A One-Dimensional Minimax Algorithm Based on Biquadratic Models

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Abstract—**We exploit the biquadratic behavior w.r.t. a variable exhibited in the frequency domain by certain lumped, linear circuits. A globally convergent, extremely efficient minimax algorithm is developed to optimize the frequency response w.r.t. any circuit parameter. The algorithm converges to the global minimax optimum and the rate of convergence is at least of second order.**

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TABLE I
INTERVALS DEFINED BY (3) AND (4)

$\mathrm{sgn}(\delta - \frac{C_i}{E_i})$	$\mathrm{sgn}[(B_i-\delta D_i)^2 - (C_i-\delta E_i)(A_i-\delta)]$	$\mathrm{sgn}(B_iE_i-C_iD_i)$	$\mathrm{sgn}(A_iE_i-C_i)$	Intervals (points) of ϕ such that $e_i(\phi) < \delta$	$e_i(\phi) > \delta$	$e_i(\phi) = \delta$
1	1 or 0	*	*	$(-\infty,r_1)\vee(r_2,\infty)$	$\underline{(r_1,r_2)}$	r_1,r_2
1	-1	*	*	$\underline{(-\infty,\infty)}$	-	-
-1	1 or 0	*	*	$\underline{(r_1,r_2)}$	$(-\infty,r_1)\vee(r_2,\infty)$	r_1,r_2
-1	-1	*	*	-	$\underline{(-\infty,\infty)}$	-
0	*	1	*	$\underline{(-\infty,r)}$	(r,∞)	r
0	*	-1	*	$\underline{(r,\infty)}$	$(-\infty,r)$	r
0	*	0	1	-	$\underline{(-\infty,\infty)}$	-
0	*	0	-1	$\underline{(-\infty,\infty)}$	-	-

* denotes values of no interest,

$r_1 \le r_2$ denote the two real roots of equation (4),

r denotes single real root of equation (4) when $\delta = C_i/E_i$.

I. INTRODUCTION

A number of researchers have considered properties of response functions w.r.t. one designable variable at a time in the context of the prediction of worst cases in design centering and tolerance assignment [1]–[8]. The bilinear behavior of certain linear circuits has been used to derive relationships between, e.g., first-order and large change sensitivities. In the tolerance problem, attempts have been made to find conditions which satisfy the common assumption that worst cases occur at extremes of parameter uncertainty intervals.

Here, we consider the resulting biquadratic function obtained from the modulus squared of the bilinear function. We determine parameter intervals which are utilized in a globally convergent and extremely efficient one-dimensional minimax algorithm. The algorithm is based on the linearization of some functions at the extreme points of these intervals. Examples employing a realistic tunable active filter demonstrate the optimization of the frequency response w.r.t. a circuit parameter.

II. THE ONE-DIMENSIONAL MINIMAX ALGORITHM

We consider the problem

$$\underset{\phi}{\text{minimize}} \max_{1 \le i \le m} e_i(\phi) \tag{1}$$

where the functions $e_i(\phi)$ are biquadratic of the form

$$e_i(\phi) = \frac{A_i + 2B_i\phi + C_i\phi^2}{1 + 2D_i\phi + E_i\phi^2}. \tag{2}$$

We assume that the functions (2) have nonnegative denominators and are irreducible. Bounds on the range of ϕ can easily be taken into account. Now, suppose we are interested in finding, for certain i, the values of ϕ such that

$$e_i(\phi) = \delta \tag{3}$$

where δ is a given number. Such values can be obtained from the equation

$$(C_i - \delta E_i)\phi^2 + 2(B_i - \delta D_i)\phi + (A_i - \delta) = 0. \tag{4}$$

Table I presents all relevant cases. According to Table I we notice that, due to the properties of biquadratic functions, it is always possible to define a *continuous* interval $R_{i\delta}$ such that either

$$e_i(\phi) \le \delta, \quad \text{for all } \phi \in R_{i\delta}$$
$$\text{and } e_i(\phi) > \delta, \quad \text{for all } \phi \notin R_{i\delta} \tag{5}$$

or

$$e_i(\phi) \ge \delta, \quad \text{for all } \phi \in R_{i\delta}$$
$$\text{and } e_i(\phi) < \delta, \quad \text{for all } \phi \notin R_{i\delta}. \tag{6}$$

Typically, the interval $R_{i\delta}$ is unique. For particular cases, however, we can find two continuous intervals such that one of them fits the situation of (5) and the other one satisfies (6). In such cases we decide to consider the interval which is underlined in Table I. (Boundary points are included in $R_{i\delta}$.) To indicate the type of the interval $R_{i\delta}$ we will use a logical variable $t_{i\delta}$ which is set to "true" or "false" if $R_{i\delta}$ satisfies (5) or (6), respectively.

Now, consider the set of error functions $e_i(\phi)$, $i = 1,2,\cdots,m$. The region R_δ defined as

$$R_\delta = \{\phi \mid e_i(\phi) \le \delta, i = 1,2,\cdots,m\} \tag{7}$$

can be calculated as

$$R_\delta = \bigcap_{t_{i\delta} = \text{true}} R_{i\delta} - \bigcup_{t_{i\delta} = \text{false}} (R_{i\delta} - \mathrm{Fr}(R_{i\delta})) \tag{8}$$

where $\mathrm{Fr}(R_{i\delta})$ denotes the boundary of $R_{i\delta}$.

It is to be noted that R_δ is not necessarily a continuous interval. In general,

$$R_\delta = \bigcup_{l=1}^{k} [\check{\phi}_l, \hat{\phi}_l] \tag{9}$$

where k is the number of separate intervals denoted $[\check{\phi}_l, \hat{\phi}_l]$. An efficient algorithm is proposed in [9] to provide k and the intervals $[\check{\phi}_l, \hat{\phi}_l]$, $l = 1,2,\cdots,k$, as well as the indexes of the functions e_i which actually define the extreme points of each interval. These indexes are denoted $\check{i}_l$ and $\hat{i}_l$ for the lower and upper extremes, respectively. In the algorithm we first determine the intersections which appear in (8) and then we subtract consecutive intervals of the second term of (8).

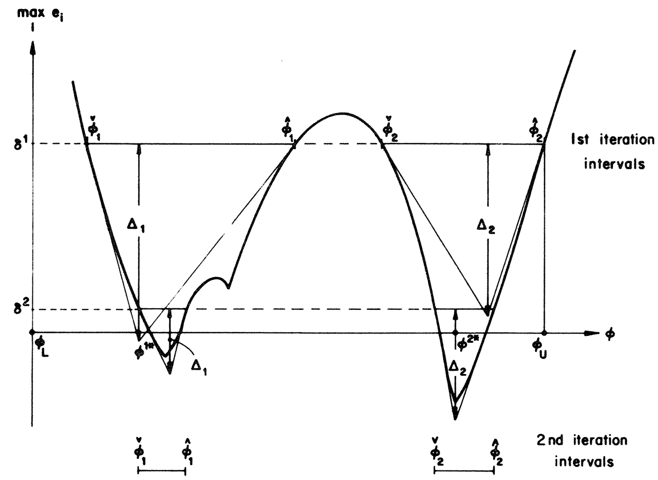

Fig. 1. Illustration of the behavior of the one-dimensional minimax algorithm. Note that the algorithm switches from interval 1 to interval 2, based on predictions of the decrease in the maximum.

The algorithm for solving problem (1) is illustrated in Fig. 1. The following steps set it out in sufficient detail.

Algorithm

Step 1: Set $\delta \leftarrow \min\{\max_i e_i(\phi_L), \max_i e_i(\phi_U)\}$, where ϕ_L and ϕ_U denote bounds on ϕ being considered.

Step 2: Find valid intervals $I_l \triangleq [\check{\phi}_l, \hat{\phi}_l]$ and corresponding function indexes $\check{i}_l, \hat{i}_l, l=1,2,\cdots,k$.

Step 3: Calculate

$$\check{g}_l = \left.\frac{de_{\check{i}_l}}{d\phi}\right|_{\check{\phi}_l}, \hat{g}_l = \left.\frac{de_{\hat{i}_l}}{d\phi}\right|_{\hat{\phi}_l}.$$

Step 4: If $k=1$, set $j \leftarrow 1$ and go to Step 6.

Step 5: Find j such that $\Delta_j \geq \Delta_l, l=1,2,\cdots,k,$ (10)

where

$$\Delta_l = \begin{cases} \hat{g}_l \check{g}_l (\hat{\phi}_l - \check{\phi}_l)/(\check{g}_l - \hat{g}_l) \\ 0 \text{ if } \check{g}_l = \hat{g}_l = 0. \end{cases} \tag{11}$$

Comment: In this step we select the jth interval which appears to be the most promising one in terms of the expected improvement in the minimax optimum based on linearization. Δ_l will always be positive unless either $\check{g}_l = 0$, $\hat{g}_l = 0$ or $\check{\phi}_l = \hat{\phi}_l$.

Step 6: Set $\phi^* \leftarrow (\check{g}_j \hat{\phi}_j - \hat{g}_j \check{\phi}_j)/(\check{g}_j - \hat{g}_j)$ if $\check{i}_j \neq \hat{i}_j$ and $\Delta_j \neq 0$.

Comment: If the extremes of the jth interval are defined by two different functions, the new value of ϕ, denoted by ϕ^*, is determined by the intersection of the linear approximation to the two functions.

Step 7: Set ϕ^* to the minimizing point of the function $e_{\hat{i}_j}$ if $\check{i}_j = \hat{i}_j$.

Step 8: Set $\phi^* \leftarrow (\check{\phi}_j + \hat{\phi}_j)/2$ if $\phi^* \notin (\check{\phi}_j, \hat{\phi}_j)$ or $\Delta_j = 0$.

Comment: This is a default value to obviate any numerical problem which may arise in Step 5 or Step 6, for example, $\hat{g}_j = 0$.

Step 9: Find $\delta = \max_i e_i(\phi^*)$.

Step 10: Stop if $k=1$ and if $(\hat{\phi}_1 - \check{\phi}_1)$ is sufficiently small.

Step 11: Go to Step 2.

In the following, superscript n will denote the index of iteration of the algorithm. The convergence properties of the algorithm are stated by the following theorem.

Theorem .If $I^n \triangleq [\check{\phi}^n, \hat{\phi}^n]$ is a unique interval such that $e_i(\phi) \leq \delta^n$ for $i=1,2,\cdots,m$, then $|\hat{\phi}^n - \check{\phi}^n| \to 0$ as $n \to \infty$. The rate of convergence is at least of second order.

Proof of the theorem is presented in [9].

Now, we will show that the algorithm is guaranteed to converge to the global minimax optimum. This is due to switching from one interval I_l to another one in Step 5. Full details are available [10].

According to the comment after Step 5, Δ_l^n is always postive if $|\hat{\phi}_l^n - \check{\phi}_l^n| > 0$. We can omit the cases when $\hat{g}_l^n = 0$ and/or $\check{g}_l^n = 0$, since $\check{g}_l^n < 0$ and $\hat{g}_l^n > 0$ almost everywhere and Step 8 secures us against these situations. Moreover, it is easy to notice that $\Delta_l^n \to 0$ if $|\hat{\phi}_l^n - \check{\phi}_l^n| \to 0$.

Let us consider two intervals I_1^n and I_2^n which are found by the algorithm in the nth iteration. Let us assume that $\bar{\phi} \in I_2^n$ is a unique global minimax optimum. According to (11) and using the following notation:

$$a_i^n = \min(-\check{g}_i^n, \hat{g}_i^n);\ b_i^n = \max(-\check{g}_i^n, \hat{g}_i^n) \tag{12}$$

where $i=1,2$ is the index of the interval, we have

$$\Delta_1^n = \frac{a_1^n b_1^n}{a_1^n + b_1^n}(\hat{\phi}_1^n - \check{\phi}_1^n) \leq \frac{b_1^n}{2}(\hat{\phi}_1^n - \check{\phi}_1^n) \tag{13}$$

and

$$\Delta_2^n = \frac{a_2^n b_2^n}{a_2^n + b_2^n}(\hat{\phi}_2^n - \check{\phi}_2^n) \geq \frac{a_2^n}{2}(\hat{\phi}_2^n - \check{\phi}_2^n). \tag{14}$$

Thus

$$\frac{\Delta_1^n}{\Delta_2^n} \leq \frac{b_1^n}{a_2^n} \cdot \frac{\hat{\phi}_1^n - \check{\phi}_1^n}{\hat{\phi}_2^n - \check{\phi}_2^n}. \tag{15}$$

Since $\bar{\phi} \in I_2^n$ is a unique global minimax optimum $|\hat{\phi}_2^n - \check{\phi}_2^n| \to \text{const} \neq 0$ if $|\hat{\phi}_1^n - \check{\phi}_1^n| \to 0$ so that $(\hat{\phi}_1^n - \check{\phi}_1^n)/(\hat{\phi}_2^n - \check{\phi}_2^n) \to 0$. The left-hand side of (15) can converge to a value different from zero only if $a_2^n \to 0$ when $|\hat{\phi}_1^n - \check{\phi}_1^n| \to 0$. But this means that there is a local minimum of at least one of the functions $e_{\check{i}_2}(\phi)$ or $e_{\hat{i}_2}(\phi)$ of value equal to the local minimum value at $\phi_{1\min} \in I_1^n$ so that $\bar{\phi} \in I_2^n$ is not the unique global minimax optimum. Otherwise, since $\Delta_1^n/\Delta_2^n \to 0$, the algorithm will select the second interval according to (10).

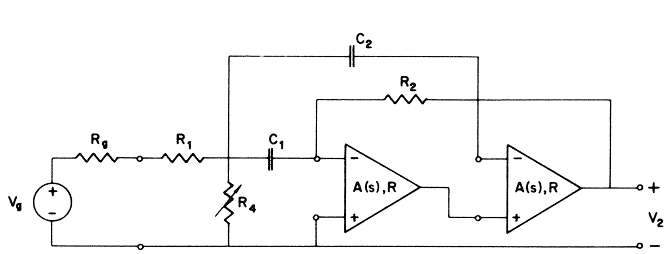

Fig. 2. Tunable active filter.

III. Example

A tunable active filter [11] has been chosen to implement the theory and algorithms. The filter is shown in Fig. 2. The specifications w.r.t. frequency on the modulus squared of the transfer function $F = |V_2/V_g|^2$ are $F \leq 0.5$ for $f/f_0 \leq 1 - 10/f_0$ or $f/f_0 \geq 1 + 10/f_0$, $F \leq 1.21$ for $1 - 10/f_0 \leq f/f_0 \leq 1 + 10/f_0$, $F \geq 0.5$ for $1 - 8/f_0 \leq f/f_0 \leq 1 + 8/f_0$, $F \geq 1$ for $f = f_0$ Hz, where f_0 is the center frequency. The one-pole rolloff model for the operational amplifiers described by the dc gain A_0 and the 3-dB rad bandwidth ω_a was used.

Based on two consecutive analyses a biquadratic model in R_4 was obtained at each sample frequency. The normalized sample frequencies are taken as 1 and $1 \pm 10/f_0$ for the relevant upper specifications, 1 and $1 \pm 8/f_0$ for the relevant lower specifications. This leads to six error functions e_i, $i=1,2,\cdots,6$. The range of R_4 for which the specifications are satisfied is that for which $e_i \leq 0$,

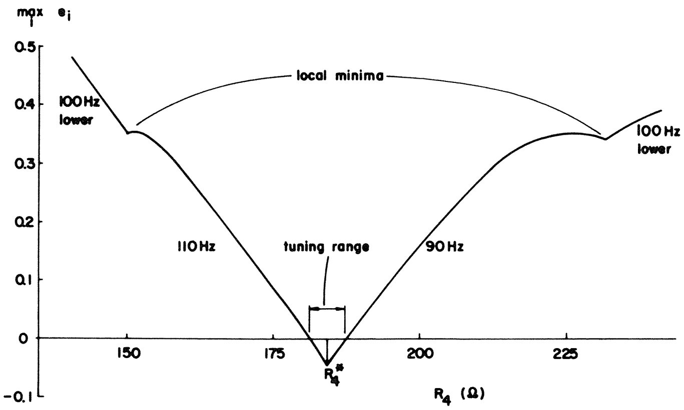

Fig. 3. $\mathrm{Max}_{1\leq i\leq 6} e_i$ versus the tuning resistor R_4 for specifications defined around $f_0 = 100$ Hz indicating the active functions (and hence active frequency points).

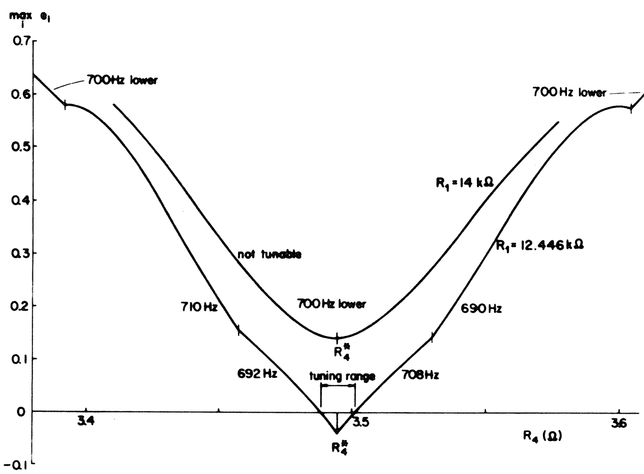

Fig. 4. $\mathrm{Max}_{1\leq i\leq 6} e_i$ versus R_4 for specifications defined around $f_0 = 700$ Hz for two cases: (a) $R_1 = 12.446$ kΩ, (b) $R_1 = 14$ kΩ.

$i = 1, 2, \cdots, 6$. The maximum of the error functions e_i versus R_4 is shown in Fig.3 . A single run of our program indicated that the filter is tunable for the specifications defined at a center frequency of 100 Hz. It meets these specifications if

$$R_4 \in [181.126, \quad 187.166]$$

and with other circuit parameters fixed at values given in Table II. It is also tunable around a center frequency of 700 Hz (see Fig. 4) and meets the specifications if

$$R_4 \in [3.4881, \quad 3.5012].$$

To find

$$\min_{R_4} \max_i e_i$$

we are faced with the local minima in Fig. 3. The convergence of other algorithms [12] to the global minimum depends upon the starting point. For the proposed algorithm the results are shown in Table III for different starting points and at different center frequencies. Note how few iterations are required.

When R_1 was altered to the value 14 kΩ the filter is not tunable as is determined by one run of the program. The optimum value of R_4, however, was obtained in only two iterations (see Table III). In fact, the algorithm converged in the first iteration since the optimum is defined by one function, however, the second iteration was performed to satisfy the stopping criterion.

TABLE II
CIRCUIT PARAMETERS

R_g	=	50.000 Ω	C_1	=	0.728556 µF
R_1	=	12.446 kΩ	C_2	=	0.728556 µF
R_2	=	26.500 kΩ	A_0	=	2×10^5
R_3	=	75.000 Ω	ω_a	=	12π rad/s

TABLE III
MINIMAX OPTIMUM OF R_4

Center Frequency (Hz)	R_4 (Ω) Starting	R_4 (Ω) Optimum	Optimum δ	N.O.I.*
100	100.0	184.3998	-0.0458	3
	300.0	184.3998	-0.0458	3
	∞	184.3998	-0.0458	3
700	10.0	3.4946	-0.0403	3
	200.0**	3.4946	-0.0403	3
	200.0	3.4940	0.1434	2

* N.O.I. = number of iterations

** R_1 was altered to 14.0 kΩ and the filter is not tunable since $\delta > 0$.

Step 9 of the algorithm was used to initialize δ at starting values of R_4. Running times per example on a CDC 6400 computer were about 0.1 s.

IV. CONCLUSIONS

The bilinear behavior of certain linear circuits in the frequency domain have been exploited. The explicit determination of the points defining the boundary of the feasible region w.r.t. one parameter leads to a simple check on the tunability of an outcome of the manufacturing process by adjusting a single parameter at a time. Convergence to the global minimax optimum from different starting points is assured in a few iterations.

The algorithm presented, after minor modifications, can be used for a broader class of functions than biquadratic ones of the form of (2). Only mild assumptions on first- and second-order derivative behavior are required to obtain the same convergence properties. From the point of view of computational effort related to the interval finding procedure, invertibility of the functions is desirable.

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