sion we have the largest bandwidth when dealing with non unity aspect ratios and low dielectric constant. In contrast, to obtain good wave guiding ability and low radiation from discontinuities a higher dielectric constant is required. If only one of these conditions is critical we can adjust the parameters of the guide to suit that particular requirement and obtain the largest bandwidth. If all the conditions are equally important we are limited to a small range of dielectric constants, e.g., for the contraints discussed an image line with unit aspect ratio has the maximum bandwidth at $\epsilon_R = 2.5$

We have used for the upper limit the first higher order mode cutoff frequency. In practice it seems that it is possible to maintain a single mode operation slightly above this frequency since the first higher order mode has extensive external fields at its cutoff frequency and is unlikely to be propagated in any circuit until a frequency is sufficiently high enough to reduce the evanescent fields down to a manageable size.

The purpose of this paper has been to give some guidance about the many parameters involved in the bandwidth of image guide in order to help in the design of broad-band circuits.

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Tolerance Analysis of Cascaded Structures

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***Abstract*—This paper presents an analysis scheme to obtain the response of a cascaded network and its first-order sensitivities wrt design variables at the vertices of the tolerance region in an efficient and systematic way. This information is needed in worst-case search algorithms to identify the worst vertex or in a general tolerance assignment. A substantial saving in computational effort is achieved by using the new approach over the basic approach of reanalyzing the circuit at every vertex.**

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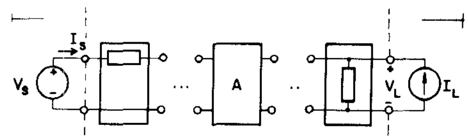

Fig. 1. Cascaded network with appropriate terminations.

I. Introduction

A recent approach for the analysis of cascaded networks (using the chain matrix) has been used efficiently to perform response evaluation as well as simultaneous and arbitrary large-change sensitivity evaluation [1]. This paper presents an analysis and sensitivity evaluation scheme using the recent approach in a form suitable for tolerance analysis of such networks.

In tolerance assignment each parameter ϕ has a tolerance associated with it so that it can have a value lying between $\phi^0+\epsilon$ and $\phi^0-\epsilon$, where ϕ^0 is the nominal value and ϵ is the tolerance. The number of vertices of the tolerance region is 2^k, where k is the number of the toleranced parameters, which includes all different combinations of parameter values.

The tolerances on elements as well as the nominal parameter values are optimized, consequently the response and its first-order sensitivity at the vertices of the tolerance region [2] are needed by the optimization algorithms. This information is particularly useful if a worst-case search algorithm [3] has to identify the worst vertex [4], [5].

A specific algorithm designed for evaluating the response and its sensitivities at the vertices of a tolerance region is presented. An example is given along with a comparison between the new approach and the conventional way (the reanalysis) for evaluating the response and its sensitivities at the vertices.

II. Necessary Background

The analysis approach is based on two types of analyses. The first is the forward analysis which consists of initializing a $\bar{\boldsymbol{u}}^T$ row vector as either $\boldsymbol{e}_1^T, \boldsymbol{e}_2^T$, where

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or a suitable linear combination and successively premultiplying each constant chain matrix by the resulting row vector until an element of interest or a termination is reached. The second is the reverse analysis, which is similar to conventional analysis of cascaded networks, proceeds by initializing a $\boldsymbol{v}$ column vector as either $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ or a suitable linear combination and successively postmultiplying each constant matrix by the resulting column vector, again until either an element of interest or a termination is reached.

Consider the network in Fig. 1. Applying forward and reverse analyses up to $\boldsymbol{A}$, we obtain the expression

$$Q \triangleq \bar{\boldsymbol{u}}^T \boldsymbol{A} \boldsymbol{v} \tag{1}$$

where

$\boldsymbol{A}$ transmission or chain matrix of the element of interest,

$\bar{\boldsymbol{u}}$ vector obtained from forward analysis (initiated at the source and ending at the input port of $\boldsymbol{A}$),

$\boldsymbol{v}$ vector obtained from reverse analysis (initiated at the load and ending at the output port of $\boldsymbol{A}$).

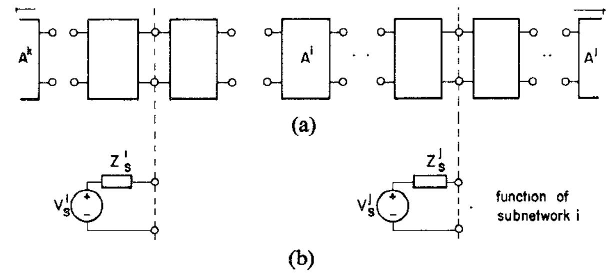

Fig. 2. (a) Subnetwork i cascaded with subnetworks k (at source end) and j (at load end). (b) Thevenin voltages and impedances for subnetworks k and i.

TABLE I
NOTATION AND IMPLIED INITIAL CONDITIONS

Initialization		Factor	Identification	Initial Conditions Forward	Reverse
$u \Rightarrow e_1$	$v \Rightarrow e_1$	$\bar{u}_1^T\,(*)\,v_1$	$(\dagger)_{11}$	voltage	voltage
$u \Rightarrow e_1$	$v \Rightarrow e_2$	$\bar{u}_1^T\,(*)\,v_2$	$(\dagger)_{12}$	voltage	current
$u \Rightarrow e_2$	$v \Rightarrow e_1$	$\bar{u}_2^T\,(*)\,v_1$	$(\dagger)_{21}$	current	voltage
$u \Rightarrow e_2$	$v \Rightarrow e_2$	$\bar{u}_2^T\,(*)\,v_2$	$(\dagger)_{22}$	current	current

(*) denotes either A, $\partial A/\partial\phi$ or ΔA

(†) denotes Q, Q' or ΔQ, as taken from (1), (2) or (3), respectively

With minimal additional effort we can obtain the partial derivative

$$Q' = \bar{u}^T \frac{\partial A}{\partial \phi} v \tag{2}$$

and the large change

$$\Delta Q = \bar{u}^T \Delta A v \tag{3}$$

where the parameter ϕ is contained in A. Note that Q relates output and input variables, namely, voltage and/or currents depending on how $\bar{u}^T$ and v are initialized (see Table I).

Usually more than one element in the cascade is considered. The network is divided into subnetworks by reference planes which are chosen so that no more than one element is to be explicitly considered between any pair of reference planes (see Fig. 2(a) where the superscripts denote the subnetworks). Forward and reverse analyses are initiated at these reference planes.

Equivalent Thevenin sources of the subnetworks can be determined by performing a sequence of forward analyses (equivalent Norton sources can be determined by reverse analyses) [1]. The Thevenin voltage for the ith subnetwork (see Fig. 2) is given by [1]

$$V_S^j = V_L^i = \frac{V_S^i}{(\bar{u}_1 + Z_S^i \bar{u}_2)^T A^i v_1} = \frac{V_S^i}{Q_{11}^i + Z_S^i Q_{21}^i} \tag{4}$$

where

- V_S^i source voltage for subnetwork i,
- Z_S^i source impedance of subnetwork i,
- V_L^i open circuit voltage at the load end of subnetwork i,
- V_S^j equivalent Thevenin source of subnetwork i, which is considered the source voltage for subnetwork j,
- $\bar{u}_1$ vector obtained from forward analysis when $\bar{u}$ is initialized as e_1,
- $\bar{u}_2$ vector obtained from forward analysis when $\bar{u}$ is initialized as e_2,
- v_1 vector obtained from reverse analysis when v is initialized as e_1,

and

$$Q_{11}^i \triangleq \bar{u}_1^T A^i v_1$$

$$Q_{21}^i \triangleq \bar{u}_2^T A^i v_1.$$

Note that we have omitted the superscript i from $\bar{u}$ and v (initialized at reference planes bounding subnetwork i) for simplicity. The output impedance of the subnetwork i is given by [1]

$$Z_S^j = \frac{(\bar{u}_1 + Z_S^i \bar{u}_2)^T A^i v_2}{(\bar{u}_1 + Z_S^i \bar{u}_2)^T A^i v_1} = \frac{Q_{12}^i + Z_S^i Q_{22}^i}{Q_{11}^i + Z_S^i Q_{21}} \tag{5}$$

which is considered as the source impedance to subnetwork j. The terms Q_{12}^i and Q_{22}^i are defined as

$$Q_{12}^i \triangleq \bar{u}_1^T A^i v_2$$

$$Q_{22}^i \triangleq \bar{u}_2^T A^i v_2.$$

A special case of (4) applicable to Fig. 1 ($Z_S = Y_L = I_L = 0$, where Y_L is the load admittance and I_L is the current source at the load end) is

$$V_L = \frac{V_S}{\bar{u}_1^T A v_1} = \frac{V_S}{Q_{11}}. \tag{6}$$

III. THE EVALUATION OF V_L AND ITS SENSITIVITIES WRT DESIGN PARAMETERS AT ALL VERTICES OF THE TOLERANCE REGION

Assume that we have partitioned the network by reference planes into subnetworks such that each subnetwork contains one chain matrix containing a variable parameter. Each reference plane is chosen to fall immediately after a variable element.

The Thevenin voltage/impedance of the ith subnetwork is considered as the source voltage/impedance of the $(i+1)$th subnetwork, given by (4) and (5), respectively, where $j = i+1$. We have to note here that the terms Q_{11}^i, Q_{21}^i, Q_{12}^i, and Q_{22}^i are as defined previously with v_1 and v_2 set to e_1 and e_2, respectively, since the appropriate reference plane immediately follows the element A^i. The number of pairs of terms V_S^{i+1} and Z_S^{i+1} to be evaluated is 2^i, since each subnetwork contains one variable element with two extreme values (assuming that each A^i contains only one variable parameter).

Differentiating (4) wrt ϕ_h, where ϕ_h does not belong to A^i, but V_S^i and Z_S^i are functions of ϕ_h (i.e., ϕ_h is in a subnetwork h before the ith subnetwork) we get

$$\frac{\partial V_S^{i+1}}{\partial \phi_h} = \frac{(Q_{11}^i + Z_S^i Q_{21}^i)\dfrac{\partial V_S^i}{\partial \phi_h} - V_S^i \dfrac{\partial Z_S^i}{\partial \phi_h} Q_{21}^i}{(Q_{11}^i + Z_S^i Q_{21}^i)^2} \tag{7}$$

and differentiating (5) wrt ϕ_h, we get

$$\frac{\partial Z_S^{i+1}}{\partial \phi_h} = \frac{\partial Z_S^i}{\partial \phi_h} \frac{(Q_{11}^i Q_{22}^i - Q_{12}^i Q_{21}^i)}{(Q_{11}^i + Z_S^i Q_{21}^i)^2}. \tag{8}$$

On the other hand, the derivatives wrt ϕ_i which is contained in A^i (Z_S^i and V_S^i are not functions of ϕ_i), are

$$\frac{\partial V_S^{i+1}}{\partial \phi_i} = \frac{-V_S^i\left(\dfrac{\partial Q_{11}^i}{\partial \phi_i} + Z_S^i \dfrac{\partial Q_{21}^i}{\partial \phi_i}\right)}{(Q_{11}^i + Z_S^i Q_{21}^i)^2} \tag{9}$$

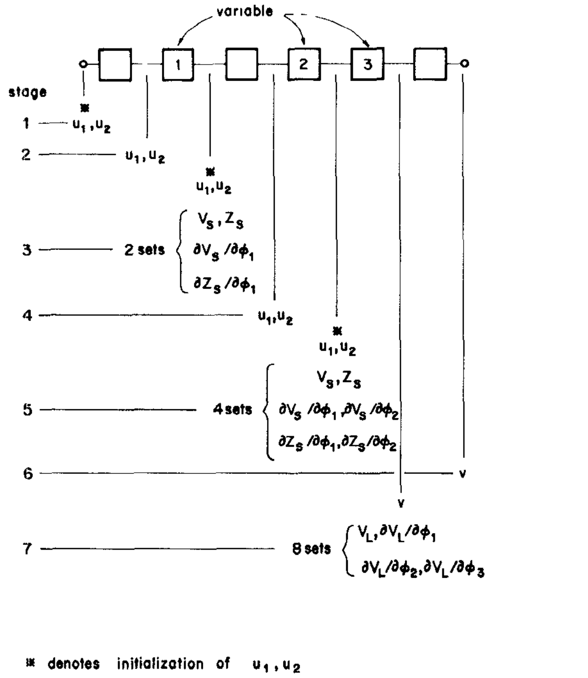

Fig. 3. Illustration of the principal stages of the algorithm. Here we are considering a cascade of six elements, where the three variables ϕ_1, ϕ_2, and ϕ_3 are in elements number 2, 4, and 5, respectively.

and

$$\frac{\partial Z_S^{i+1}}{\partial \phi_i} = \frac{(Q_{11}^i + Z_S^i Q_{21}^i)\left(\frac{\partial Q_{12}^i}{\partial \phi_i} + Z_S^i \frac{\partial Q_{22}^i}{\partial \phi_i}\right) - (Q_{12}^i + Z_S^i Q_{22}^i)\left(\frac{\partial Q_{11}^i}{\partial \phi_i} + Z_S^i \frac{\partial Q_{21}^i}{\partial \phi_i}\right)}{(Q_{11}^i + Z_S^i Q_{21}^i)^2} \tag{10}$$

where $\frac{\partial Q_{11}^i}{\partial \phi_i}$, $\frac{\partial Q_{21}^i}{\partial \phi_i}$, $\frac{\partial Q_{12}^i}{\partial \phi_i}$, and $\frac{\partial Q_{22}^i}{\partial \phi_i}$ correspond to (2) and Table I. This sensitivity information is carried out through the analysis for each subnetwork. The number of variables for which sensitivities of V_S^{i+1} and Z_S^{i+1} exist at the $(i+1)$th subnetwork is i so that $2^i \cdot i$ sensitivity calculations are performed. The expression relating V_L and the last sets of V_S and Z_S, is given by (4), so that 2^k values for V_L and its sensitivities can be obtained from appropriate values of V_S, Z_S, and A.

Fig. 3 shows an example of the stages involved in the following algorithm to obtain the response and its sensitivities at the vertices (3 variables$\Rightarrow$8 vertices) of the tolerance region.

Algorithm

Step 1: Initialize u_1, u_2 and v_1. (See Table I).
Set $i \leftarrow 1, m \leftarrow 1, j \leftarrow n$.

Comment: n is the total number of elements in the cascade, and m is a counter for the variable elements. In the example of Fig. 3 $n=6$.

Step 2: If $i = l_m$ go to Step 6.

Comment: l_m is an element of L, an index set containing superscripts of the k matrices containing the k variable parameters and ordered consecutively. It is assumed that each matrix contains only one variable. In the example of Fig. 3 $k=3$ and $L=\{2,4,5\}$.

Step 3: $\bar{u}_1^T \leftarrow \bar{u}_1^T A^i$.
$\bar{u}_2^T \leftarrow \bar{u}_2^T A^i$.
Set $i \leftarrow i+1$.

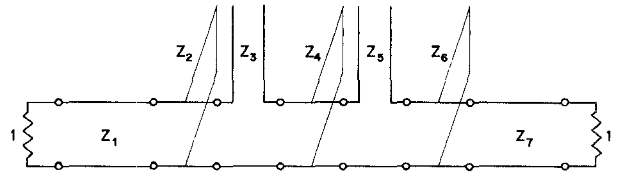

Fig. 4. Seven-section filter containing unit elements and stubs [6]. All sections are quarter-wave at 2.175 GHz.

Step 4: If $i = l_m$ go to Step 5.
Go to Step 3.

Comment: Step 2 and Step 4 check whether or not we have reached a chain matrix containing a variable parameter. If we reached a variable we calculate the Thevenin voltages and impedances and their sensitivities, otherwise we proceed with the forward analysis performed in Step 3.

Step 5: If $m = k$ go to Step 7.

Comment: This step checks if we have reached the last chain matrix containing a variable or not.

Step 6: Calculate V_S, Z_S
$\frac{\partial V_S}{\partial \phi_1}, \cdots, \frac{\partial V_S}{\partial \phi_m}$
$\frac{\partial Z_S}{\partial \phi_1}, \cdots, \frac{\partial Z_S}{\partial \phi_m}$
2^m sets all together.
Set $m \leftarrow m+1$.
$i \leftarrow i+1$.
Initialize u_1 and u_2 and go to Step 4.

Comment: This step is referred to at stage 3 and stage 5 in Fig. 3.

Step 7: If $n = l_k$ go to Step 10.

Comment: k is the number of variables. In the example of Fig. 3, $k=3$ and $l_k=5$.

Step 8: $v_1 = A^j v_1$.
Set $j \leftarrow j-1$.

Step 9: If $j = l_k$ go to Step 10.
Go to Step 8.

Step 10: Calculate $Q, \partial Q/\partial \phi_1, \cdots, \partial Q/\partial \phi_k$ 2^k times.
Stop.

Comment: This step is referred to at stage 7 in Fig. 3.

IV. Example

The cascaded seven-section bandpass filter shown in Fig. 4 [6] was considered. All sections are quarter-wave at 2.175 GHz. The optimal minimax characteristic impedances [7] are taken as nominal values. They are

$$Z_1 = Z_7 = 0.606595$$
$$Z_2 = Z_6 = 0.303547$$
$$Z_3 = Z_5 = 0.722287$$
$$Z_4 = 0.235183.$$

A tolerance of ± 0.03 on Z_1, Z_4, and Z_5 was chosen. The algorithm was used to evaluate V_L, $\partial V_L/\partial Z_1$, $\partial V_L/\partial Z_4$, and $\partial V_L/\partial Z_5$ at the eight vertices of the tolerance region (2^3 vertices where 3 is the number of toleranced variables). The results are

TABLE II
THE RESPONSE V_L AND ITS SENSITIVITIES AT THE VERTICES OF THE TOLERANCE REGION AT NORMALIZED FREQUENCY 0.7

Vertex	V_L	$\partial V_L/\partial Z_1$	$\partial V_L/\partial Z_4$	$\partial V_L/\partial Z_5$	Sign of Tolerance Extreme		
1	0.49135+j0.02351	-0.02450+j0.05953	0.26004-j1.15934	0.02549+j0.32944	-	-	-
2	0.48819+j0.02571	-0.07761+j0.01588	0.28346-j1.05326	0.00954+j0.34878	+	-	-
3	0.49679-j0.04862	0.03751+j0.15916	-0.06631-j0.94430	0.04534+j0.29165	-	+	-
4	0.49677-j0.04046	-0.03384+j0.11417	-0.00426-j0.87724	0.03578+j0.31848	+	+	-
5	0.49209+j0.04341	-0.04367+j0.08072	0.29407-j1.19530	-0.00103+j0.33324	-	-	+
6	0.48786+j0.04670	-0.09378+j0.03123	0.32067-j1.07952	-0.02042+j0.35007	+	-	+
7	0.49889-j0.03101	0.02608+j0.18868	-0.05742-j0.97346	0.02462+j0.29494	-	+	+
8	0.49818-j0.02127	-0.04526+j0.13735	0.01132-j0.90191	0.01113+j0.32057	+	+	+

tabulated in Table II. They were checked individually by reanalyzing the circuit at each vertex.

V. CONCLUSIONS

The algorithm for evaluating the response and its sensitivities at the vertices of the tolerance region proved to be very efficient. The seven-section filter example was run with tolerances on the characteristic impedances of the stubs and transmission lines (all seven). It took 0.269 s CPU time, on a CDC 6400 computer, to evaluate the response (only) at the $128(2^7)$ vertices. Using the conventional method of reanalyzing the circuit for different component values would take $0.074\times 128=9.472$ s CPU, where one analysis is performed in approximately 0.074 s. For the case of evaluating the response and its sensitivities at vertices discussed in Section IV, it took 0.118 s CPU time compared with $8\times 0.074=0.592$ s for eight analyses. The savings in computational effort is substantial.

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