CIRCUIT THEORY AND APPLICATIONS, VOL. 8, 445–455 (1980)

THE IMPACT OF GENERALIZED SYMMETRY ON COMPUTER-AIDED DESIGN OF CASCADED STRUCTURES*

J. W. BANDLER, R. M. BIERNACKI AND M. R. M. RIZK

Group on Simulation, Optimization and Control and Department of Electrical and Computer Engineering, McMaster University, Hamilton, Canada L8S 4L7

SUMMARY

New theoretical and computational tools are presented which deal with generalized symmetry concepts related to the computer-aided design of cascaded structures. The presentation includes the computational implications of networks consisting of symmetrically located reverse adjoint subnetworks with and without scaling, as well as antisymmetry. Formulas presented are designed to be used in simulation, sensitivity and tolerance analyses as well as in optimal design.

INTRODUCTION

Symmetry or antisymmetry is one of the most pervasive features in microwave circuit design.[1] To date, when using efficient optimization methods, there has hardly been any systematic exploration of the constraints imposed on response and sensitivity evaluation by such features, nor has there been any general attempt to exploit them in reducing computational effort.

Bandler *et al.*[2,3] have recently provided the theoretical tools for handling an important class of networks in optimal design, namely cascaded structures. Their framework, as will be shown, permits the special constraints imposed by symmetry (interpreted generally) to be embodied directly into an overall computational scheme. Advantages to be expected from this include reducing the sizes of design problems in terms of numbers of independent variables and constraints with the attendant reduction in computation cost.

This paper presents an important new definition of generalized symmetry best described by the term symmetrically located reverse adjoint subnetworks with scaling. This is interpreted and its use in network analysis is explained. Antisymmetrical structures are also treated in an analogous manner.

THEORETICAL BACKGROUND

Consider the two-port element shown in Figure 1. Taking $\mathbf{A}$ as the transmission or chain matrix of the element, $\mathbf{y}$ as the output vector and $\bar{\mathbf{y}}$ as the input vector containing in the first and second rows, respectively, the voltage and current,

$$\bar{\mathbf{y}} = \mathbf{A}\mathbf{y} \tag{1}$$

is termed the basic iteration.[2] Reverse analysis, the conventional form, is expressed by

$$\bar{\mathbf{v}} = \mathbf{A}\mathbf{v} \tag{2}$$

where

$$\mathbf{y} = c\mathbf{v} \Rightarrow \bar{\mathbf{y}} = c\bar{\mathbf{v}} \tag{3}$$

* This work was supported by the Natural Sciences and Engineering Research Council of Canada under Grant A7239. This paper was presented at the 1979 IEEE International Microwave Symposium, Orlando, Florida, 30 April–2 May 1979.

0098–9886/80/0408–0445 $01.00

Received 6 December 1979
Revised 26 May 1980

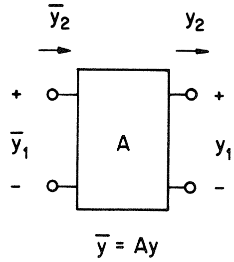

Figure 1. Two-port element showing reference directions for the voltage and current variables

Forward analysis is carried out by

$$\bar{\mathbf{u}}^{\mathrm{T}}\mathbf{A} = \mathbf{u}^{\mathrm{T}} \tag{4}$$

and involves carrying forward through a cascaded network the voltage–current pair. It is clear from the foregoing definitions, for example, that

$$\bar{\mathbf{u}}^{\mathrm{T}}\bar{\mathbf{y}} = \bar{\mathbf{u}}^{\mathrm{T}}\mathbf{A}\mathbf{y} = \mathbf{u}^{\mathrm{T}}\mathbf{y} \tag{5}$$

For a cascade of n two-ports we have

$$\bar{\mathbf{y}}^1 = \mathbf{y}^0 = \mathbf{A}^1\mathbf{A}^2 \ldots \mathbf{A}^i \ldots \mathbf{A}^n\mathbf{y}^n \tag{6}$$

so that, analysing the network in a forward direction to the input of $\mathbf{A}^i$ and in a reverse direction to the output of $\mathbf{A}^i$ we obtain[2]

$$d = \bar{\mathbf{u}}^{1\mathrm{T}}\bar{\mathbf{y}}^1 = c\,\bar{\mathbf{u}}^{i\mathrm{T}}\mathbf{A}^i\mathbf{v}^i \tag{7}$$

where we take

$$\mathbf{y}^n = c\,\mathbf{v}^n \tag{8}$$

and c and d are constants associated with the output and input, respectively. As Bandler *et al.*[2,3] have shown, these concepts lead to formulas for sensitivity evaluation, partial derivative evaluation, tolerance and tolerance sensitivity analyses, and large-change sensitivity evaluation, and hence to algorithms.

Voltage and current selectors

We describe

$$\mathbf{e}_1 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9}$$

as voltage and current selectors, respectively, because $\mathbf{e}_1^{\mathrm{T}}\mathbf{y}$ selects the voltage variable and $\mathbf{e}_2^{\mathrm{T}}\mathbf{y}$ selects the current variable in $\mathbf{y}$. Taking

$$\mathbf{v}_j^n = \mathbf{e}_j, \qquad j \in \{1, 2\} \tag{10}$$

we have

$$\mathbf{v}_j^i = \mathbf{A}^{i+1}\mathbf{A}^{i+2} \ldots \mathbf{A}^n\mathbf{e}_j \tag{11}$$

and, letting

$$\mathbf{u}_j^0 = \bar{\mathbf{u}}_j^1 = \mathbf{e}_j, \qquad j \in \{1, 2\} \tag{12}$$

we have

$$\mathbf{e}_j^{\mathrm{T}}\mathbf{A}^1\mathbf{A}^2 \ldots \mathbf{A}^{i-1} = \bar{\mathbf{u}}_j^{i\mathrm{T}} \tag{13}$$

In the ensuing discussion, superscripts i may be dropped when there is no resulting confusion and when our attention is directed to a particular element **A**.

Transformations

Some essential transformations which will be applied to the chain matrices are defined in this section. As will be seen, these are similarity transformations.

Consider first a rotation matrix given by

$$\mathbf{R} \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{14}$$

whose features are evident from

$$\mathbf{R} = \mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}} \tag{15}$$

The effect on a vector **a** can be expressed by

$$\mathbf{a}^r \triangleq \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{R}\mathrm{a} \tag{16}$$

Thus, **R** interchanges the rows of a vector.

We can also consider a similarity transformation which we will refer to as antitransposition. It is given by

$$\mathbf{A}^{\mathrm{R}} = \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \mathbf{R}\mathbf{A}^{\mathrm{T}}\mathbf{R} \tag{17}$$

Important properties of this transformation are

$$(\mathbf{A}^1\mathbf{A}^2)^{\mathrm{R}} = (\mathbf{A}^2)^{\mathrm{R}}(\mathbf{A}^1)^{\mathrm{R}} \tag{18}$$

and

$$(\mathbf{A}^{\mathrm{R}})^{\mathrm{R}} = \mathbf{A} \tag{19}$$

Secondly, consider a scaling matrix given by

$$\mathrm{S} \triangleq \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \tag{20}$$

whose inverse is obviously

$$\mathbf{S}^{-1} = \begin{bmatrix} 1/\alpha & 0 \\ 0 & 1 \end{bmatrix} \tag{21}$$

This transformation allows us to define a similar matrix

$$\mathbf{A}_\alpha \triangleq \begin{bmatrix} a_{11} & \alpha a_{12} \\ a_{21}/\alpha & a_{22} \end{bmatrix} = \mathbf{S}\mathbf{A}\mathbf{S}^{-1} \tag{22}$$

Important implications of this are

$$(\mathbf{A}^1\mathbf{A}^2)_\alpha = (\mathbf{A}^1)_\alpha(\mathbf{A}^2)_\alpha \tag{23}$$

$$(\mathbf{A}^{\mathrm{T}})_\alpha = (\mathbf{A}_{1/\alpha})^{\mathrm{T}} \tag{24}$$

$$(\mathbf{A}_\alpha)_{1/\alpha} = \mathbf{A} \tag{25}$$

and

$$(\mathbf{A}^{\mathrm{R}})_\alpha = (\mathbf{A}_\alpha)^{\mathrm{R}} = \mathbf{A}_\alpha^{\mathrm{R}} \tag{26}$$

GENERALIZED SYMMETRY

Consider a network with a pair of elements placed as shown in Figure 2. Consider, furthermore that the left subnetwork is being analysed in the forward direction and the right subnetwork is being analysed in the reverse direction. This means that we are simultaneously considering for $i, j \in \{1, 2\}$ the iteration pair

$$\text{forward:} \quad \bar{\mathbf{u}}_i^{\mathrm{T}}\mathbf{A} = \mathbf{u}_i^{\mathrm{T}} \tag{27}$$

$$\text{reverse:} \quad \bar{\mathbf{v}}_j = \mathbf{A}_\alpha^{\mathrm{R}}\mathbf{v}_j \tag{28}$$

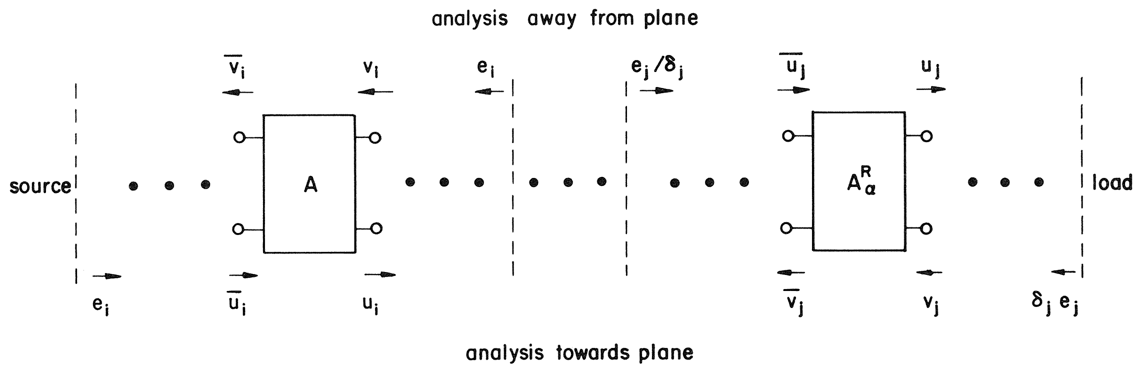

Figure 2. A cascaded network with a pair of elements under consideration for symmetry. The directions indicate those of analysis and the subscripts correspond to the type of initialization shown in (9) with $i \neq j$

Suppose that the following conditions exist between the output of the right element and the input of the left element:

$$\mathbf{v}_j = \mathbf{SR}\bar{\mathbf{u}}_i, \qquad i \neq j \tag{29}$$

Using (15) we can derive

$$\begin{aligned}\bar{\mathbf{v}}_j &= \mathbf{A}_\alpha^{\mathrm{R}}\mathbf{v}_j \\ &= (\mathbf{SR})\mathbf{A}^{\mathrm{T}}(\mathbf{SR})^{-1}\mathbf{v}_j \\ &= (\mathbf{SR})\mathbf{A}^{\mathrm{T}}(\mathbf{SR})^{-1}\mathbf{SR}\bar{\mathbf{u}}_i \\ &= \mathbf{SR}\mathbf{u}_i\end{aligned}$$

Summarizing, we have shown that

$$\mathbf{v}_j = \mathbf{SR}\bar{\mathbf{u}}_i \Leftrightarrow \bar{\mathbf{v}}_j = \mathbf{SR}\mathbf{u}_i, \qquad i \neq j \tag{30}$$

This means that the condition which was assumed at the beginning of the iteration is preserved for the next iteration.

Similarly, it is easy to show that the following holds for the left subnetwork analyzed in the reverse direction simultaneously with the right subnetwork analyzed in the forward direction:

$$\mathbf{RS}\bar{\mathbf{u}}_j = \mathbf{v}_i \Leftrightarrow \mathbf{RS}\mathbf{u}_j = \bar{\mathbf{v}}_i, \qquad i \neq j \tag{31}$$

The broad implications of the foregoing presentation are that if $\mathbf{A}$ and $\mathbf{A}_\alpha^{\mathrm{R}}$ represent elements in a network then a forward/reverse iteration for $\mathbf{A}$ simultaneously supplies the results of a reverse/forward iteration for $\mathbf{A}_\alpha^{\mathrm{R}}$ and vice versa. The **RS** transformation must, of course, naturally hold between the results of any previous iteration or, alternatively, it may be forced.

Initializations at the source and load of a network, for example, may be related by

$$[\delta_j \mathbf{e}_j] = \mathbf{SRe}_i, \qquad i \neq j \tag{32}$$

where $\mathbf{e}_j$ refers to the load end and $\mathbf{e}_i$ to the source end; for the opposite case

$$\mathbf{RS}\begin{bmatrix} \dfrac{\mathbf{e}_j}{\delta_j} \end{bmatrix} = \mathbf{e}_i, \qquad i \neq j \tag{33}$$

where

$$\delta_j = \begin{cases} \alpha & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} \tag{34}$$

As illustrated by Figure 2 the term in square brackets provides the initialization for the right subnetwork, the differences depending on whether the analysis is towards the plane of symmetry or away from it.

ANTISYMMETRY WITH SCALING

Following an approach similar to the previous one (see Figure 3) we replace (27) and (28) by

$$\text{forward:} \quad \bar{\mathbf{u}}_i^{\mathrm{T}} \mathbf{A} = \mathbf{u}_i^{\mathrm{T}} \tag{35}$$

$$\text{reverse:} \quad \bar{\mathbf{v}}_i = (\mathbf{A}^{\mathrm{T}})_\alpha \mathbf{v}_i \tag{36}$$

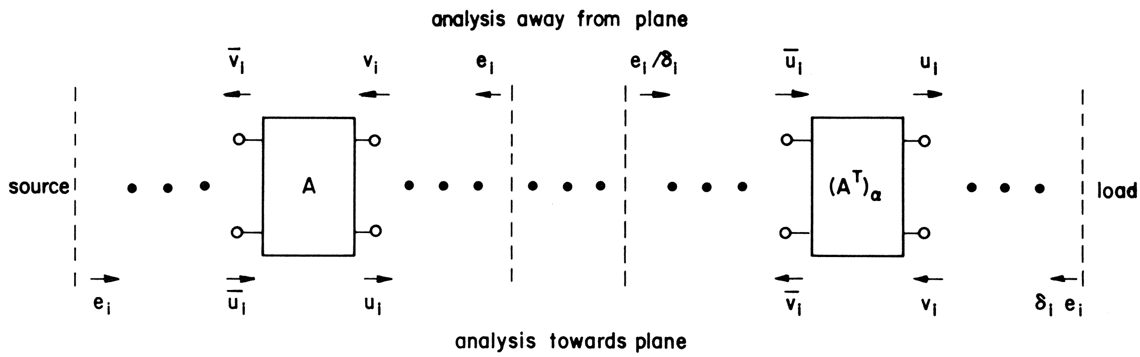

Figure 3. A cascaded network with a pair of elements under consideration for antisymmetry. The directions indicate those of analysis and the subscripts correspond to the type of initialization shown in (9) for $i \in \{1, 2\}$

and let

$$\mathbf{v}_i = \mathbf{S}\bar{\mathbf{u}}_i \tag{37}$$

then

$$\begin{aligned} \bar{\mathbf{v}}_i &= \mathbf{SA}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{v}_i \\ &= \mathbf{SA}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{S}\bar{\mathbf{u}}_i \\ &= \mathbf{Su}_i \end{aligned} \tag{38}$$

Proceeding in the opposite direction of analysis we have

$$\text{reverse:} \quad \bar{\mathbf{v}}_i = \mathbf{A}\mathbf{v}_i \tag{39}$$

$$\text{forward:} \quad \bar{\mathbf{u}}_i^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})_\alpha = \mathbf{u}_i^{\mathrm{T}} \tag{40}$$

and we let

$$\mathbf{S}\bar{\mathbf{u}}_i = \mathbf{v}_i \tag{41}$$

from which it follows that

$$\begin{aligned}\mathbf{S}\mathbf{u}_i &= \mathbf{S}\mathbf{S}^{-1}\mathbf{A}\mathbf{S}\bar{\mathbf{u}}_i \\ &= \mathbf{A}\mathbf{v}_i \\ &= \bar{\mathbf{v}}_i\end{aligned} \tag{42}$$

In summary, for antisymmetry we have

$$\mathbf{v}_i = \mathbf{S}\bar{\mathbf{u}}_i \Leftrightarrow \bar{\mathbf{v}}_i = \mathbf{S}\mathbf{u}_i \tag{43}$$

for analyses towards the plane of antisymmetry and

$$\mathbf{S}\bar{\mathbf{u}}_i = \mathbf{v}_i \Leftrightarrow \mathbf{S}\mathbf{u}_i = \bar{\mathbf{v}}_i \tag{44}$$

for analyses away from the plane of antisymmetry.

Initializations corresponding to (32) and (33) are, respectively, given by

$$[\delta_i \mathbf{e}_i] = \mathbf{S}\mathbf{e}_i \tag{45}$$

and

$$\mathbf{S}\left[\frac{\mathbf{e}_i}{\delta_i}\right] = \mathbf{e}_i, \tag{46}$$

where the terms in square brackets are the appropriate initializations for the right subnetwork. See Figure 3 for illustrations.

EXAMPLES AND INTERPRETATION

Reverse adjoint subnetworks

Suppose we have a cascade of two-ports, the analysis of which is represented by (6). Let

$$\mathbf{A}^{n-j} = (\mathbf{A}^{j+1})^{\mathrm{R}}_{\alpha}, \qquad j = 0, 1, \ldots, n/2-1 \tag{47}$$

Then

$$\mathbf{A}^{n-j}\mathbf{A}^{n-j+1}\ldots\mathbf{A}^{n} = (\mathbf{A}^1\mathbf{A}^2\ldots\mathbf{A}^{j+1})^{\mathrm{R}}_{\alpha} \tag{48}$$

This situation may be described by the phrase symmetrically located reverse adjoint subnetworks with scaling. The following is an interpretation which may help in clarifying this designation.

A given element is described by

$$\mathbf{A} \triangleq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{49}$$

Its adjoint is given by[4]

$$\begin{aligned}\hat{\mathbf{A}} &\triangleq \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix} \\ &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\end{aligned} \tag{50}$$

Interchanging the ports of the adjoint, i.e. reversing the element, we obtain

$$\begin{aligned}\hat{\mathbf{A}}^{\text{rev}} &= \frac{1}{\hat{a}_{11}\hat{a}_{22}-\hat{a}_{12}\hat{a}_{21}}\begin{bmatrix}\hat{a}_{22} & \hat{a}_{12}\\ \hat{a}_{21} & \hat{a}_{11}\end{bmatrix}\\ &= \begin{bmatrix}a_{22} & a_{12}\\ a_{21} & a_{11}\end{bmatrix}\\ &= \mathbf{A}^{\text{R}}\end{aligned} \tag{51}$$

by definition (17). Finally, applying the scaling transformation (22) we obtain the final result.

Figure 4 shows an equivalent network representation of a pair of symmetrically located reverse adjoint subnetworks or elements with scaling. Notice that, unlike the adjoint element which requires interchanging

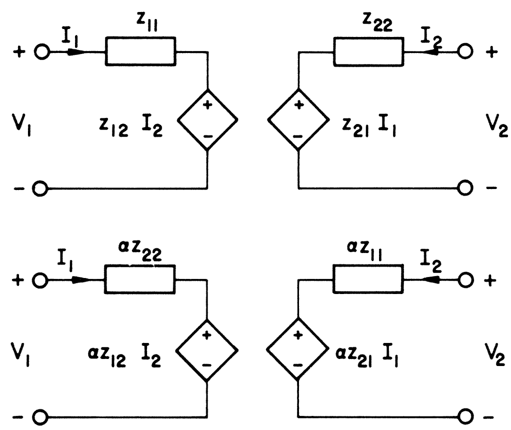

Figure 4. Equivalent network (impedance matrix) representation of a pair of reverse adjoint subnetworks with scaling

of controlling and controlled branches of non-reciprocal elements and unlike the reversed element which would, for example, reverse the direction of amplification of amplifiers, the symmetrical element preserves the directions of any non-reciprocal propagation caused by its companion.

Special cases

A common special case of interest to filter designers and microwave engineers is that of a network consisting of symmetrically located identical pairs of elements, each element of which is itself symmetrical. Such networks are characterized by

$$\text{identical pairs:}\quad \mathbf{A}^{n-j} = \mathbf{A}^{j+1}, \qquad j = 0, 1, \ldots, n-1 \tag{52}$$

$$\text{symmetrical elements:}\quad \mathbf{A}^{i} = (\mathbf{A}^{i})^{\text{R}}, \qquad i = 1, 2, \ldots, n \tag{53}$$

Notice that no assumption is implied about reciprocity.

Another interesting special case is one for reciprocal networks: a symmetrically decomposed symmetrical network implied by

$$\text{symmetrical pairs:}\quad \mathbf{A}^{n-j} = (\mathbf{A}^{j+1})^{\text{R}}, \qquad j = 0, 1, \ldots, n-1 \tag{54}$$

$$\text{symmetrical decomposition:}\quad (\mathbf{A}^{i})^{\text{R}} = (\mathbf{A}^{i})^{\text{rev}}, \qquad i = 1, 2, \ldots, n \tag{55}$$

where the designation rev implies a reversed element.

Antisymmetrical networks

Antisymmetrical networks are also of wide interest. For such networks

$$\mathbf{A}^{n-j} = ((\mathbf{A}^{j+1})^{\mathrm{T}})_{\alpha}, \qquad j = 0\ 1, \ldots, n-1 \tag{56}$$

In this case

$$\mathbf{A}^{n-j}\mathbf{A}^{n-j+1} \ldots \mathbf{A}^{n} = ((\mathbf{A}^{1}\mathbf{A}^{2} \ldots \mathbf{A}^{j+1})^{\mathrm{T}})_{\alpha} \tag{57}$$

Ladder network and stepped impedance transmission-line networks are well-known examples as illustrated in Figure 5.

For a series impedance (Figure 5a)

$$\mathbf{A} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \Rightarrow (\mathbf{A}^{\mathrm{T}})_{\alpha} = \begin{bmatrix} 1 & 0 \\ Z/\alpha & 1 \end{bmatrix}$$

For a shunt admittance (Figure 5a)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \Rightarrow (\mathbf{A}^{\mathrm{T}})_{\alpha} = \begin{bmatrix} 1 & \alpha Y \\ 0 & 1 \end{bmatrix}$$

For a transmission line,

$$\mathbf{A} = \begin{bmatrix} \cos\theta & jZ\sin\theta \\ j\dfrac{\sin\theta}{Z} & \cos\theta \end{bmatrix} \Rightarrow (\mathbf{A}^{\mathrm{T}})_{\alpha} = \begin{bmatrix} \cos\theta & j\dfrac{\alpha\sin\theta}{Z} \\ j\dfrac{Z\sin\theta}{\alpha} & \cos\theta \end{bmatrix}$$

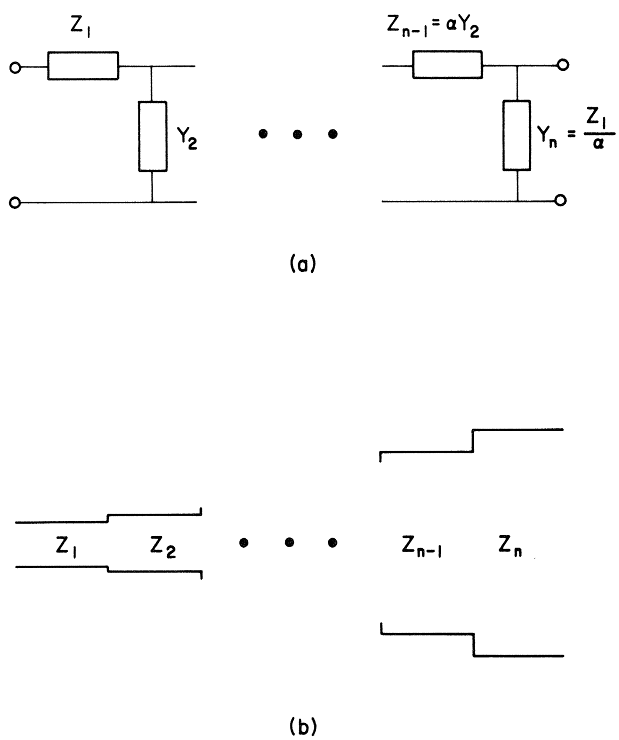

Figure 5. Examples of antisymmetry. (a) Ladder network, (b) stepped transmission-line network with $Z_{n-j}Z_{j+1} = \alpha$, $j = 0, 1, \ldots, n-1$

This implies that the product of the corresponding characteristic impedances of the transmission lines is α, as shown in Figure 5b.

NETWORK SIMULATION AND DESIGN EXPLOITING SYMMETRY

Bandler *et al.*[2] have shown that

$$Q_{kl} = \bar{u}_k^{\mathrm{T}} \mathbf{A} \mathbf{v}_l, \qquad k, l \in \{1, 2\} \tag{58}$$

embodies all the information required in the calculation of responses, first- and higher-order sensitivities as well as large-change sensitivities for that subnetwork (or network) containing $\mathbf{A}$, where the $\mathbf{u}_k$ and $\mathbf{v}_l$ variables are associated with forward and reverse analyses, respectively, in that subnetwork initiated by appropriate unit vectors.

The corresponding term for a subnetwork containing $\mathbf{A}_\alpha^{\mathrm{R}}$ is given by

$$P_{kl} = \delta_k[(\mathbf{RS})^{-1}\mathbf{v}_m]^{\mathrm{T}} \mathbf{A}_\alpha^{\mathrm{R}} [\mathbf{SR}\bar{\mathbf{u}}_n]/\delta_l$$
$$k, l, m, n \in \{1, 2\} \tag{59}$$
$$k \neq m \qquad l \neq n$$

Notice that the forward vector must be multiplied by δ_k and the reverse vector divided by δ_l so that the formula is valid for unit initializations on the symmetrical subnetwork (Figure 2). Notice also that (59) already incorporates the constraints (30) and (31). Upon substituting for $\mathbf{A}_\alpha^{\mathrm{R}}$ we have

$$P_{kl} = \delta_k \mathbf{v}_m^{\mathrm{T}} \mathbf{A}^{\mathrm{T}} \bar{\mathbf{u}}_n/\delta_l = \delta_k \bar{\mathbf{u}}_n^{\mathrm{T}} \mathbf{A} \mathbf{v}_m/\delta_l \tag{60}$$

From (58) and (59) we have

$$P_{11} = Q_{22} \tag{61}$$

$$P_{12} = \alpha Q_{12} \tag{62}$$

$$P_{21} = Q_{21}/\alpha \tag{63}$$

$$P_{22} = Q_{11} \tag{64}$$

Formulas (61) to (64) can be used directly in *any* recursive formulas already derived by Bandler *et al.*[2,3] in all their computations.

NETWORK SIMULATION AND DESIGN EXPLOITING ANTISYMMETRY

Following analogous considerations to the foregoing the term corresponding to (59) for a subnetwork containing $(\mathbf{A}^{\mathrm{T}})_\alpha$ is given by

$$P_{kl} = \delta_k[\mathbf{S}^{-1}\mathbf{v}_k]^{\mathrm{T}} (\mathbf{A}^{\mathrm{T}})_\alpha [\mathbf{S}\bar{\mathbf{u}}_l]/\delta_l \tag{65}$$

Substituting for $(\mathbf{A}^{\mathrm{T}})_\alpha$ we have

$$P_{kl} = \delta_k \mathbf{v}_k^{\mathrm{T}} \mathbf{A}^{\mathrm{T}} \bar{\mathbf{u}}_l/\delta_l = \delta_k \bar{\mathbf{u}}_l^{\mathrm{T}} \mathbf{A} \mathbf{v}_k/\delta_l \tag{66}$$

hence

$$P_{11} = Q_{11} \tag{67}$$

$$P_{12} = \alpha Q_{21} \tag{68}$$

$$P_{21} = Q_{12}/\alpha \tag{69}$$

$$P_{22} = Q_{22} \tag{70}$$

which can be utilized in all algorithms involving responses, sensitivities and tolerances (as before) to reduce effort.

NUMERICAL EXAMPLE

We consider as a numerical example the optimization over 100 per cent relative bandwidth of the response of a quarter-wave transformer having $n = 6$ and terminated resistively in 1 Ω at the source and 100 Ω at the load[1] (see table 1). Obviously, $\alpha = 100$. Optimization w.r.t. all variables: 6 lengths and 6 characteristic impedances was carried out in two ways. The first follows Bandler *et al.*[2] in which the 12 variables were treated independently. The second uses the results of the present paper, in which the appropriate constraint is imposed before the analysis and only half the network is analysed. Only 6 variables are optimized in this case. Least pth optimization by a new package called FLOPT5 was used.[5] The sequence of p was 2, 20, 1000, 50,000 and 1,000,000. Twenty-one sample points were used during optimization. At each least pth optimum quadratic interpolation at 101 points identified maxima in the reflection coefficient and appropriate sample points in the set of 21 were replaced. The length variable is normalized to the quarter wave length at centre frequency. The incredible consistency of the solutions (only one digit differs in the two approaches) was achieved by FLOPT5 on a CDC 6400 in 266 response evaluations (40 s CPU time) and 140 response evaluations (8 s CPU time).

Table 1. The results of optimization of the six section transformer

Section	Parameter	Value	
		Start	Optimum
1	Z	1·2	1·2960244
	l	0·8	1·0000000
2	Z	2·4	2·3894713
	l	1·1	1·0000000
3	Z	6·1	5·9778006
	l	1·5	1·0000000
4	Z	100/6·1	16·728561
	l	1·5	1·0000000
5	Z	100/2·4	41·850262(3)
	l	1·1	1·0000000
6	Z	100/1·2	77·159040
	l	0·8	1·0000000

CONCLUSIONS

This paper deals with the exploitation of symmetry and antisymmetry in cascaded networks. The principal goal is to save computational effort in computer-aided design and optimization of such structures. Theoretical descriptions of symmetry and antisymmetry were given for the general case of non-reciprocal as well as scaled networks. Our generalization permits us to consider active and scaled networks with similar computational savings as expected for classical symmetrical and antisymmetrical networks.

The properties of symmetry and antisymmetry were examined from the perspective of forward and reverse analysis in the spirit of the work of Bandler *et al.*[2,3] Hence, the implementation of the current results together with the previous algorithms is straightforward. Desired symmetry is, therefore, easily forced in any computer-aided optimization process.

REFERENCES

1. G. L. Matthaei, L. Young and E. M. T. Jones, *Microwave Filters Impedance-matching Networks, and Coupling Structures*, McGraw-Hill, New York, 1964.

2. J. W. Bandler, M. R. M. Rizk and H. L. Abdel-Malek, 'New results in network simulation, sensitivity and tolerance analysis for cascaded structures', *IEEE Trans. Microwave Theory Tech.*, **MTT-26**, 963–972 (1978).
3. J. W. Bandler and M. R. M. Rizk, 'Algorithms for tolerance and second-order sensitivities of cascaded structures', *Proc. IEEE Int. Symp. on Circuits and Systems*, Tokyo, Japan, July 1979, pp. 1056–1059.
4. J. W. Bandler and R. E. Seviora, 'Current trends in network optimization', *IEEE Trans. Microwave Theory Tech.*, **MTT-18**, 1159–1170 (1970).
5. J. W. Bandler and D. Sinha, 'FLOPT5—a program for minimax optimization using the accelerated least *p*th algorithm', Faculty of Engineering, McMaster University, Hamilton, Canada, *Report SOC-218*, 1978.