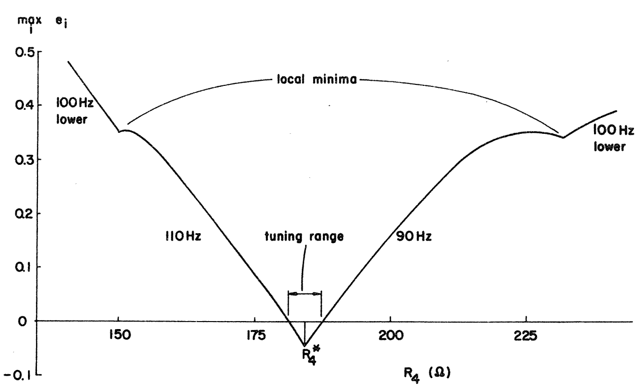

Fig. 2 Max e_i versus the tuning resistor R_4 for specifications defined around f_0 = 100 Hz indicating the active functions (and hence active frequency points).

is that for which $e_i \leq 0$, $i = 1, 2, \ldots, 6$. A single run of a computer program indicated that the filter is tunable for the specifications defined at a center frequency of 100 Hz. It meets these specifications if $R_4 \in [181.126, 187.166]$ and with other circuit parameters fixed at $R_g = 50\ \Omega$, $C_1 = 0.728556\ \mu F$, $R_1 = 12.446\ k\Omega$, $C_2 = 0.728556\ \mu F$, $R_2 = 26.5\ k\Omega$, $A_0 = 2 \times 10^5$, $R_3 = 75\ \Omega$, $\omega_a = 12\ \pi$ rad/s. It is also tunable around a center frequency of 700 Hz (see Fig. 3) and meets the specifications if $R_4 \in [3.4881, 3.5012]$.

Observe the local minima in Fig. 2. Convergence of other algorithms to the global minimum depends upon the starting point. For our algorithm the results are shown in Table I for

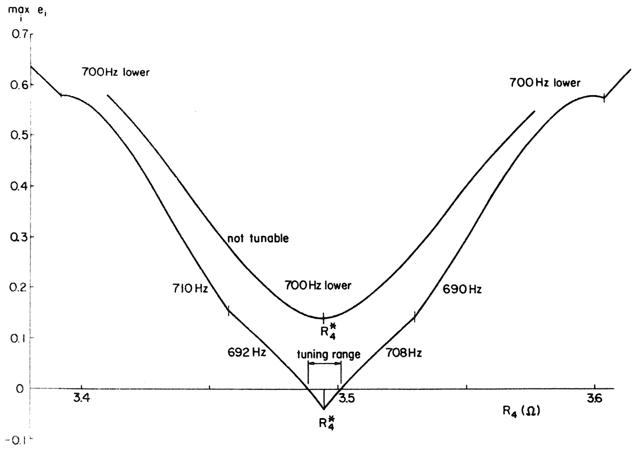

Fig. 3 Max e_i versus R_4 for specifications defined around f_0 = 700 Hz for two cases (a) R_1 = 12.446 kΩ, (b) R_1 = 14 kΩ.

different starting points and at different center frequencies. Note the small number of iterations required.

TABLE I
MINIMAX OPTIMUM OF TUNING RESISTOR R_4

Center Frequency (Hz)	R_4 (Ω) Starting	Optimum	Optimum δ	N.O.I.*	CDC Time (s)
100	100.0	184.3998	-0.0458	3	0.14
	300.0	184.3998	-0.0458	3	0.14
	∞	184.3998	-0.0458	3	0.14
700	10.0	3.4946	-0.0403	3	0.14
	200.0	3.4946	-0.0403	3	0.14
	200.0**	3.4940	0.1434	2	0.14

* N.O.I. = number of iterations
** R_1 was altered to 14.0 kΩ and the filter is not tunable since δ>0.

CONCLUSIONS

The explicit determination of the points defining the boundary of the feasible region w.r.t. one parameter led to results on centering and tolerance assignment as well as a simple check on tunability. Detection of worst cases within an interval for any circuit parameter, of course, is also facilitated.

Our minimax algorithm is not only extremely efficient but is also globally convergent. It requires few iterations to reach the global minimax optimum from any starting point. There are no difficulties arising out of multiple local minima unlike a one-dimensional version of the minimax algorithm of Madsen et al. [12].

ACKNOWLEDGEMENTS

The authors are very grateful to K. Madsen of the Technical University of Denmark and R.M. Biernacki, on leave from the Warsaw Technical University, Poland, for their comments.

REFERENCES

[1] J.K. Fidler, "Network sensitivity calculation," IEEE TRANS. CIRCUITS AND SYSTEMS, Vol. CAS-23, pp. 567-571: 1976.

[2] K.H. Leung and R. Spence, "Multiparameter large-change sensitivity analysis and systematic exploration," IEEE TRANS. CIRCUITS AND SYSTEMS, Vol. CAS-22, pp. 796-804: 1975.

[3] K. Geher, THEORY OF NETWORK TOLERANCES. Budapest, Hungary: Akademiai Kiado, 1971.

[4] J.W. Bandler, "Optimization of design tolerances using nonlinear programming," J. OPTIMIZATION THEORY AND APPLICATIONS, Vol. 14, pp. 99-114: 1974.

[5] J.W. Bandler and P.C. Liu, "Some implications of biquadratic functions in the tolerance problem," IEEE TRANS. CIRCUITS AND SYSTEMS, Vol. CAS-22, pp. 385-390: 1975.

[6] R.K. Brayton, A.J. Hoffman and T.R. Scott, "A theorem of inverses of convex sets of real matrices with application to the worst-case DC problem," IEEE TRANS. CIRCUITS AND SYSTEMS, Vol. CAS-24, pp. 409-415: 1977.

[7] H. Tromp, "The generalized tolerance problem and worst case search," PROC. CONF. ON COMPUTER AIDED DESIGN OF ELECTRONIC AND MICROWAVE CIRCUITS AND SYSTEMS (Hull, England, July 1977), pp. 72-77.

[8] H.L. Abdel-Malek and J.W. Bandler, "Centering, tolerancing, tuning and minimax design employing biquadratic models", Faculty of Engineering, McMaster University, Hamilton, Canada, Report SOC-211, 1978.

[9] E.M. Butler, "Realistic design using large-change sensitivities and performance contours," IEEE TRANS. CIRCUIT THEORY, Vol. CT-18, pp. 58-66: 1971.

[10] H.L. Abdel-Malek, J.W. Bandler and R.M. Biernacki, "Proof of global convergence and rate of convergence for a one-dimensional minimax algorithm", Faculty of Engineering, McMaster University, Hamilton, Canada, Report SOC-229, 1979.

[11] J.W. Bandler, H.L. Abdel-Malek, P. Dalsgaard, Z.S. El-Razaz and M.R.M. Rizk, "Optimization and design centering of active and nonlinear circuits including component tolerances and model uncertainties," PROC. INT. SYMP. LARGE ENGINEERING SYSTEMS (Waterloo, Canada, May 1978), pp. 127-132.

[12] K. Madsen, H. Schjaer-Jacobsen and J. Voldby, "Automated minimax design of networks," IEEE TRANS. CIRCUITS AND SYSTEMS, Vol. CAS-22, pp. 791-796: 1975.