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NEWTON'S LOAD FLOW IN COMPLEX MODE

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Abstract

This paper investigates direct solution techniques for an unfamiliar form of linear complex equations expressed in terms of a set of complex variables and their complex conjugate. This complex form may represent linearized power network equations. The well-known Newton-Raphson method is described and applied, with the aid of a novel elimination technique, in a compact complex form, to the load flow problem described in power system studies.

1. INTRODUCTION

The load flow problem [1] consists of determining power flows and voltages of a linear power network for specified terminal or bus conditions. Load flow calculations are performed in a wide variety of applications including power system planning, operational planning, optimal power flow analysis and sensitivities, outage security assessment and stability.

Unlike the analysis of typical linear electronic circuits, in which the equations describing the system are linear, the load flow analysis comprises a set of nonlinear equations. The a.c. electronic circuit analysis implies a solution of a set of complex linear equations to be solved exploiting the advantages realized by retaining the complex mode of the equations [2] to reduce the required computer memory by about 50%. The a.c. nonlinear load flow equations, on the other hand, are usually solved by iterative methods.

Recently, a variety of iterative numerical techniques for solving the load flow problem have been described [1]. While Gauss-Seidel and other nonderivative-based methods [1,3] have been described in the complex mode, the Newton-Raphson method [1,3,4] has been basically described [4] in the real mode. The Newton-Raphson method, however, can be interpreted formally in terms of first-order changes of problem variables. In this paper, we invoke this interpretation to describe the Newton-Raphson method in the more compact complex mode, and we utilize some theoretical derivations given in [5,6] to relate analytical aspects of the resulting form of equations to those of other familiar forms.

We set aside one section of the paper to describe, with the aid of a suitably developed notation, an elimination procedure which, in conjunction with the well-known forward Gaussian elimination technique, provides a suitable method for solving the resulting equations in complex mode, directly. Modifications required to preserve the complex mode in application to practical systems are also investigated.

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In this paper, we use the cartesian coordinate system in formulating the equations. It should be noticed that the Taylor series expansion of the load flow problem in cartesian coordinates involves terms up to the second order only, and the use of first-order variations of the complex variables is equivalent, as stated in our paper, to eliminating the last term of Taylor series. Our paper is concerned mainly with the fundamental formulation and resulting eliminating technique. All expected subsequent improvements regarding efficient sparsity programmed ordered elimination [7], however, can follow.

2. PROBLEM FORMULATION

The power network performance equations [3] are written, using the bus frame of reference, in the admittance form

$$\underset{\sim}{Y}_T \underset{\sim}{V}_M = \underset{\sim}{I}_M, \tag{1}$$

where

$$\underset{\sim}{Y}_T = \underset{\sim}{Y}_{T1} + j\,\underset{\sim}{Y}_{T2} \tag{2}$$

is the bus admittance matrix of the network,

$$\underset{\sim}{V}_M = \underset{\sim}{V}_{M1} + j\,\underset{\sim}{V}_{M2} \tag{3}$$

is a column vector of the bus voltages, and

$$\underset{\sim}{I}_M = \underset{\sim}{I}_{M1} + j\,\underset{\sim}{I}_{M2} \tag{4}$$

is a vector of bus currents. The bus loading equations are also written in the matrix form

$$\underset{\sim}{E}_M^* \underset{\sim}{I}_M = \underset{\sim}{S}_M^*, \tag{5}$$

where $\underset{\sim}{E}_M$ is a diagonal matrix of components of $\underset{\sim}{V}_M$ in corresponding order, i.e.,

$$\underset{\sim}{E}_M \underset{\sim}{v} = \underset{\sim}{V}_M, \tag{6}$$

$\underset{\sim}{v}$ is defined as

$$\underset{\sim}{v} \overset{\Delta}{=} \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \tag{7}$$

$\underset{\sim}{S}_M$ is a vector of the bus powers given by

$$S_M \overset{\Delta}{=} P_M + j\,Q_M \qquad (8)$$

and * denotes the complex conjugate. Substituting (1) into (5), we get

$$E_M^* \, Y_T \, V_M = S_M^*. \qquad (9)$$

The system of nonlinear equations (9) represents the typical load flow problem.

We write (9) in the perturbed form

$$K^S \, \delta V_M + \bar{K}^S \, \delta V_M^* = \delta S_M^*, \qquad (10)$$

where δV_M, δV_M^* and δS_M^* represent first-order changes of V_M, V_M^* and S_M^*, respectively,

$$K^S = E_M^* \, Y_T \qquad (11)$$

and $\bar{K}^S$ is a diagonal matrix of components of I_M, i.e.,

$$\bar{K}^S \, v = I_M. \qquad (12)$$

The form (10) rigorously represents a set of linear equations to be solved in the context of the Newton-Raphson iterative method. The form (10) and related forms will be used throughout the paper while bearing in mind that the equation corresponding to the slack bus may be eliminated.

3. NEWTON-RAPHSON ITERATION IN COMPLEX MODE

The familiar form of the Newton-Raphson iteration in the real mode [3] is obtained by separating (10) into real and imaginary parts and collecting the terms, appropriately, using the perturbed forms of (3) and (8), to get

$$\begin{bmatrix} (K_1^S + \bar{K}_1^S) & (-K_2^S + \bar{K}_2^S) \\ -(K_2^S + \bar{K}_2^S) & (-K_1^S + \bar{K}_1^S) \end{bmatrix} \begin{bmatrix} \delta V_{M1} \\ \delta V_{M2} \end{bmatrix} = \begin{bmatrix} \delta P_M \\ \delta Q_M \end{bmatrix}, \qquad (13)$$

where we have set

$$K^S = K_1^S + j\,K_2^S \qquad (14)$$

and

$$\bar{K}^S = \bar{K}_1^S + j\,\bar{K}_2^S. \qquad (15)$$

The 2n x 2n matrix of coefficients in (13), n denoting the number of buses in the power network, constitutes the Jacobian matrix of the load flow problem.

On the other hand, equation (10) can be written in the consistent form

$$\begin{bmatrix} K^S & \bar{K}^S \\ \bar{K}^{S*} & K^{S*} \end{bmatrix} \begin{bmatrix} \delta V_M \\ \delta V_M^* \end{bmatrix} = \begin{bmatrix} \delta S_M^* \\ \delta S_M \end{bmatrix}. \qquad (16)$$

It can be shown [5,6] that the matrix of coefficients of (16) has the same rank as that of (13) and the system of equations (16) is consistent if and only if the system (13) is consistent.

Now, the system of complex equations (16) is equivalent to the more compact system of complex equations

$$\tilde{K}^S \, \delta V_M = \tilde{d}^S, \qquad (17)$$

where we have defined

$$\tilde{K}^S = \bar{K}^{S*} - K^{S*} \, (\bar{K}^S)^{-1} \, K^S \qquad (18)$$

and

$$\tilde{d}^S = \delta S_M - K^{S*} \, (\bar{K}^S)^{-1} \, \delta S_M^*. \qquad (19)$$

In the jth iteration of the Newton-Raphson method in the complex mode, we solve the system of equations (17) with

$$\delta V_M = V_M^{j+1} - V_M^j, \qquad (20)$$

$$\delta S_M^* = S_{M(\text{scheduled})}^* - K^S \, V_M^j \qquad (21)$$

and the matrices K^S and $\bar{K}^S$ are calculated at V_M^j.

A trade off between the direct use of forms (13) and (17) must take into account the sparsity of the matrix of coefficients. While the matrix K^S of (14) has the same sparsity as the bus admittance matrix Y_T, the matrix of coefficients $\tilde{K}^S$ of (17) is as sparse as the matrix

$$Y_{TT} \overset{\Delta}{=} Y_T \, Y_T. \qquad (22)$$

In other words, the advantage of the direct use of the compact nxn complex matrix $\tilde{K}^S$ rather than the 2nx2n real matrix of coefficients of (13) may be restricted by the relative sparsity of the matrices Y_T and Y_{TT}, the factor which obviously depends on the graph of the network. To illustrate this point, we consider, in Fig. 1, three special structures for which the sparsity coefficients [8] of Y_T and Y_{TT} are compared.

We remark here that in the analysis of typical linear electronic circuits, a system of complex linear equations of form (1) is solved for the unknown node voltages V_M with known excitations I_M. The conjugate vector of variables V_M^* is not involved in the set of equations. The analysis, hence, does not imply the previous restriction and the use of the complex mode is undoubtedly advantageous [2].

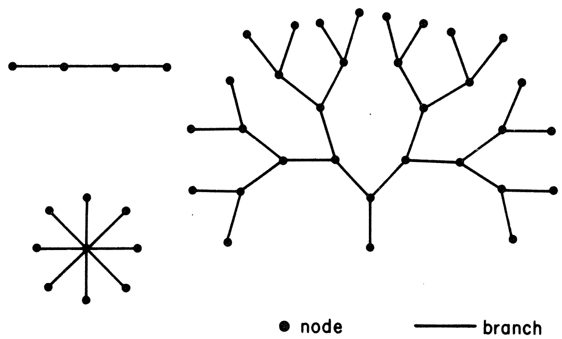
4. THE CONJUGATE ELIMINATION TECHNIQUE

In the previous section we have defined the matrix $\tilde{K}^S$ and used it to transform the original problem (10) into the form (17). This transformation enjoys the advantage of the compact complex-mode analysis and, at the same time, provides a form (17) which is suitable for ordinary methods of solving a set of linear equations. In this section we present an alternative approach to the problem. Instead of applying the ordinary elimination methods to the more dense matrix of coefficients $\tilde{K}^S$ of (17), we use a special technique in order to handle, directly, the original form (10).

In order to facilitate the derivations, we introduce the following notation. First, we define the term

$$\{k_{ij}, \bar{k}_{ij}\}\, x_j \triangleq k_{ij}\, x_j + \bar{k}_{ij}\, x_j^*, \qquad (23)$$

where k_{ij} and $\bar{k}_{ij}$ stand, for example, for general elements of the matrices K^S and $\bar{K}^S$. We call the set of elements a of {a,b} the basic set and the set of elements b the conjugate set. Then we state the following basic rules which can be easily verified.



Network	Sparsity Coefficient	
	Y_T	Y_{TT}
(a) simple chain	$1 - (3n-2)/n^2$	$1 - (5n-6)/n^2$
(b) simple star	$1 - (3n-2)/n^2$	0
(c) simple tree	$1 - (3n-2)/n^2$	$1 - (6n-8)/n^2$

n = number of nodes = order of Y_T or Y_{TT}

Fig. 1 Sparsity coefficients of Y_T and Y_{TT} for simple networks

Rule 1

$$\{k_{ij}, \bar{k}_{ij}\}\, x_j = \{\bar{k}_{ij}, k_{ij}\}\, x_j^*. \qquad (24)$$

Rule 2

$$(\{k_{ij},\bar{k}_{ij}\}x_j)^* = \{k_{ij}^*,\bar{k}_{ij}^*\}x_j^* = \{\bar{k}_{ij}^*,k_{ij}^*\}x_j. \qquad (25)$$

Rule 3

$$\mu\{k_{ij}, \bar{k}_{ij}\}\, x_j = \{\mu\, k_{ij}, \mu\, \bar{k}_{ij}\}\, x_j = \{k_{ij}, \bar{k}_{ij}\}\, (\mu\, x_j), \qquad (26)$$

where μ is a complex scalar.

Rule 4

$$\{k_{ij}, \bar{k}_{ij}\}\, x_j + \mu\, \{k_{\ell j}, \bar{k}_{\ell j}\}\, x_j = \{(k_{ij} + \mu\, k_{\ell j}), (\bar{k}_{ij} + \mu\, \bar{k}_{\ell j})\}\, x_j. \qquad (27)$$

The above notation may be exploited in developing suitable methods for solving systems of the form (10). Here, we invoke this notation to describe a technique which allows the forward Gaussian elimination process to be directly applied to the form (10). The system of equations (10) is written, using (23), as

$$\sum_{j=1}^{n} \{k_{ij}, \bar{k}_{ij}\}\, x_j = b_i, \; i = 1, \ldots, n, \qquad (28)$$

where x_j and b_i are elements of $x = \delta V_M$ and $b = \delta S_M^*$, respectively. Since

$$\bar{k}_{ij} = 0, \text{ for } i \neq j, \qquad (29)$$

equations (28) can be written as

$$\{k_{ii}, \bar{k}_{ii}\}\, x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \{k_{ij}, 0\}\, x_j = b_i, \quad i = 1, \ldots, n. \qquad (30)$$

We assume that the order of elimination [7] has been taken into account by applying suitable permutations to (30).

At the first iteration, we write the first equation of (30) as

$$\{k_{11}, \bar{k}_{11}\}\, x_1 + \sum_{j=2}^{n} \{k_{1j}, 0\}\, x_j = b_1 \qquad (31)$$

or, using (25),

$$\{\bar{k}_{11}^*, k_{11}^*\}\, x_1 + \sum_{j=2}^{n} \{0, k_{1j}^*\}\, x_j = b_1^*. \qquad (32)$$

Multiplying (32) by $\mu_1^{(1)*}$, where

$$\mu_i^{(1)} \triangleq - k_{1i}^*/\bar{k}_{ii} \qquad (33)$$

and adding to (31), we get, using (27),

$$\{0, \bar{k}_{11}+\mu_1^{(1)*}\, k_{11}^*\}\, x_1 + \sum_{j=2}^{n} \{k_{1j}, \mu_1^{(1)*}\, k_{1j}^*\}\, x_j = b_1+\mu_1^{(1)*} b_1^* \qquad (34)$$

or, using (25),

$$\{\bar{k}_{11}^* + \mu_1^{(1)}\, k_{11}, 0\}\, x_1 + \sum_{j=2}^{n} \{\mu_1^{(1)}\, k_{1j}, k_{1j}^*\}\, x_j = b_1^* + \mu_1^{(1)}\, b_1. \qquad (35)$$

Multiplying the ith equation of (30), i = 2, ..., n, by $\mu_i^{(1)}$ of (33) and adding to (35), we get

$$\sum_{j=1}^{n} \{k_{1j}^{(1)}, 0\}\, x_j = b_1^{(1)}, \qquad (36)$$

where

$$k_{11}^{(1)} = \bar{k}_{11}^* + \sum_{i=1}^{n} \mu_i^{(1)}\, k_{i1}, \qquad (37a)$$

$$k_{1j}^{(1)} = \sum_{i=1}^{n} \mu_i^{(1)}\, k_{ij}, \; j = 2, \ldots, n \qquad (37b)$$

and

$$b_1^{(1)} = b_1^* + \sum_{i=1}^{n} \mu_i^{(1)} b_i . \tag{38}$$

Equation (36) together with equations 2, 3, ..., n of (30) represent a set of equations ready for applying the first iteration of a forward Gaussian elimination to the matrix $\tilde{K}^{S(1)}$ which is obtained by replacing the elements of the first row of $\tilde{K}^S$ by the elements of (37). Observe that we have evacuated the conjugate set of the first equation.

In general, at the mth iteration, and with $k_{ij}^{(m-1)}$ and $b_i^{(m-1)}$ denoting the current elements of $\tilde{K}^S$ and $\tilde{b}$, respectively, we replace the elements of the mth row of $\tilde{K}^S$ by the elements

$$k_{mm}^{(m)} = \bar{k}_{mm}^* + \sum_{i=m}^{n} \mu_i^{(m)} k_{im}^{(m-1)} \tag{39a}$$

and

$$k_{mj}^{(m)} = \sum_{i=m}^{n} \mu_i^{(m)} k_{ij}^{(m-1)}, \; j = m+1, \ldots, n \tag{39b}$$

and we replace $b_m^{(m-1)}$ by

$$b_m^{(m)} = b_m^{(m-1)*} + \sum_{i=m}^{n} \mu_i^{(m)} b_i^{(m-1)}, \tag{40}$$

where

$$\mu_i^{(m)} \triangleq - k_{mi}^{(m-1)*} / \bar{k}_{ii} . \tag{41}$$

We shall call the special elimination process described by (39)-(40) the conjugate elimination in which the coefficients of the conjugate variables are successively eliminated. A tableau representation of the combined elimination process is shown in Table I for n = 3, and corresponding numerical results are shown in Table II where the solution of the arbitrary system of equations

$$x_1 - jx_2 + 2x_3 + 2x_1^* = 5$$

$$jx_1 - x_2 + jx_3 - x_2^* = j$$

$$2x_1 + jx_2 - x_3 + x_3^* = 0$$

is investigated. The backward substitution results in the solution

$$\tilde{x} = \begin{bmatrix} 1 \\ j2 \\ 0 \end{bmatrix} .$$

5. COMPLEX FORMULATION FOR PRACTICAL SYSTEMS

In the system of equations of form (10), it is assumed that all buses other than the slack bus are of the same type, namely a load bus type for which the active and reactive powers are known. In practice, voltage-controlled or generator-type buses must be considered and modelled appropriately. For generator type buses, the magnitude of the bus voltage and the active power are specified. This situation obviously impedes the direct use of the complex form (10). In the following, we present a special technique of formulation which allows the generator-type buses to be included while preserving the complex mode of (10).

Consider the equation of (10) corresponding to a generator bus g. We define the complex quantity

$$\tilde{S}_g^0 \triangleq P_g + j\,|V_g| , \tag{42}$$

hence

$$\delta\tilde{S}_g^0 = \delta P_g + j\,\delta|V_g| . \tag{43}$$

Since

$$2P_g = V_g I_g^* + V_g^* I_g , \tag{44}$$

then

$$2\delta P_g = V_g\,\delta I_g^* + I_g^*\,\delta V_g + V_g^*\,\delta I_g + I_g\,\delta V_g^* . \tag{45}$$

Using (1), we write I_g in the form

$$I_g = \tilde{y}_g^T \tilde{V}_M , \tag{46}$$

where $\tilde{y}_g^T$ represents the corresponding row of the bus admittance matrix $\tilde{Y}_T$, hence

$$\delta I_g = \tilde{y}_g^T\,\delta\tilde{V}_M . \tag{47}$$

Also,

$$\delta|V_g| = \delta(V_g V_g^*)^{1/2}$$

$$= (V_g\,\delta V_g^* + V_g^*\,\delta V_g)/(2|V_g|) . \tag{48}$$

Using (45)-(48) it is straightforward to show that $\delta\tilde{S}_g^0$ of (43) is given by

$$\delta\tilde{S}_g^0 = \tilde{k}_g^{0T}\,\delta\tilde{V}_M + \tilde{\bar{k}}_g^{0T}\,\delta\tilde{V}_M^* , \tag{49}$$

where $\tilde{k}_g^{0T}$, which replaces the row of $\tilde{K}^S$ of (10) corresponding to the generator bus g, has elements defined as

$$k_{gj}^0 \triangleq V_g^* Y_{gj}/2, \; j \neq g \tag{50a}$$

and

$$k_{gg}^0 \triangleq j\,V_g^*/(2|V_g|) + (V_g^* Y_{gg} + I_g^*)/2, \tag{50b}$$

Y_{ij} denoting elements of $\tilde{Y}_T$, and $\tilde{\bar{k}}_g^{0T}$, which replaces the row of $\bar{K}^S$ of (10) corresponding to the generator bus g, has elements defined as

$$\bar{k}_{gj}^0 \triangleq V_g Y_{gj}^*/2, \; j \neq g \tag{51a}$$

and

$$\bar{k}_{gg}^0 \triangleq j\,V_g/(2|V_g|) + (V_g Y_{gg}^* + I_g)/2. \tag{51b}$$

The above formulation results in an equation of (30) for i = g of the form

$$\{k_{gg}^0, \bar{k}_{gg}^0\}\, x_g + \sum_{\substack{j=1 \\ j\neq g}}^{n} \{k_{gj}^0, \bar{k}_{gj}^0\}\, x_j = b_g^0, \tag{52}$$

where b_g^0 stands for $\delta\tilde{S}_g^0$.

In order to prepare the original conjugate tableau of (10) to be suitable for applying the technique described in the previous section, we multiply equation i, i≠g, of (30) by the factor

TABLE I
THE COMBINED ELIMINATION TECHNIQUE

Iteration No.	Basic Tableau			Conjugate Tableau			$\underset{\sim}{b}$	Type of Elimination
0	k_{11}	k_{12}	k_{13}	$\bar{k}_{11}$	0	0	b_1	Original tableau
	k_{21}	k_{22}	k_{23}	0	$\bar{k}_{22}$	0	b_2	
	k_{31}	k_{32}	k_{33}	0	0	$\bar{k}_{33}$	b_3	
1a	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Conjugate elimination
	k_{21}	k_{22}	k_{23}	0	$\bar{k}_{22}$	0	b_2	
	k_{31}	k_{32}	k_{33}	0	0	$\bar{k}_{33}$	b_3	
1b	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Gaussian forward elimination
	0	$k_{22}^{(1)}$	$k_{23}^{(1)}$	0	$\bar{k}_{22}$	0	$b_2^{(1)}$	
	0	$k_{32}^{(1)}$	$k_{33}^{(1)}$	0	0	$\bar{k}_{33}$	$b_3^{(1)}$	
2a	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Conjugate elimination
	0	$k_{22}^{(2)}$	$k_{23}^{(2)}$	0	0	0	$b_2^{(2)}$	
	0	$k_{32}^{(1)}$	$k_{33}^{(1)}$	0	0	$\bar{k}_{33}$	$b_3^{(1)}$	
2b	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Gaussian forward elimination
	0	$k_{22}^{(2)}$	$k_{23}^{(2)}$	0	0	0	$b_2^{(2)}$	
	0	0	$k_{33}^{(2)}$	0	0	$\bar{k}_{33}$	$b_3^{(2)}$	
3a	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Conjugate elimination
	0	$k_{22}^{(2)}$	$k_{23}^{(2)}$	0	0	0	$b_2^{(2)}$	
	0	0	$k_{33}^{(3)}$	0	0	0	$b_3^{(3)}$	

TABLE II
EXAMPLE OF ELIMINATION TABLEAU

Iteration No.	Basic Tableau			Elements of Conjugate Tableau	$\underset{\sim}{b}$
0	1	-j	2	2	5
	j	-1	j	-1	j
	2	j	-1	1	0
1	7	j5	0	0	-3
	0	-2/7	j	-1	j10/7
	0	-j3/7	-1	1	6/7
2	7	j5	0	0	-3
	0	12/7	j9/2	0	j24/7
	0	0	-17/8	1	0
3	7	j5	0	0	-3
	0	12/7	j9/2	0	j24/7
	0	0	-225/136	0	0

$$\mu_i^0 \triangleq -\bar{k}_{gi}^0/\bar{k}_{ii}. \qquad (53)$$

The sum of the resulting equations is added to (52) to obtain, putting $\bar{k}_{gg} = \bar{k}_{gg}^0$,

$$\{k_{gg}, \bar{k}_{gg}\}\, x_g + \sum_{\substack{j=1 \\ j\neq g}}^{n} \{k_{gj}, 0\}\, x_j = b_g, \qquad (54)$$

where

$$k_{gj} \triangleq k_{gj}^0 + \sum_{\substack{i=1 \\ i\neq g}}^{n} \mu_i^0\, k_{ij},\ j = 1, 2, \ldots, n, \qquad (55)$$

and

$$b_g \triangleq b_g^0 + \sum_{\substack{i=1 \\ i\neq g}}^{n} \mu_i^0\, b_i. \qquad (56)$$

Equation (54), hence, represents the gth equation of (30).

Note that $|V_g|$ of (42) could be replaced, for example, by $|V_g|^2$. Moreover, one could equally well replace the elements of $\delta \underset{\sim}{V}_M$ and $\delta \underset{\sim}{V}_M^*$, namely, δV_i and δV_i^*, i = 1, ..., n by the relative quantities $\delta V_i/|V_i|$ and $\delta V_i^*/|V_i|$, respectively. In this case, the elements k_{ij} and $\bar{k}_{ij}$ of the ith row of the coefficient matrices are replaced by $|V_j| k_{ij}$ and $|V_j| \bar{k}_{ij}$, respectively.

6. CONCLUSIONS

We have presented a suitable approach for solving, in complex mode, the load flow problem described for power system studies using the well known Newton-Raphson method. The unfamiliar form of the resulting complex equations has been directly handled by a new elimination technique. Other similar forms of complex equations which are expressed in terms of conjugate pairs of variables may be handled. We have synthesized appropriate complex variables comprising the practical adjustable variables associated with voltage-controlled buses. Hence, the complex mode of the resulting perturbed power flow equations is preserved.

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