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LAGRANGIAN VS TELLEGEN APPROACHES TO NETWORK SENSITIVITY ANALYSIS - A UNIFIED, COMPREHENSIVE COMPARISON

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Invited Paper

Abstract

We present a comprehensive comparison between the widely used Lagrange multipliers and Tellegen's theorem approaches to sensitivity calculations in electrical networks. The two approaches are described on a unified basis, hence different aspects of comparison can be clearly investigated.

1. INTRODUCTION

Sensitivity calculations are performed routinely in electrical network analysis and design to supply first-order changes and gradients of functions of interest w.r.t. practically defined control or design variables.

Two approaches, namely the Lagrange multiplier approach [1,2] and Tellegen's theorem approach [3,4], are intensively used for sensitivity calculations in both electronic and power networks. Methods based on the two approaches have been described and applied [1-4] on an individual basis. A combination between the two approaches has been proposed in [5].

The material presented in this paper aims at investigating relationships between the two approaches. This investigation is accomplished by employing common bases of description and analysis through which the required aspects of comparison can be clearly stated.

We state the notation used and the basic formulation in Section 2. In Sections 3 and 4, we describe, on a unified basis, the application of the Lagrange multiplier and the Tellegen's theorem approaches to sensitivity analysis of electrical networks. A comprehensive discussion of some aspects of comparison is then presented in Section 5.

2. BASIC FORMULATION

We denote by f a single valued continuous real or complex function of n_x system state variables $\underset{\sim}{x}$ and n_u control variables $\underset{\sim}{u}$ which may be real or complex, $\underset{\sim}{x}$ and $\underset{\sim}{u}$ being column vectors. We also denote by $\underset{\sim}{h}$ a set of n_x real or complex equality constraints relating $\underset{\sim}{x}$ to $\underset{\sim}{u}$.

The first-order change of f is written as

$$\delta f = \underset{\sim}{f}_x^T \, \delta \underset{\sim}{x} + \underset{\sim}{f}_u^T \, \delta \underset{\sim}{u}, \tag{1}$$

where δ denotes first-order change, T denotes transposition and f_x and f_u denote $\partial f/\partial \underset{\sim}{x}$ and $\partial f/\partial \underset{\sim}{u}$, respectively. Also, the first-order change of $\underset{\sim}{h}$ is written as

$$\delta \underset{\sim}{h} = \underset{\sim}{H}_x \, \delta \underset{\sim}{x} + \underset{\sim}{H}_u \, \delta \underset{\sim}{u} = \underset{\sim}{0}, \tag{2}$$

where $\underset{\sim}{H}_x$ and $\underset{\sim}{H}_u$ stand for $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{x})^T$ and $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{u})^T$, respectively.

In the case of complex variables, $\underset{\sim}{x}$ and $\underset{\sim}{u}$ may contain complex conjugate pairs [4] and $\underset{\sim}{f}_x$, $\underset{\sim}{f}_u$, $\underset{\sim}{H}_x$ and $\underset{\sim}{H}_u$ of (1) and (2) may represent formal [5] partial derivatives w.r.t. the complex variables $\underset{\sim}{x}$

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and $\underset{\sim}{u}$.

When dealing with electrical networks, $\underset{\sim}{x}$ and $\underset{\sim}{u}$ may be classified [4] into 2-component subvectors $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$, respectively, associated with different element (branch) types, b denoting the bth branch. In general, $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ constitute node branch variables $\underset{\sim}{x}_m$ and $\underset{\sim}{u}_m$ and line branch variables $\underset{\sim}{x}_t$ and $\underset{\sim}{u}_t$. For example, $\underset{\sim}{x}_m$ may represent node voltages in a typical linear electronic network. In this case the components of $\underset{\sim}{x}_m$ are, e.g., V_{m1} and V_{m2} or [6] V_m and V_m^*, where V_{m1} and V_{m2} are, respectively, the real and imaginary parts of V_m and * denotes the complex conjugate.

In power networks $\underset{\sim}{x}_m$ and $\underset{\sim}{u}_m$ are further classified [4] into vectors associated with load $(\underset{\sim}{x}_\ell, \underset{\sim}{u}_\ell)$, generator $(\underset{\sim}{x}_g, \underset{\sim}{u}_g)$ and slack generator $(\underset{\sim}{x}_n, \underset{\sim}{u}_n)$ branches.

In general, we write

$$\underset{\sim}{x} = \{\underset{\sim}{x}_b\} = \{\underset{\sim}{x}_m, \underset{\sim}{x}_t\} \tag{3}$$

and

$$\underset{\sim}{u} = \{\underset{\sim}{u}_b\} = \{\underset{\sim}{u}_m, \underset{\sim}{u}_t\}. \tag{4}$$

In the above formulation, we have assumed that the number of state or control variables defined is $2n_B$, n_B denoting number of branches in the network. This assumption is made to simplify the comparison between Lagrange multiplier and Tellegen's theorem approaches performed in the following sections. Both of these approaches can be applied [2,5] for a general number of state variables.

3. LAGRANGE MULTIPLIER APPROACH

In this approach, we use (2) to write the first-order change δf of (1) in the form

$$\delta f = (\underset{\sim}{f}_u - \underset{\sim}{H}_u^T \underset{\sim}{\lambda})^T \delta \underset{\sim}{u}, \tag{5}$$

where $\underset{\sim}{\lambda}$ is a vector of the n_x Lagrange multipliers obtained by solving the adjoint equations

$$\underset{\sim}{H}_x^T \underset{\sim}{\lambda} = \underset{\sim}{f}_x. \tag{6}$$

Hence, from (5)

$$\frac{df}{d\underset{\sim}{u}} = \underset{\sim}{f}_u - \underset{\sim}{H}_u^T \underset{\sim}{\lambda}. \tag{7}$$

In practice, we solve the n_x adjoint equations (6) for the Lagrange multipliers $\underset{\sim}{\lambda}$ which are then substituted into (7) to obtain the required total derivatives of f w.r.t. control variables.

For use later, we now describe the approach in a slightly different way. We employ the classifications of (3) and (4) to define the change of an element-local Lagrangian term as

$$\delta L_b \triangleq (\underset{\sim}{\lambda}^T \underset{\sim}{H}_{bx})\, \delta \underset{\sim}{x}_b + (\underset{\sim}{\lambda}^T \underset{\sim}{H}_{bu})\, \delta \underset{\sim}{u}_b, \tag{8}$$

where

$$\underset{\sim}{H}_x = [\underset{\sim}{H}_{1x} \cdots \underset{\sim}{H}_{n_B x}] \tag{9}$$

and

$$\underset{\sim}{H}_u \triangleq [\underset{\sim}{H}_{1u} \cdots \underset{\sim}{H}_{n_B u}], \tag{10}$$

$\underset{\sim}{H}_{bx}$ and $\underset{\sim}{H}_{bu}$ being $2n_B x2$ submatrices.

We also define

$$\delta L \triangleq \sum_b \delta L_b, \tag{11}$$

hence, from (2) and (8)

$$\delta L = 0. \tag{12}$$

Using (8), (12) and

$$\delta f = \sum_b (\underset{\sim}{f}_{bx}^T\, \delta \underset{\sim}{x}_b + \underset{\sim}{f}_{bu}^T\, \delta \underset{\sim}{u}_b) \tag{13}$$

we may write, from (11)

$$\delta L = \delta f - \sum_b [(\underset{\sim}{f}_{bx}^T - \underset{\sim}{\lambda}^T \underset{\sim}{H}_{bx}) \delta \underset{\sim}{x}_b + (\underset{\sim}{f}_{bu}^T - \underset{\sim}{\lambda}^T \underset{\sim}{H}_{bu}) \delta \underset{\sim}{u}_b]. \tag{14}$$

Observe that when $\underset{\sim}{\lambda}$ of (14) satisfies (6), namely

$$\underset{\sim}{H}_{bx}^T \underset{\sim}{\lambda} = \underset{\sim}{f}_{bx}, \text{ for all b}, \tag{15}$$

then (14) reduces to

$$\delta L = \delta f - \sum_b (\underset{\sim}{f}_{bu} - \underset{\sim}{H}_{bu}^T \underset{\sim}{\lambda})^T \delta \underset{\sim}{u}_b, \tag{16}$$

hence, from (12)

$$\delta f = \sum_b (\underset{\sim}{f}_{bu} - \underset{\sim}{H}_{bu}^T \underset{\sim}{\lambda})^T \delta \underset{\sim}{u}_b \tag{17}$$

so that

$$\frac{df}{d\underset{\sim}{u}_b} = \underset{\sim}{f}_{bu} - \underset{\sim}{H}_{bu}^T \underset{\sim}{\lambda}, \tag{18}$$

which is a form of (7).

4. TELLEGEN'S THEOREM APPROACH

In this approach, the application of Tellegen's theorem [4] results in the identity

$$\delta T = 0, \qquad (19)$$

where

$$\delta T \triangleq \sum_b \delta T_b, \qquad (20)$$

the element-local Tellegen term δT_b is defined as

$$\delta T_b \triangleq \hat{\eta}_{bx}^T \, \delta x_b + \hat{\eta}_{bu}^T \, \delta u_b \qquad (21)$$

and the 2-component vectors $\hat{\eta}_{bx}$ and $\hat{\eta}_{bu}$ are linear functions of the formulated adjoint network current variables $\hat{I}_b$ and voltage variables $\hat{V}_b$ (and their complex conjugate). Hence, the $\hat{\eta}_{bx}$ and $\hat{\eta}_{bu}$ are related through Kirchhoff's current and voltage laws formulating 2n real network equations, n denoting the number of nodes (or buses) in the original network. Using (13) and (21), we may write, from (20)

$$\delta T = \delta f - \sum_b [(f_{bx}^T - \hat{\eta}_{bx}^T)\delta x_b + (f_{bu}^T - \hat{\eta}_{bu}^T)\delta u_b]. \qquad (22)$$

The adjoint network is defined by setting

$$\hat{\eta}_{bx} = f_{bx}, \qquad (23)$$

hence (22) reduces to

$$\delta T = \delta f - \sum_b (f_{bu} - \hat{\eta}_{bu})^T \, \delta u_b, \qquad (24)$$

from which

$$\frac{df}{du_b} = f_{bu} - \hat{\eta}_{bu}. \qquad (25)$$

In practice, we formulate the adjoint network using (23) and solve the 2n adjoint network equations to get $\hat{\eta}_{bu}$ which are then substituted into (21) to obtain the required total derivatives of f w.r.t. control variables.

5. ANALOGY AND COMPARISON

In the last two sections, we have described both the Lagrange multiplier and Tellegen's theorem approaches to sensitivity calculations in electrical networks. In this section, we investigate the analogous features of the two approaches and state a general comparison between them.

First, we remark on the resemblance between the element-local Lagrangian term δL_b of (8) and the element-local Tellegen term δT_b of (21). We also remark on the resemblance between equation (12) formed to satisfy (2), namely, the network equations and equation (19) formed by applying Tellegen's theorem. The δf of (14) and (22) is expressed solely in terms of the control variables via defining, respectively, the adjoint systems (15) and (23). The solution of the adjoint network is then used to obtain the total derivatives df/du_b from (18) and (25), respectively.

In the Lagrange multiplier approach, the adjoint system of equations to be solved for the adjoint variables (Lagrange multipliers) λ constitutes a $2n_B \times 2n_B$ matrix of coefficients. In general, when other state variables are defined [2], the order of the matrix of coefficients is determined by the total number of state variables defined. On the other hand, the adjoint system of equations in the Tellegen's theorem approach represents a set of network equations and constitutes only a 2n x 2n matrix of coefficients.

The compactness of the adjoint system formulation in the Tellegen's theorem approach is afforded in essence by realizing, when formulating the adjoint equations, Kirchhoff's relations between the different adjoint variables which constitute a fictitious electrical network.

Assuming that the effort required is divided into formulation and solution parts of the adjoint system, we immediately see that the Tellegen's theorem approach sweeps the major effort into the formulation part and results in only 2n adjoint equations to be solved. In contrast, the Lagrange multiplier approach requires almost nothing to formulate the adjoint system which then constitutes n_x adjoint equations to be solved.

Note that if we formulate the vectors I and V to contain all branch current and voltage variables, respectively, and consider [7] the perturbed relationships

$$\delta I = H_{ix} \, \delta x + H_{iu} \, \delta u, \qquad (26)$$

$$\delta V = H_{vx} \, \delta x + H_{vu} \, \delta u = A^T \, \delta V_M, \qquad (27)$$

$$A \, \delta I = 0 \qquad (28)$$

and (1), where A is a form of incidence matrix and

V_M contains node (bus) voltage variables, it is straightforward to show that a vector $\hat{\eta}_u$, which contains all the $\hat{\eta}_{bu}$ of (21), is given by

$$\hat{\eta}_u = H_{vu}^T \lambda_i + H_{iu}^T \lambda_v, \quad (29)$$

where λ_i and λ_v satisfy KCL and KVL, respectively, and the relationship

$$H_{vx}^T \lambda_i + H_{ix}^T \lambda_v = f_x. \quad (30)$$

6. CONCLUSIONS

The two widely used approaches to sensitivity calculations in electrical networks, namely the Lagrange multiplier and the Tellegen's theorem approaches have been described and compared. The description has been performed on a unified basis where we have defined and employed element-local terms in formulating the two approaches so that different aspects of comparison are clearly investigated. The resemblance in formulating the adjoint systems of the two approaches has been discussed.

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