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A GENERALIZED, COMPLEX ADJOINT APPROACH TO POWER NETWORK SENSITIVITIES

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SUMMARY

A unified study of the class of adjoint network approaches to power system sensitivity analysis which exploits the Jacobian matrix of the load flow solution is presented. Generalized sensitivity expressions which are easily derived, compactly described and effectively used for calculating first-order changes and gradients of functions of interest are obtained. These generalized sensitivity expressions are common to all modes of formulating the power flow equations, e.g. polar and Cartesian. The approach exploits a special complex notation and complex matrix manipulations to define the adjoint system and to derive the sensitivity formulae. The approach is applicable to both real and complex function sensitivities.

INTRODUCTION

Two kinds of analysis can be distinguished in power system operation and planning studies. In the first kind, which implies the load flow solution[1,2] of the power network, the system states are obtained with the control (independent) variables fixed at particular values. The solution obtained describes the power system steady state behaviour associated with these particular values of the control variables. The second kind of analysis deals with variations in control variables and the resulting effect on either system states or, in general, on a particular function of interest.[3–5] This analysis is usually referred to as sensitivity analysis. The importance of sensitivity analysis has been recognized[6,7] in power system operation and planning studies to supply first-order changes of functions of interest and their gradients required for effective optimization techniques.

The class of adjoint network approaches[6,8–10] incorporating the method of Lagrange multipliers provides the advantage of using the transpose of the Jacobian of the load flow problem as an adjoint matrix of coefficients. When describing adjoint network approaches which exploit the Jacobian of the load flow problem, the sensitivity expressions for different elements are derived according to the mode of formulation used, e.g. polar or Cartesian. Different forms of sensitivity expressions have been presented for different studies. A unified sensitivity study for this class of adjoint network approaches has not, however, been previously described.

The impact of the conjugate notation,[10,11] which describes the first-order changes of general complex functions in terms of formal derivatives w.r.t. complex system variables, provides a useful tool for describing a generalized adjoint network sensitivity approach, as presented in this paper, where generalized sensitivity expressions are easily derived, compactly described and effectively used subject to any mode of formulation. The adjoint matrix of coefficients is always the transpose of the Jacobian of the original load flow problem and, regardless of the formulation, these generalized sensitivity expressions can be used.

In the first few sections, we briefly describe the notation used and illustrate the problem formulation. For the detailed analytical aspects of the conjugate notation, the reader is referred to References 10 and 11. We then derive the complex transformation matrices relating different modes of formulating the power flow equations to a standard complex form. This standard complex form is employed in the

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subsequent sections to define and analyse the adjoint system and to derive the generalized sensitivity formulae. In order to illustrate the novel concepts, two examples of the simplest 2-bus sample power system are employed throughout the paper. Numerical results for a 6-bus sample power system are also presented. The formulae derived, however, are general and can be directly programmed for a general power system of practical size.

NOTATION

In the conjugate notation[10,11] a complex variable

$$\zeta_i = \zeta_{i1} + j\zeta_{i2} \tag{1}$$

and its complex conjugate ζ_i^* replace, as independent quantities, the real and imaginary parts of the variable. Hence, we may express the first-order change of a continuous function of a set of complex variables arranged in a column vector $\boldsymbol{\zeta}$,

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}_1 + j\boldsymbol{\zeta}_2 \tag{2}$$

and their complex conjugate $\boldsymbol{\zeta}^*$ in the form

$$\delta f = \left(\frac{\partial f}{\partial \boldsymbol{\zeta}}\right)^{\mathrm{T}} \delta\boldsymbol{\zeta} + \left(\frac{\partial f}{\partial \boldsymbol{\zeta}^*}\right)^{\mathrm{T}} \delta\boldsymbol{\zeta}^* \tag{3}$$

where δ denotes first-order change, T denotes transposition and $\partial f/\partial\boldsymbol{\zeta}$ and $\partial f/\partial\boldsymbol{\zeta}^*$ are column vectors representing the formal[12] partial derivatives of f w.r.t. $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}^*$, respectively.

It can be shown[10] that, for a real function f, we may write

$$\frac{\partial f}{\partial \boldsymbol{\zeta}^*} = \left(\frac{\partial f}{\partial \boldsymbol{\zeta}}\right)^* \tag{4}$$

BASIC FORMULATION

Load flow equations

The electric power network can be represented by a system of node equations in the form

$$\mathbf{Y}_{\mathrm{T}}\mathbf{V}_{\mathrm{M}} = \mathbf{I}_{\mathrm{M}} \tag{5}$$

where

$$\mathbf{Y}_{\mathrm{T}} = \mathbf{Y}_{\mathrm{T1}} + j\mathbf{Y}_{\mathrm{T2}} \tag{6}$$

is the bus admittance matrix of the power network,

$$\mathbf{V}_{\mathrm{M}} = \mathbf{V}_{\mathrm{M1}} + j\mathbf{V}_{\mathrm{M2}} \tag{7}$$

is a column vector of the bus voltages, and

$$\mathbf{I}_{\mathrm{M}} = \mathbf{I}_{\mathrm{M1}} + j\mathbf{I}_{\mathrm{M2}} \tag{8}$$

is a vector of bus currents.

We write the bus loading equations in the matrix form

$$\mathbf{E}_{\mathrm{M}}^*\mathbf{I}_{\mathrm{M}} = \mathbf{S}_{\mathrm{M}}^* \tag{9}$$

where $\mathbf{E}_{\mathrm{M}}$ is a diagonal matrix of components of $\mathbf{V}_{\mathrm{M}}$ in corresponding order, i.e.

$$\mathbf{E}_{\mathrm{M}}\mathbf{v} = \mathbf{V}_{\mathrm{M}} \tag{10}$$

where $\mathbf{v}$ is given by

$$\mathbf{v} \triangleq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{11}$$

and $\mathbf{S}_{\mathrm{M}}$ is a vector of the injected bus powers given by

$$\mathbf{S}_{\mathrm{M}} \triangleq \mathbf{P}_{\mathrm{M}} + j\mathbf{Q}_{\mathrm{M}} \tag{12}$$

Substituting (5) into (9), we get

$$\mathbf{E}_{\mathrm{M}}^{*}\mathbf{Y}_{\mathrm{T}}\mathbf{V}_{\mathrm{M}} = \mathbf{S}_{\mathrm{M}}^{*} \tag{13}$$

The system of non-linear equations (13) represents the typical load flow problem, whose solution is required.

Complex perturbed form

The system (13) may be written in the perturbed form

$$\mathbf{K}^{\mathrm{S}}\delta\mathbf{V}_{\mathrm{M}} + \bar{\mathbf{K}}^{\mathrm{S}}\delta\mathbf{V}_{\mathrm{M}}^{*} = \delta\mathbf{S}_{\mathrm{M}}^{*} - \mathbf{E}_{\mathrm{M}}^{*}\delta\mathbf{Y}_{\mathrm{T}}\mathbf{V}_{\mathrm{M}} \tag{14}$$

where $\delta\mathbf{V}_{\mathrm{M}}$, $\delta\mathbf{V}_{\mathrm{M}}^{*}$, $\delta\mathbf{S}_{\mathrm{M}}^{*}$ and $\delta\mathbf{Y}_{\mathrm{T}}$ represent first-order changes of $\mathbf{V}_{\mathrm{M}}$, $\mathbf{V}_{\mathrm{M}}^{*}$, $\mathbf{S}_{\mathrm{M}}^{*}$ and $\mathbf{Y}_{\mathrm{T}}$, respectively,

$$\mathbf{K}^{\mathrm{S}} \triangleq \mathbf{E}_{\mathrm{M}}^{*}\mathbf{Y}_{\mathrm{T}} \tag{15}$$

and $\bar{\mathbf{K}}^{\mathrm{S}}$ is a diagonal matrix of components of $\mathbf{I}_{\mathrm{M}}$, i.e.

$$\bar{\mathbf{K}}^{\mathrm{S}}\mathbf{v} = \mathbf{I}_{\mathrm{M}} \tag{16}$$

We write (14) in the form

$$\mathbf{K}^{\mathrm{S}}\delta\mathbf{V}_{\mathrm{M}} + \bar{\mathbf{K}}^{\mathrm{S}}\delta\mathbf{V}_{\mathrm{M}}^{*} = \mathbf{d}^{\mathrm{S}} \tag{17}$$

where we have defined

$$\mathbf{d}^{\mathrm{S}} \triangleq \delta\mathbf{S}_{\mathrm{M}}^{*} - \mathbf{E}_{\mathrm{M}}^{*}\delta\mathbf{Y}_{\mathrm{T}}\mathbf{V}_{\mathrm{M}} \tag{18}$$

Note that for constant $\mathbf{Y}_{\mathrm{T}}$, $\mathbf{d}^{\mathrm{S}}$ of (18) is simply $\delta\mathbf{S}_{\mathrm{M}}^{*}$, and (17) rigorously represents a set of linear equations to be solved by the well-known Newton–Raphson iterative method.

Slack bus

The equation of (17) corresponding to the slack bus of specified voltage is replaced by

$$\mathbf{k}_{n}^{\mathrm{T}}\delta\mathbf{V}_{\mathrm{M}} + \bar{\mathbf{k}}_{n}^{\mathrm{T}}\delta\mathbf{V}_{\mathrm{M}}^{*} = \delta V_{n}^{*} \tag{19}$$

where we have assigned the last bus, namely the nth bus, as a slack bus,

$$\mathbf{k}_{n} = \mathbf{0} \tag{20}$$

and

$$\bar{\mathbf{k}}_{n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{21}$$

Observe that in the special application to the load flow solution, the equation corresponding to the slack bus may be eliminated.

Generator buses

Consider the equation of (17) corresponding to a voltage-controlled or generator bus g. Let

$$\tilde{S}_g \triangleq P_g + j|V_g| \tag{22}$$

hence

$$\delta\tilde{S}_g^* = \delta P_g - j\delta|V_g| \tag{23}$$

Since

$$2P_g = V_g I_g^* + V_g^* I_g \tag{24}$$

then

$$2\delta P_g = V_g \delta I_g^* + I_g^* \delta V_g + V_g^* \delta I_g + I_g \delta V_g^* \tag{25}$$

Using (5), we write I_g as

$$I_g = \mathbf{y}_g^{\mathrm{T}} \mathbf{V}_{\mathrm{M}} \tag{26}$$

where $\mathbf{y}_g^{\mathrm{T}}$ represents the corresponding row of the bus admittance matrix $\mathbf{Y}_{\mathrm{T}}$, hence

$$\delta I_g = \mathbf{y}_g^{\mathrm{T}} \delta \mathbf{V}_{\mathrm{M}} + \mathbf{V}_{\mathrm{M}}^{\mathrm{T}} \delta \mathbf{y}_g \tag{27}$$

Also,

$$\delta|V_g| = \delta(V_g V_g^*)^{1/2} = (V_g \delta V_g^* + V_g^* \delta V_g)/(2|V_g|) \tag{28}$$

Using (25)–(28), it is straightforward to show that $\delta\tilde{S}_g^*$ of (23) is given by

$$\delta\tilde{S}_g^* = \mathbf{k}_g^{\mathrm{T}} \delta \mathbf{V}_{\mathrm{M}} + \bar{\mathbf{k}}_g^{\mathrm{T}} \delta \mathbf{V}_{\mathrm{M}}^* + V_g^* \mathbf{V}_{\mathrm{M}}^{\mathrm{T}} \delta \mathbf{y}_g/2 + V_g \mathbf{V}_{\mathrm{M}}^{*\mathrm{T}} \delta \mathbf{y}_g^*/2 \tag{29}$$

where

$$\mathbf{k}_g \triangleq (V_g^*/2)\mathbf{y}_g + [\mathbf{y}_g^{*\mathrm{T}} \mathbf{V}_{\mathrm{M}}^*/2 - jV_g^*/(2|V_g|)]\boldsymbol{\mu}_g \tag{30}$$

and

$$\bar{\mathbf{k}}_g \triangleq (V_g/2)\mathbf{y}_g^* + [\mathbf{y}_g^{\mathrm{T}} \mathbf{V}_{\mathrm{M}}/2 - jV_g/(2|V_g|)]\boldsymbol{\mu}_g \tag{31}$$

and where $\boldsymbol{\mu}_g$ is a column vector of unity gth element and zero other elements. Using (29), the equation of (17) corresponding to the gth bus is replaced by

$$\mathbf{k}_g^{\mathrm{T}} \delta \mathbf{V}_{\mathrm{M}} + \bar{\mathbf{k}}_g^{\mathrm{T}} \delta \mathbf{V}_{\mathrm{M}}^* = d_g \tag{32}$$

where

$$d_g = \delta P_g - j\delta|V_g| - V_g^* \mathbf{V}_{\mathrm{M}}^{\mathrm{T}} \delta \mathbf{y}_g/2 - V_g \mathbf{V}_{\mathrm{M}}^{*\mathrm{T}} \delta \mathbf{y}_g^*/2 \tag{33}$$

Standard complex form

We write (17), including (19) for slack bus and (33) for generator buses, in the form

$$\mathbf{K}\delta\mathbf{V}_{\mathrm{M}} + \bar{\mathbf{K}}\delta\mathbf{V}_{\mathrm{M}}^* = \mathbf{d} \tag{34}$$

Note that the elements of $\delta\mathbf{V}_{\mathrm{M}}$ and $\delta\mathbf{V}_{\mathrm{M}}^*$, namely, δV_i and δV_i^*, $i = 1, \ldots, n$ can be replaced by the relative quantities $\delta V_i/|V_i|$ and $\delta V_i^*/|V_i|$, respectively. In this case the elements k_{ij} and $\bar{k}_{ij}$ of the ith row of the coefficient matrices $\mathbf{K}$ and $\bar{\mathbf{K}}$ are replaced by $|V_j|k_{ij}$ and $|V_j|\bar{k}_{ij}$, respectively. Note also that we could equally well specify $|V_g|^2$ instead of $|V_g|$ for a generator bus. In this case $|V_g|^2$ replaces $|V_g|$ in (22) as a control variable and the required modifications for subsequent derivation can be performed in a straightforward manner.

MODES OF FORMULATION

In the previous section, we have considered the complex formulation of power system equations. We shall exploit this formulation to derive compact forms of sensitivity expressions. In this section, we investigate, via suitable transformations, the relationship between the complex formulation and other formulations. This investigation provides the possibility of formulating the adjoint equations to be solved in the same mode as the original load flow problem. Hence, the available Jacobian of the load flow may be used in solving the adjoint system.

Transformation for rectangular formulation

We define the transformation matrix

$$\mathbf{L}^{q} \triangleq \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_1^* \\ \mathbf{L}_2 & \mathbf{L}_2^* \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{j} & \mathbf{j} \end{bmatrix} \tag{35}$$

where $\mathbf{1}$ is the identity matrix of order n and

$$\mathbf{j} \triangleq j\mathbf{1} \tag{36}$$

hence

$$(\mathbf{L}^{q})^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{j} \\ \mathbf{1} & -\mathbf{j} \end{bmatrix} \tag{37}$$

n denoting the number of buses in the power network. It follows, using

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{j} & \mathbf{j} \end{bmatrix}\begin{bmatrix} \zeta \\ \zeta^* \end{bmatrix} \tag{38}$$

and (7), that

$$\begin{bmatrix} \mathbf{V}_{M1} \\ \mathbf{V}_{M2} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_1^* \\ \mathbf{L}_2 & \mathbf{L}_2^* \end{bmatrix}\begin{bmatrix} \mathbf{V}_M \\ \mathbf{V}_M^* \end{bmatrix} \tag{39}$$

hence

$$\begin{bmatrix} \delta\mathbf{V}_{M1} \\ \delta\mathbf{V}_{M2} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_1^* \\ \mathbf{L}_2 & \mathbf{L}_2^* \end{bmatrix}\begin{bmatrix} \delta\mathbf{V}_M \\ \delta\mathbf{V}_M^* \end{bmatrix} \tag{40}$$

Using the perturbed form (40), it is straightforward to show that (34) can be written in the form

$$\begin{bmatrix} (\mathbf{K}_1+\bar{\mathbf{K}}_1) & (-\mathbf{K}_2+\bar{\mathbf{K}}_2) \\ -(\mathbf{K}_2+\bar{\mathbf{K}}_2) & (-\mathbf{K}_1+\bar{\mathbf{K}}_1) \end{bmatrix}\begin{bmatrix} \delta\mathbf{V}_{M1} \\ \delta\mathbf{V}_{M2} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ -\mathbf{d}_2 \end{bmatrix} \tag{41}$$

where we have set

$$\mathbf{K} = \mathbf{K}_1 + j\mathbf{K}_2 \tag{42}$$

$$\bar{\mathbf{K}} = \bar{\mathbf{K}}_1 + j\bar{\mathbf{K}}_2 \tag{43}$$

and

$$\mathbf{d} = \mathbf{d}_1 + j\mathbf{d}_2 \tag{44}$$

The $2n \times 2n$ matrix of coefficients in (41), denoted by $\mathbf{K}^{\mathrm{crt}}$, constitutes the well-known Jacobian matrix of the flow problem in rectangular form. Moreover, writing (34) in the form

$$[\mathbf{K}\ \bar{\mathbf{K}}]\begin{bmatrix} \delta\mathbf{V}_M \\ \delta\mathbf{V}_M^* \end{bmatrix} = \mathbf{d} \tag{45}$$

it follows that

$$[\mathbf{K}\ \bar{\mathbf{K}}]=[\mathbf{K}^{q}\ \bar{\mathbf{K}}^{q}]\begin{bmatrix}\mathbf{L}_1 & \mathbf{L}_1^*\\ \mathbf{L}_2 & \mathbf{L}_2^*\end{bmatrix} \tag{46}$$

where $\mathbf{K}^q$ and $\bar{\mathbf{K}}^q$ are formed from the Jacobian of (41) as

$$\mathbf{K}^q=(\mathbf{K}_1+\bar{\mathbf{K}}_1)+j(\mathbf{K}_2+\bar{\mathbf{K}}_2) \tag{47}$$

and

$$\bar{\mathbf{K}}^q=(-\mathbf{K}_2+\bar{\mathbf{K}}_2)-j(-\mathbf{K}_1+\bar{\mathbf{K}}_1) \tag{48}$$

Observe that (46) relates the Jacobian of the complex formulation (34) to the Jacobian of the rectangular formulation (41).

Transformation for polar formulation

For polar formulation, we set

$$V_i=|V_i|\angle\delta_i, \qquad i=1,\ldots,n \tag{49}$$

where V_i are elements of $\mathbf{V}_M$, and we define the vectors

$$|\mathbf{V}|\triangleq\begin{bmatrix}|V_1|\\ \vdots\\ |V_n|\end{bmatrix} \tag{50}$$

and

$$\boldsymbol{\delta}\triangleq\begin{bmatrix}\delta_1\\ \vdots\\ \delta_n\end{bmatrix} \tag{51}$$

Then, we define the transformation matrix

$$\mathbf{L}^p\triangleq\begin{bmatrix}\mathbf{L}_\delta & \mathbf{L}_\delta^*\\ \mathbf{L}_v & \mathbf{L}_v^*\end{bmatrix} \tag{52}$$

where $\mathbf{L}_\delta$, $\mathbf{L}_\delta^*$, $\mathbf{L}_v$ and $\mathbf{L}_v^*$ are diagonal matrices whose elements represent the formal partial derivatives $\partial\delta_i/\partial V_i$, $\partial\delta_i/\partial V_i^*$, $\partial|V_i|/\partial V_i$ and $\partial|V_i|/\partial V_i^*$, respectively, hence

$$\mathbf{L}_\delta\triangleq \text{diag}\{L_{\delta i}\} \tag{53}$$

and

$$\mathbf{L}_v\triangleq \text{diag}\{L_{vi}\} \tag{54}$$

where

$$L_{\delta i}=-j/(2V_i) \tag{55}$$

and

$$L_{vi}=V_i^*/(2|V_i|) \tag{56}$$

The inverse of $\mathbf{L}^p$ is given by

$$(\mathbf{L}^p)^{-1}=\begin{bmatrix}\tilde{\mathbf{L}}_\delta & \tilde{\mathbf{L}}_v\\ \tilde{\mathbf{L}}_\delta^* & \tilde{\mathbf{L}}_v^*\end{bmatrix} \tag{57}$$

where $\tilde{\mathbf{L}}_\delta$, $\tilde{\mathbf{L}}_\delta^*$, $\tilde{\mathbf{L}}_v$ and $\tilde{\mathbf{L}}_v^*$ are diagonal matrices whose elements are the partial derivatives $\partial V_i/\partial\delta_i$, $\partial V_i^*/\partial\delta_i$, $\partial V_i/\partial|V_i|$ and $\partial V_i^*/\partial|V_i|$, respectively, hence

$$\tilde{\mathbf{L}}_\delta \triangleq \text{diag}\,\{\tilde{L}_{\delta i}\} \tag{58}$$

and

$$\tilde{\mathbf{L}}_v \triangleq \text{diag}\,\{\tilde{L}_{vi}\} \tag{59}$$

where

$$\tilde{L}_{\delta i} = jV_i \tag{60}$$

and

$$\tilde{L}_{vi} = V_i/|V_i| \tag{61}$$

Similarly to (40), we may write

$$\begin{bmatrix} \delta\boldsymbol{\delta} \\ \delta|\mathbf{V}| \end{bmatrix} = \begin{bmatrix} \mathbf{L}_\delta & \mathbf{L}_\delta^* \\ \mathbf{L}_v & \mathbf{L}_v^* \end{bmatrix} \begin{bmatrix} \delta\mathbf{V}_{\mathrm{M}} \\ \delta\mathbf{V}_{\mathrm{M}}^* \end{bmatrix} \tag{62}$$

Using the perturbed form (62), it is straightforward to show that (34) can also be written in the form

$$\begin{bmatrix} \mathbf{K}_1^{\mathrm{p}} & \bar{\mathbf{K}}_1^{\mathrm{p}} \\ -\mathbf{K}_2^{\mathrm{p}} & -\bar{\mathbf{K}}_2^{\mathrm{p}} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\delta} \\ \delta|\mathbf{V}| \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ -\mathbf{d}_2 \end{bmatrix} \tag{63}$$

where we have set

$$\mathbf{K}^{\mathrm{p}} = \mathbf{K}_1^{\mathrm{p}} + j\mathbf{K}_2^{\mathrm{p}} \tag{64}$$

and

$$\bar{\mathbf{K}}^{\mathrm{p}} = \bar{\mathbf{K}}_1^{\mathrm{p}} + j\bar{\mathbf{K}}_2^{\mathrm{p}} \tag{65}$$

and where the matrices $\mathbf{K}^{\mathrm{p}}$ and $\bar{\mathbf{K}}^{\mathrm{p}}$ are related to $\mathbf{K}$ and $\bar{\mathbf{K}}$ through the relationship

$$[\mathbf{K}\ \ \bar{\mathbf{K}}] = [\mathbf{K}^{\mathrm{p}}\ \bar{\mathbf{K}}^{\mathrm{p}}] \begin{bmatrix} \mathbf{L}_\delta & \mathbf{L}_\delta^* \\ \mathbf{L}_v & \mathbf{L}_v^* \end{bmatrix} \tag{66}$$

The $2n \times 2n$ matrix of coefficients in (63), denoted by $\mathbf{K}^{\mathrm{plr}}$, constitutes the well-known Jacobian matrix of the load flow problem in polar form. Observe that (66) relates the Jacobian of the complex formulation (34) to the Jacobian of the polar formulation (63), where $\mathbf{K}^{\mathrm{p}}$ and $\bar{\mathbf{K}}^{\mathrm{p}}$ are formed directly from the Jacobian of (63).

At the end of this section, we illustrate the foregoing concepts by two simple examples.

Example 1

Consider first the 2-bus sample power system of Figure 1 which consists of a load bus and a slack bus. The solution of the load flow equations (13) is given by

$$V_1 = 0{\cdot}7352 - j\,0{\cdot}2041$$

and

$$S_2 = 5{\cdot}6705 + j\,1{\cdot}0706$$

Note that S_2 is the injected power at bus 2. The matrices $\mathbf{K}$ and $\bar{\mathbf{K}}$ of (45) are given by

$$\mathbf{K} = \begin{bmatrix} (8{\cdot}0852 - j\,12{\cdot}0097) & (-8{\cdot}4934 + j\,13{\cdot}4802) \\ 0 & 0 \end{bmatrix}$$

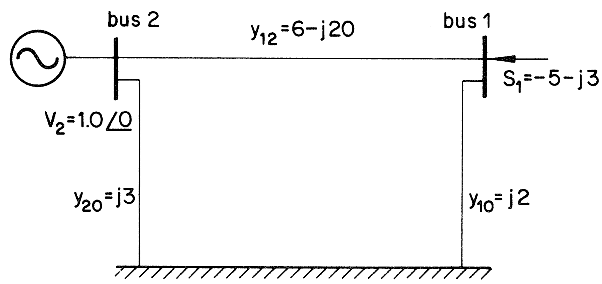

Figure 1. Two-bus load–slack sample power system

and

$$\bar{\mathbf{K}} = \begin{bmatrix} (-5{\cdot}2623 + j\,5{\cdot}5411) & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, using Cartesian co-ordinates, the matrix of coefficients of (41) has, using (42) and (43), the form

$$\mathbf{K}^{\text{crt}} = \begin{bmatrix} 2{\cdot}8229 & -8{\cdot}4934 & 17{\cdot}5508 & -13{\cdot}4802 \\ 0 & 1 & 0 & 0 \\ 6{\cdot}4686 & -13{\cdot}4802 & -13{\cdot}3475 & 8{\cdot}4934 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is the Jacobian of the load flow problem in Cartesian co-ordinates when the slack bus equations are included.

For the polar formulation, the matrices $\tilde{\mathbf{L}}_\delta$ and $\tilde{\mathbf{L}}_v$ of (57) are given by

$$\tilde{\mathbf{L}}_\delta = \begin{bmatrix} (0{\cdot}2041 + j\,0{\cdot}7352) & 0 \\ 0 & j \end{bmatrix}$$

and

$$\tilde{\mathbf{L}}_v = \begin{bmatrix} (0{\cdot}9636 - j\,0{\cdot}2675) & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, using (52), (57) and (66), the matrices $\mathbf{K}^{\text{p}}$ and $\bar{\mathbf{K}}^{\text{p}}$ are given by

$$\mathbf{K}^{\text{p}} = \begin{bmatrix} (13{\cdot}4802 + j\,8{\cdot}4934) & (-13{\cdot}4802 - j\,8{\cdot}4934) \\ 0 & -j \end{bmatrix}$$

and

$$\bar{\mathbf{K}}^{\text{p}} = \begin{bmatrix} (-1{\cdot}9745 - j\,9{\cdot}8031) & (-8{\cdot}4934 + j\,13{\cdot}4802) \\ 0 & 1 \end{bmatrix}$$

from which the matrix of coefficients of (63) has the form

$$\mathbf{K}^{\text{plr}} = \begin{bmatrix} 13{\cdot}4802 & -13{\cdot}4802 & -1{\cdot}9745 & -8{\cdot}4934 \\ 0 & 0 & 0 & 1 \\ -8{\cdot}4934 & 8{\cdot}4934 & 9{\cdot}8031 & -13{\cdot}4802 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is the Jacobian of the load flow problem in polar co-ordinates when the slack bus equations are included.

Example 2

Now, consider the 2-bus sample power system of Figure 2 which consists of a generator bus and a slack bus. The solution of the load flow equations (13) is given by

$$\delta_1 = -0{\cdot}1995 \text{ rad}$$

$$Q_1 = 1{\cdot}9929$$

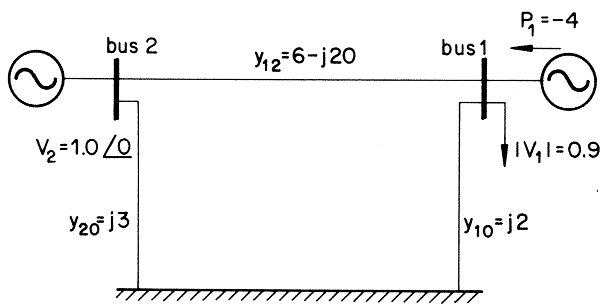

Figure 2. Two-bus generator–slack sample power system

and

$$S_2 = 4{\cdot}2742 - j\,1{\cdot}7131$$

The matrices **K** and $\bar{\mathbf{K}}$ of (45) are given by

$$\mathbf{K} = \begin{bmatrix} (2{\cdot}3920 - j\,9{\cdot}4199) & (-4{\cdot}4300 + j\,8{\cdot}2864) \\ 0 & 0 \end{bmatrix}$$

and

$$\bar{\mathbf{K}} = \begin{bmatrix} (2{\cdot}1938 + j\,8{\cdot}4398) & (-4{\cdot}4300 - j\,8{\cdot}2864) \\ 0 & 1 \end{bmatrix}.$$

Hence, using Cartesian co-ordinates, the matrix of coefficients of (41) has, using (42) and (43), the form

$$\mathbf{K}^{\text{crt}} = \begin{bmatrix} 4{\cdot}5858 & -8{\cdot}8600 & 17{\cdot}8597 & -16{\cdot}5729 \\ 0 & 1 & 0 & 0 \\ 0{\cdot}9802 & 0 & -0{\cdot}1982 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is the Jacobian of the load flow problem in Cartesian co-ordinates when the slack bus equations are included.

For the polar formulation, the matrices $\tilde{\mathbf{L}}_\delta$ and $\tilde{\mathbf{L}}_v$ of (57) are given by

$$\tilde{\mathbf{L}}_\delta = \begin{bmatrix} (0{\cdot}1784 + j\,0{\cdot}8822) & 0 \\ 0 & j \end{bmatrix}$$

and

$$\tilde{\mathbf{L}}_v = \begin{bmatrix} (0{\cdot}9802 - j\,0{\cdot}1982) & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, using (52), (57) and (66), the matrices $\mathbf{K}^{\mathrm{p}}$ and $\bar{\mathbf{K}}^{\mathrm{p}}$ are given by

$$\mathbf{K}^{\mathrm{p}} = \begin{bmatrix} 16{\cdot}5729 & -16{\cdot}5729 \\ 0 & -j \end{bmatrix}$$

and

$$\bar{\mathbf{K}}^{\mathrm{p}} = \begin{bmatrix} 0{\cdot}9556 - j\,1{\cdot}0 & -8{\cdot}8600 \\ 0 & 1 \end{bmatrix}$$

from which the matrix of coefficients of (63) has the form

$$\mathbf{K}^{\mathrm{plr}} = \begin{bmatrix} 16{\cdot}5729 & -16{\cdot}5729 & 0{\cdot}9556 & -8{\cdot}8600 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is the Jacobian of the load flow problem in polar co-ordinates when the slack bus equations are included.

COMPLEX ADJOINT ANALYSIS

In this section, we derive the required sensitivity expressions using the compact complex form (34). We exploit the relationships derived in the previous section to provide flexibility in solving the resulting adjoint system of equations in other modes of formulation. We have shown that, using Cartesian co-ordinates, (34) has the form

$$\mathbf{K}^{\mathrm{crt}} \begin{bmatrix} \delta \mathbf{V}_{\mathrm{M1}} \\ \delta \mathbf{V}_{\mathrm{M2}} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ -\mathbf{d}_2 \end{bmatrix} \tag{67}$$

where the $2n \times 2n$ matrix of coefficients $\mathbf{K}^{\mathrm{crt}}$ which constitutes the Jacobian matrix of the load flow problem in rectangular form is given from (41). Also, using polar co-ordinates, (34) has the form

$$\mathbf{K}^{\mathrm{plr}} \begin{bmatrix} \delta \boldsymbol{\delta} \\ \delta |\mathbf{V}| \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ -\mathbf{d}_2 \end{bmatrix} \tag{68}$$

where the $2n \times 2n$ matrix of coefficients $\mathbf{K}^{\mathrm{plr}}$ which constitutes the Jacobian matrix of the load flow problem in polar form is given from (63).

Standard complex form

We write (34) in the form

$$\begin{bmatrix} \mathbf{K} & \bar{\mathbf{K}} \\ \bar{\mathbf{K}}^* & \mathbf{K}^* \end{bmatrix} \begin{bmatrix} \delta \mathbf{V}_{\mathrm{M}} \\ \delta \mathbf{V}_{\mathrm{M}}^* \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ \mathbf{d}^* \end{bmatrix} \tag{69}$$

It can be shown[8] that the matrix of coefficients of (69), denoted by $\mathbf{K}^{\mathrm{cmp}}$, has the same rank as that of (67) and the system of equations (69) is consistent if and only if the system (67) is consistent.

For a real function f, we may write, using (3) and (4)

$$\delta f = [\hat{\boldsymbol{\mu}}^{\mathrm{T}} \;\; \hat{\boldsymbol{\mu}}^{*\mathrm{T}}] \begin{bmatrix} \delta \mathbf{V}_{\mathrm{M}} \\ \delta \mathbf{V}_{\mathrm{M}}^* \end{bmatrix} + \delta f_p \tag{70}$$

where we have defined

$$\hat{\boldsymbol{\mu}} \triangleq \frac{\partial f}{\partial \mathbf{V}_{\mathrm{M}}} \tag{71}$$

and used

$$\frac{\partial f}{\partial \mathbf{V}_{\mathrm{M}}} = \left(\frac{\partial f}{\partial \mathbf{V}_{\mathrm{M}}^*}\right)^* \tag{72}$$

δf_ρ denoting the change in f due to changes in other variables in terms of which f may be explicitly expressed. Hence, from (69)

$$\delta f = [\hat{\boldsymbol{\mu}}^{\mathrm{T}}\ \hat{\boldsymbol{\mu}}^{*\mathrm{T}}]\begin{bmatrix}\mathbf{K} & \bar{\mathbf{K}}\\ \bar{\mathbf{K}}^* & \mathbf{K}^*\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{d}\\ \mathbf{d}^*\end{bmatrix} + \delta f_\rho \tag{73}$$

or

$$\delta f = [\hat{\mathbf{V}}^{\mathrm{T}}\ \hat{\mathbf{V}}^{*\mathrm{T}}]\begin{bmatrix}\mathbf{d}\\ \mathbf{d}^*\end{bmatrix} + \delta f_\rho \tag{74}$$

where

$$\begin{bmatrix}\mathbf{K}^{\mathrm{T}} & \bar{\mathbf{K}}^{*\mathrm{T}}\\ \bar{\mathbf{K}}^{\mathrm{T}} & \mathbf{K}^{*\mathrm{T}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{V}}\\ \hat{\mathbf{V}}^*\end{bmatrix} = \begin{bmatrix}\hat{\boldsymbol{\mu}}\\ \hat{\boldsymbol{\mu}}^*\end{bmatrix} \tag{75}$$

or, simply

$$[\mathbf{K}^{\mathrm{T}}\ \bar{\mathbf{K}}^{*\mathrm{T}}]\begin{bmatrix}\hat{\mathbf{V}}\\ \hat{\mathbf{V}}^*\end{bmatrix} = \hat{\boldsymbol{\mu}} \tag{76}$$

Hence, the first-order change of the real function f and corresponding gradients can be evaluated by solving (75) and substituting into (74).

Cartesian co-ordinates

Similarly to (70), we may write, using the rectangular formulation

$$\delta f = [\hat{\boldsymbol{\mu}}_{\mathrm{r}}^{\mathrm{T}}\ \hat{\boldsymbol{\mu}}_{\mathrm{s}}^{\mathrm{T}}]\begin{bmatrix}\delta \mathbf{V}_{\mathrm{M1}}\\ \delta \mathbf{V}_{\mathrm{M2}}\end{bmatrix} + \delta f_\rho \tag{77}$$

where we have defined

$$\hat{\boldsymbol{\mu}}_{\mathrm{r}} \triangleq \frac{\partial f}{\partial \mathbf{V}_{\mathrm{M1}}} \tag{78}$$

and

$$\hat{\boldsymbol{\mu}}_{\mathrm{s}} \triangleq \frac{\partial f}{\partial \mathbf{V}_{\mathrm{M2}}} \tag{79}$$

Hence, from (41)

$$\delta f = [\hat{\mathbf{V}}_{\mathrm{r}}^{\mathrm{T}}\ \hat{\mathbf{V}}_{\mathrm{s}}^{\mathrm{T}}]\begin{bmatrix}\mathbf{d}_1\\ -\mathbf{d}_2\end{bmatrix} + \delta f_\rho \tag{80}$$

where

$$\begin{bmatrix}(\mathbf{K}_1+\bar{\mathbf{K}}_1)^{\mathrm{T}} & -(\mathbf{K}_2+\bar{\mathbf{K}}_2)^{\mathrm{T}}\\ (-\mathbf{K}_2+\bar{\mathbf{K}}_2)^{\mathrm{T}} & (-\mathbf{K}_1+\bar{\mathbf{K}}_1)^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{V}}_{\mathrm{r}}\\ \hat{\mathbf{V}}_{\mathrm{s}}\end{bmatrix} = \begin{bmatrix}\hat{\boldsymbol{\mu}}_{\mathrm{r}}\\ \hat{\boldsymbol{\mu}}_{\mathrm{s}}\end{bmatrix} \tag{81}$$

Observe that the matrix of coefficients of (81) is the transpose of the Jacobian matrix of the load flow problem in rectangular form (67).

Theorem 1

(a) The solution vectors $\hat{\mathbf{V}}_r$ and $\hat{\mathbf{V}}_s$ of the adjoint system of equations (81) are given by

$$\hat{\mathbf{V}}_r = 2\,\mathrm{Re}\,\{\hat{\mathbf{V}}\}$$

and

$$\hat{\mathbf{V}}_s = 2\,\mathrm{Im}\,\{\hat{\mathbf{V}}\}$$

where $\hat{\mathbf{V}}$ is given from (75).

(b) The RHS vectors $\hat{\boldsymbol{\mu}}_r$ and $\hat{\boldsymbol{\mu}}_s$ of the adjoint system of equations (81) are given by

$$\hat{\boldsymbol{\mu}} = \mathbf{L}_1^T\hat{\boldsymbol{\mu}}_r + \mathbf{L}_2^T\hat{\boldsymbol{\mu}}_s$$

where $\hat{\boldsymbol{\mu}}$ is given by (71) and $\mathbf{L}_1$ and $\mathbf{L}_2$ are given by (35).

Proof. Comparing (74) and (80), and using (66), we get

$$\hat{\mathbf{V}} = (\hat{\mathbf{V}}_r + j\hat{\mathbf{V}}_s)/2 \tag{82}$$

From (82), the first part of the theorem is proved. Now, multiplying (81) from the left by the transpose of $\mathbf{L}^q$ of (35) and using the relation

$$2\begin{bmatrix} (\mathbf{K}_1+\bar{\mathbf{K}}_1)^T & -(\mathbf{K}_2+\bar{\mathbf{K}}_2)^T \\ (-\mathbf{K}_2+\bar{\mathbf{K}}_2)^T & (-\mathbf{K}_1+\bar{\mathbf{K}}_1)^T \end{bmatrix} = \begin{bmatrix} \mathbf{K}^{qT} & \mathbf{K}^{q*T} \\ \bar{\mathbf{K}}^{qT} & \bar{\mathbf{K}}^{q*T} \end{bmatrix}\begin{bmatrix} \mathbf{1} & \mathbf{j} \\ \mathbf{1} & -\mathbf{j} \end{bmatrix} \tag{83}$$

it follows from (46) and (82) that

$$\begin{bmatrix} \mathbf{K}^T & \bar{\mathbf{K}}^{*T} \\ \bar{\mathbf{K}}^T & \mathbf{K}^{*T} \end{bmatrix}\begin{bmatrix} \hat{\mathbf{V}} \\ \hat{\mathbf{V}}^* \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1^T & \mathbf{L}_2^T \\ \mathbf{L}_1^{*T} & \mathbf{L}_2^{*T} \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{\mu}}_r \\ \hat{\boldsymbol{\mu}}_s \end{bmatrix} \tag{84}$$

hence, from (75)

$$\begin{bmatrix} \hat{\boldsymbol{\mu}} \\ \hat{\boldsymbol{\mu}}^* \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1^T & \mathbf{L}_2^T \\ \mathbf{L}_1^{*T} & \mathbf{L}_2^{*T} \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{\mu}}_r \\ \hat{\boldsymbol{\mu}}_s \end{bmatrix} \tag{85}$$

or, simply

$$\hat{\boldsymbol{\mu}} = [\mathbf{L}_1^T \;\; \mathbf{L}_2^T]\begin{bmatrix} \hat{\boldsymbol{\mu}}_r \\ \hat{\boldsymbol{\mu}}_s \end{bmatrix} \qquad \square \tag{86}$$

The relationship (86) could also be derived by applying, formally, the chain rule of differentiation using the definitions (71), (78) and (79).

Observe that equation (82) relates the solution of the adjoint system (81) to that of (76), and equation (86) relates the RHS of (81) to that of (76).

Polar co-ordinates

Using the polar formulation, we may write

$$\delta f = [\hat{\boldsymbol{\mu}}_\sigma^T \;\; \hat{\boldsymbol{\mu}}_v^T]\begin{bmatrix} \delta\boldsymbol{\delta} \\ \delta|\mathbf{V}| \end{bmatrix} + \delta f_\rho \tag{87}$$

where we have defined

$$\hat{\boldsymbol{\mu}}_\sigma \triangleq \frac{\partial f}{\partial \boldsymbol{\delta}} \tag{88}$$

and

$$\hat{\boldsymbol{\mu}}_v \triangleq \frac{\partial f}{\partial |\mathbf{V}|} \tag{89}$$

Hence, from (68)

$$\delta f = [\hat{\mathbf{V}}_\delta^{\mathrm{T}} \ \hat{\mathbf{V}}_v^{\mathrm{T}}]\begin{bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{bmatrix} + \delta f_p \tag{90}$$

where

$$\begin{bmatrix}\mathbf{K}_1^{\mathrm{pT}} & -\mathbf{K}_2^{\mathrm{pT}}\\ \bar{\mathbf{K}}_1^{\mathrm{pT}} & -\bar{\mathbf{K}}_2^{\mathrm{pT}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{V}}_\delta\\ \hat{\mathbf{V}}_v\end{bmatrix} = \begin{bmatrix}\hat{\boldsymbol{\mu}}_\delta\\ \hat{\boldsymbol{\mu}}_v\end{bmatrix} \tag{91}$$

The matrix of coefficients of (91) is the transpose of the Jacobian matrix of the load flow problem in polar form.

Theorem 2

(a) The solution vectors $\hat{\mathbf{V}}_\delta$ and $\hat{\mathbf{V}}_v$ of the adjoint system of equations (91) are given by

$$\hat{\mathbf{V}}_\delta = 2\,\mathrm{Re}\,\{\hat{\mathbf{V}}\}$$

and

$$\hat{\mathbf{V}}_v = 2\,\mathrm{Im}\,\{\hat{\mathbf{V}}\}$$

where $\hat{\mathbf{V}}$ is given from (75).

(b) The RHS vectors $\hat{\boldsymbol{\mu}}_\delta$ and $\hat{\boldsymbol{\mu}}_v$ of the adjoint system of equations (91) are given by

$$\hat{\boldsymbol{\mu}} = \mathbf{L}_\delta^{\mathrm{T}}\hat{\boldsymbol{\mu}}_\delta + \mathbf{L}_v^{\mathrm{T}}\hat{\boldsymbol{\mu}}_v$$

where $\hat{\boldsymbol{\mu}}$ is given by (71) and $\mathbf{L}_\delta$ and $\mathbf{L}_v$ are given by (53) and (54).

Proof. Comparing (74) and (90), and using (44), we get

$$\hat{\mathbf{V}} = (\hat{\mathbf{V}}_\delta + j\hat{\mathbf{V}}_v)/2 \tag{92}$$

From (92), the first part of the theorem is proved. Now, multiplying (90) from the left by the transpose of $\mathbf{L}^{\mathrm{p}}$ of (52) and using the relation

$$2\begin{bmatrix}\mathbf{K}_1^{\mathrm{pT}} & -\mathbf{K}_2^{\mathrm{pT}}\\ \bar{\mathbf{K}}_1^{\mathrm{pT}} & -\bar{\mathbf{K}}_2^{\mathrm{pT}}\end{bmatrix} = \begin{bmatrix}\mathbf{K}^{\mathrm{pT}} & \mathbf{K}^{\mathrm{p*T}}\\ \bar{\mathbf{K}}^{\mathrm{pT}} & \bar{\mathbf{K}}^{\mathrm{p*T}}\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{j}\\ \mathbf{1} & -\mathbf{j}\end{bmatrix} \tag{93}$$

it follows from (64) and (92) that

$$\begin{bmatrix}\mathbf{K}^{\mathrm{T}} & \bar{\mathbf{K}}^{*\mathrm{T}}\\ \bar{\mathbf{K}}^{\mathrm{T}} & \mathbf{K}^{*\mathrm{T}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{V}}\\ \hat{\mathbf{V}}^*\end{bmatrix} = \begin{bmatrix}\mathbf{L}_\delta^{\mathrm{T}} & \mathbf{L}_v^{\mathrm{T}}\\ \mathbf{L}_\delta^{*\mathrm{T}} & \mathbf{L}_v^{*\mathrm{T}}\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\mu}}_\delta\\ \hat{\boldsymbol{\mu}}_v\end{bmatrix} \tag{94}$$

hence, from (85)

$$\begin{bmatrix}\hat{\boldsymbol{\mu}}\\ \hat{\boldsymbol{\mu}}^*\end{bmatrix} = \begin{bmatrix}\mathbf{L}_\delta^{\mathrm{T}} & \mathbf{L}_v^{\mathrm{T}}\\ \mathbf{L}_\delta^{*\mathrm{T}} & \mathbf{L}_v^{*\mathrm{T}}\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\mu}}_\delta\\ \hat{\boldsymbol{\mu}}_v\end{bmatrix} \tag{95}$$

or, simply

$$\hat{\boldsymbol{\mu}} = [\mathbf{L}_\delta^{\mathrm{T}} \ \mathbf{L}_v^{\mathrm{T}}]\begin{bmatrix}\hat{\boldsymbol{\mu}}_\delta\\ \hat{\boldsymbol{\mu}}_v\end{bmatrix} \quad \square \tag{96}$$

Again, the relationship (96) could also be derived by applying, formally, the chain rule of differentiation using the definitions (71), (88) and (89).

Equation (92) relates the solution of the adjoint system (91) to that of (76), and equation (96) relates the RHS of (91) to that of (76).

Remarks

We remark that using (82) or (92), the adjoint system can be formulated and solved in a convenient mode, preferably the same formulation as the original load flow problem, and the first-order change of f and corresponding gradients may be derived compactly using the adjoint variables $\hat{\mathbf{V}}$. On the other hand, the relations (86) and (96) allow the use of more elegant formal derivatives which, in many cases, facilitate the formulation. For example, consider the function

$$f = \sigma |V_i - V_j|^2 = \sigma (V_i - V_j)(V_i^* - V_j^*) \tag{97}$$

where V_i and V_j are the ith and jth components of $\mathbf{V}_M$, respectively, and σ is a real scalar or variable. Note that f of (97) may represent, for example, the power loss in line ij. For the polar formulation, $\hat{\boldsymbol{\mu}}_v$ and $\hat{\boldsymbol{\mu}}_\delta$ of (91) are calculated as follows. The ith and jth components of $\hat{\boldsymbol{\mu}}_\delta$ and $\hat{\boldsymbol{\mu}}_v$ are given by

$$\hat{\mu}_{\delta i} = \sigma[-2(|V_i| \cos \delta_i - |V_j| \cos \delta_j)|V_i| \sin \delta_i + 2(|V_i| \sin \delta_i - |V_j| \sin \delta_j)|V_i| \cos \delta_i]$$

$$\hat{\mu}_{\delta j} = \sigma[2(|V_i| \cos \delta_i - |V_j| \cos \delta_j)|V_j| \sin \delta_j - 2(|V_i| \sin \delta_i - |V_j| \sin \delta_j)|V_j| \cos \delta_j]$$

$$\hat{\mu}_{vi} = \sigma[2(|V_i| \cos \delta_i - |V_j| \cos \delta_j) \cos \delta_i + 2(|V_i| \sin \delta_i - |V_j| \sin \delta_j) \sin \delta_i]$$

and

$$\hat{\mu}_{vj} = \sigma[2(|V_i| \cos \delta_i - |V_j| \cos \delta_j) \cos \delta_j - 2(|V_i| \sin \delta_i - |V_j| \sin \delta_j) \sin \delta_j]$$

All other components are zero. On the other hand, one may calculate

$$\hat{\boldsymbol{\mu}} = \sigma \begin{bmatrix} 0 \\ \vdots \\ (V_i^* - V_j^*) \\ \vdots \\ -(V_i^* - V_j^*) \\ \vdots \\ 0 \end{bmatrix}$$

and use (95) to calculate $\hat{\boldsymbol{\mu}}_v$ and $\hat{\boldsymbol{\mu}}_\delta$, where $(\mathbf{L}^{\mathrm{PT}})^{-1}$ is the transpose of $(\mathbf{L}^{\mathrm{P}})^{-1}$ of (57). In this example, the derivation of the formal derivatives is clearly easier.

We also remark that other forms of power flow equations can be handled in a similar way. The previous theorems can be easily generalized for other formulations provided that transformations similar to (35) and (52) are defined.

We illustrate the foregoing concepts by the two simple examples considered before.

Example 3

For the first system, as shown in Figure 1, consider the function

$$f = |V_1|^2 = V_1 V_1^*$$

From (71),

$$\hat{\boldsymbol{\mu}} = \begin{bmatrix} V_1^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0{\cdot}7352 + j\,0{\cdot}2041 \\ 0 \end{bmatrix}$$

and (76) has the solution

$$\hat{\mathbf{V}} = \begin{bmatrix} 0{\cdot}0562 + j\,0{\cdot}0892 \\ 1{\cdot}6788 + j\,0{\cdot}0 \end{bmatrix}$$

Also, for the polar formulation, we have from (88) and (89)

$$\hat{\boldsymbol{\mu}}_\delta = \mathbf{0}$$

and

$$\hat{\boldsymbol{\mu}}_v = \begin{bmatrix} 2|V_1| \\ 0 \end{bmatrix} = \begin{bmatrix} 1{\cdot}5261 \\ 0 \end{bmatrix}$$

and (91) has the solution

$$\hat{\mathbf{V}}_\delta = \begin{bmatrix} 0{\cdot}1123 \\ 3{\cdot}3577 \end{bmatrix}$$

and

$$\hat{\mathbf{V}}_v = \begin{bmatrix} 0{\cdot}1783 \\ 0 \end{bmatrix}$$

Note that the $\hat{\mathbf{V}}_\delta$ and $\hat{\mathbf{V}}_v$ obtained for the polar formulation and $\hat{\mathbf{V}}$ satisfy (92).

Example 4

For the second system, as shown in Figure 2, consider the function

$$f = \delta_1 = \tan^{-1}\left[\frac{V_1 - V_1^*}{j(V_1^* + V_1)}\right]$$

From (71)

$$\hat{\boldsymbol{\mu}} = \begin{bmatrix} -j/(2V_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0{\cdot}1101 - j\,0{\cdot}5445 \\ 0 \end{bmatrix}$$

and (76) has the solution

$$\hat{\mathbf{V}} = \begin{bmatrix} 0{\cdot}0302 - j\,0{\cdot}0288 \\ 0{\cdot}2673 + j\,0{\cdot}5 \end{bmatrix}$$

Also, for the polar formulation, we have from (88) and (89)

$$\hat{\boldsymbol{\mu}}_\delta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\hat{\boldsymbol{\mu}}_v = \mathbf{0}$$

and (91) has the solution

$$\hat{\mathbf{V}}_\delta = \begin{bmatrix} 0{\cdot}0603 \\ 0{\cdot}5346 \end{bmatrix}$$

and

$$\hat{\mathbf{V}}_v = \begin{bmatrix} -0{\cdot}0577 \\ 1{\cdot}0 \end{bmatrix}$$

Observe that the $\hat{\mathbf{V}}_\delta$ and $\hat{\mathbf{V}}_v$ obtained for the polar formulation and $\hat{\mathbf{V}}$ satisfy (92).

GRADIENT CALCULATIONS

In the previous section, we have derived the adjoint systems in different modes of formulation and investigated the relationships between the corresponding excitation and solution vectors. In power system studies such as contingency analysis, the first-order change of f is of prime interest. The first-order change

δf can be calculated from (74), (80) and (90). On the other hand, the derivatives of f w.r.t. control variables are required to be calculated, for example, in planning studies.

In the following, we consider the buses to be ordered such that subscripts $l=1, 2, \ldots, n_L$ identify load buses, $g=n_L+1, \ldots, n_L+n_G$ identify generator buses and $n=n_L+n_G+1$ identifies the slack bus.

The vector $\mathbf{d}$ of (34) is now partitioned into subvectors associated with the sets of load, generator and slack buses of appropriate dimension in the form

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_L \\ \mathbf{d}_G \\ d_n \end{bmatrix} \tag{98}$$

where $\mathbf{d}_L$ has elements d_l given from (18) by

$$d_l = \delta S_l^* - V_l^* \mathbf{V}_M^T \delta \mathbf{y}_l \tag{99}$$

$\mathbf{y}_l^T$ representing the corresponding row of the bus admittance matrix $\mathbf{Y}_T$, $\mathbf{d}_G$ has elements d_g given by (33) and d_n is δV_n^* from (19). Also, the vector $\hat{\mathbf{V}}$ of (74) is partitioned correspondingly in the form

$$\hat{\mathbf{V}} = \begin{bmatrix} \hat{\mathbf{V}}_L \\ \hat{\mathbf{V}}_G \\ \hat{V}_n \end{bmatrix} \tag{100}$$

Note that the above formulation leads to expressing the vector $\mathbf{d}$ solely in terms of variations in control variables, the gradients in terms of which can be obtained by writing (74) in the form

$$\delta f = \hat{\mathbf{V}}_L^T \mathbf{d}_L + \hat{\mathbf{V}}_G^T \mathbf{d}_G + \hat{V}_n d_n + \left(\frac{\partial f}{\partial \boldsymbol{\rho}}\right)^T \delta \boldsymbol{\rho} + \hat{\mathbf{V}}_L^{*T} \mathbf{d}_L^* + \hat{\mathbf{V}}_G^{*T} \mathbf{d}_G^* + \hat{V}_n^* d_n^* + \left(\frac{\partial f}{\partial \boldsymbol{\rho}}\right)^{*T} \delta \boldsymbol{\rho}^* \tag{101}$$

The first term of (101) is given, using (99), by

$$\begin{aligned} \hat{\mathbf{V}}_L^T \mathbf{d}_L &= \sum_{l=1}^{n_L} \hat{V}_l d_l \\ &= \sum_{l=1}^{n_L} (\hat{V}_l \delta S_l^*) - \sum_{l=1}^{n_L} \sum_{m=1}^{n} (\hat{V}_l V_l^* V_m \delta Y_{lm}) \end{aligned} \tag{102}$$

where Y_{lm} is an element of $\mathbf{Y}_T$, which is assumed, for simplicity, to be a symmetric admittance matrix (the case of an unsymmetric admittance matrix can be analysed in a similar straightforward way), or

$$\hat{\mathbf{V}}_L^T \mathbf{d}_L = \sum_{l=1}^{n_L} (\hat{V}_l \delta S_l^*) + \sum_{l=1}^{n_L} \sum_{\substack{m=1 \\ m \neq l}}^{n} \hat{V}_l V_l^* (V_m - V_l) \delta y_{lm} - \sum_{l=1}^{n_L} (\hat{V}_l V_l^* V_l \delta y_{l0}) \tag{103}$$

where y_{lm} denotes the admittance of line lm connecting load bus l with bus m $(=l, g$ or $n)$, and y_{l0} is the shunt admittance at bus l. The second term of (101) is given, using (33) by

$$\begin{aligned} \hat{\mathbf{V}}_G^T \mathbf{d}_G &= \sum_{g=n_L+1}^{n-1} \hat{V}_g d_g \\ &= \sum_{g=n_L+1}^{n-1} \hat{V}_g (\delta P_g - j\delta |V_g|) - \sum_{g=n_L+1}^{n-1} \sum_{m=1}^{n} \hat{V}_g \operatorname{Re}\{V_g^* V_m \delta Y_{gm}\} \end{aligned} \tag{104}$$

or

$$\hat{\mathbf{V}}_G^T \mathbf{d}_G = \sum_{g=n_L+1}^{n-1} \hat{V}_g (\delta P_g - j\delta |V_g|) + \sum_{g=n_L+1}^{n-1} \sum_{\substack{m=1 \\ m \neq g}}^{n} \hat{V}_g \operatorname{Re}\{V_g^* (V_m - V_g) \delta y_{gm}\} - \sum_{g=n_L+1}^{n-1} \hat{V}_g \operatorname{Re}\{V_g^* V_g \delta y_{g0}\} \tag{105}$$

where y_{gm} denotes the admittance of line gm connecting generator bus g with bus m ($=l$, g or n), and y_{g0} is the shunt admittance at bus g. The third term of (101) is given, using (19), by

$$\hat{V}_n d_n = \hat{V}_n \delta V_n^* \tag{106}$$

The fourth term of (101) is simply the first-order change of f due to changes in other variables $\boldsymbol{\rho}$ in terms of which the function f may be explicitly expressed.

Equations (103)–(106) provide useful information for gradient evaluation since they provide direct expressions w.r.t. the control variables of interest. The derivatives of the function f w.r.t. the control variables are obtained as follows, where we temporarily assume that $\boldsymbol{\rho}$ does not contain such control variables.

Load bus control variables

From (103) and its complex conjugate, the derivatives of f w.r.t. the demand S_l and S_l^* at load bus l is given by

$$\frac{\mathrm{d}f}{\mathrm{d}S_l} = \hat{V}_l^* \tag{107}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}S_l^*} = \hat{V}_l \tag{108}$$

Generator bus control variables

From (105) and its complex conjugate, the derivatives of f w.r.t. the real generated power P_g and the voltage magnitude $|V_g|$ at generator bus g are given by

$$\frac{\mathrm{d}f}{\mathrm{d}\tilde{S}_g} = \hat{V}_g^* \tag{109}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}\tilde{S}_g^*} = \hat{V}_g \tag{110}$$

where $\tilde{S}_g$ is given by (22).

Slack bus control variables

From (106) and its complex conjugate, the derivatives of f w.r.t. the slack bus voltage V_n and V_n^* are given by

$$\frac{\mathrm{d}f}{\mathrm{d}V_n} = \hat{V}_n^* \tag{111}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}V_n^*} = \hat{V}_n \tag{112}$$

In practice, the phase angle of the slack bus voltage is set to zero as a reference angle. Hence, the slack bus has only one effective real control variable.

Line control variables

The derivatives of f w.r.t. line control variables $y_{ij'}$ can be obtained from (103) and (105) and their complex conjugate as follows. For $y_{ll'}$, between load buses l and l', we have from (103) and its complex conjugate

$$\frac{\mathrm{d}f}{\mathrm{d}y_{ll'}} = (\hat{V}_l V_l^* - \hat{V}_{l'} V_{l'}^*)(V_{l'} - V_l) \tag{113}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{ll'}^*} = (\hat{V}_l^* \hat{V}_l - \hat{V}_{l'}^* V_{l'})(V_{l'}^* - V_l^*) \tag{114}$$

For y_{l0} between load bus l and ground, we have from (103) and its complex conjugate

$$\frac{\mathrm{d}f}{\mathrm{d}y_{l0}} = -\hat{V}_l V_l^* V_l \tag{115}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{l0}^*} = -\hat{V}_l^* V_l V_l^* \tag{116}$$

For $y_{gg'}$ between generator buses g and g', we have from (105) and its complex conjugate

$$\frac{\mathrm{d}f}{\mathrm{d}y_{gg'}} = (\hat{V}_{g1} V_g^* - \hat{V}_{g'1} V_{g'}^*)(V_{g'} - V_g) \tag{117}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{gg'}^*} = (\hat{V}_{g1} V_g - \hat{V}_{g'1} V_{g'})(V_{g'}^* - V_g^*) \tag{118}$$

where

$$\hat{V}_m = \hat{V}_{m1} + j\hat{V}_{m2} \tag{119}$$

and m is a bus index. For y_{g0} between generator bus g and ground, we have from (105)

$$\frac{\mathrm{d}f}{\mathrm{d}y_{g0}} = \frac{\mathrm{d}f}{\mathrm{d}y_{g0}^*} = -\hat{V}_{g1} V_g^* V_g \tag{120}$$

For y_{lg} between load bus l and generator bus g, we have from (103) and (105) and their complex conjugate

$$\frac{\mathrm{d}f}{\mathrm{d}y_{lg}} = (\hat{V}_{g1} V_g^* - \hat{V}_l V_l^*)(V_l - V_g) \tag{121}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{lg}^*} = (\hat{V}_{g1} V_g - \hat{V}_l^* V_l)(V_l^* - V_g^*) \tag{122}$$

For y_{ln} between load bus l and the slack bus n, we have from (103) and its complex conjugate

$$\frac{\mathrm{d}f}{\mathrm{d}y_{ln}} = \hat{V}_l V_l^* (V_n - V_l) \tag{123}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{ln}^*} = \hat{V}_l^* V_l (V_n^* - V_l^*) \tag{124}$$

Finally, for y_{gn} between generator bus g and the slack bus n, we have from (105) and its complex conjugate

$$\frac{\mathrm{d}f}{\mathrm{d}y_{gn}} = \hat{V}_{g1} V_g^*(V_n - V_g) \tag{125}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{gn}^*} = \hat{V}_{g1} V_g(V_n^* - V_g^*) \tag{126}$$

Special considerations

If $\boldsymbol{\rho}$ of (101) contains some of the above control variables, the partial derivatives of f w.r.t. appropriate control variables must be added to the expressions obtained.

When any of the control variables u_k is a function of some real design variables we write

$$\delta u_k = \sum_i \frac{\partial u_k}{\partial \zeta_{ki}} \Delta\zeta_{ki} \tag{127}$$

where ζ_{ki} is the ith design variable associated with u_k and $\Delta\zeta_{ki}$ denotes the change in ζ_{ki}. Hence,

$$\frac{\mathrm{d}f}{\mathrm{d}\zeta_{ki}} = \frac{\mathrm{d}f}{\mathrm{d}u_k}\frac{\partial u_k}{\partial \zeta_{ki}} \tag{128}$$

The control variables associated with other power system components, e.g. transformers, which are represented in the bus admittance matrix $\mathbf{Y}_\mathrm{T}$ can be easily considered. The corresponding sensitivity expressions may be derived in a similar straightforward manner.

Equations (107)–(118) and (120)–(126) compactly define the required formal derivatives of the real function f w.r.t. complex control variables. In practice, gradients w.r.t. real and imaginary parts of the defined control variables are of direct interest. These gradients are simply obtained from

$$\frac{\mathrm{d}f}{\mathrm{d}u_{k1}} = 2\,Re\left\{\frac{\mathrm{d}f}{\mathrm{d}u_k}\right\} \tag{129}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}u_{k2}} = -2\,\mathrm{Im}\left\{\frac{\mathrm{d}f}{\mathrm{d}u_k}\right\} \tag{130}$$

where the complex control variable u_k is given by

$$u_k = u_{k1} + ju_{k2} \tag{131}$$

Table I summarizes the derived expressions of function gradients w.r.t. real control variables of practical interest.

Example 5

Using the values of $\hat{\mathbf{V}}$ obtained, we have for the first system

$$\frac{\mathrm{d}f}{\mathrm{d}P_1} = 2\hat{V}_{11} = 0{\cdot}1123$$

$$\frac{\mathrm{d}f}{\mathrm{d}Q_1} = 2\hat{V}_{12} = 0{\cdot}1783$$

$$\frac{\mathrm{d}f}{\mathrm{d}V_{21}} = 2\hat{V}_{21} = 3{\cdot}3577$$

$$\frac{\mathrm{d}f}{\mathrm{d}B_{10}} = 2|V_1|^2\hat{V}_{12} = 0{\cdot}1038$$

$$\frac{\mathrm{d}f}{\mathrm{d}G_{12}} = 2\,\mathrm{Re}\,\{\hat{V}_1 V_1^*(V_2 - V_1)\} = -0.0192$$

Table I. Derivatives of a real function f w.r.t. control variables

Control variable	Description	Derivative
P_l	demand real power	$2\hat{V}_{l1}$
Q_l	demand reactive power	$2\hat{V}_{l2}$
P_g	generator real power	$2\hat{V}_{g1}$
$\|V_g\|$	generator bus voltage magnitude	$2\hat{V}_{g2}$
V_{n1}	real component of slack bus voltage	$2\hat{V}_{n1}$
$G_{ll'}$	conductance between two load buses	$2\,\mathrm{Re}\,\{(\hat{V}_l V_l^* - \hat{V}_{l'} V_{l'}^*)(V_{l'} - V_l)\}$
$B_{ll'}$	susceptance between two load buses	$-2\,\mathrm{Im}\,\{(\hat{V}_l V_l^* - \hat{V}_{l'} V_{l'}^*)(V_{l'} - V_l)\}$
G_{l0}	shunt conductance of a load bus	$-2\|V_l\|^2 \hat{V}_{l1}$
B_{l0}	shunt susceptance of a load bus	$2\|V_l\|^2 \hat{V}_{l2}$
$G_{gg'}$	conductance between two generator buses	$2\,\mathrm{Re}\,\{(\hat{V}_{g1} V_g^* - \hat{V}_{g'1} V_{g'}^*)(V_{g'} - V_g)\}$
$B_{gg'}$	susceptance between two generator buses	$-2\,\mathrm{Im}\,\{(\hat{V}_{g1} V_g^* - \hat{V}_{g'1} V_{g'}^*)(V_{g'} - V_g)\}$
G_{g0}	shunt conductance of a generator bus	$-2\|V_g\|^2 \hat{V}_{g1}$
B_{g0}	shunt susceptance of a generator bus	0
G_{lg}	conductance between load and generator buses	$2\,\mathrm{Re}\,\{(\hat{V}_{g1} V_g^* - \hat{V}_l V_l^*)(V_l - V_g)\}$
B_{lg}	susceptance between load and generator buses	$-2\,\mathrm{Im}\,\{(\hat{V}_{g1} V_g^* - \hat{V}_l V_l^*)(V_l - V_g)\}$
G_{ln}	conductance between load and slack buses	$2\,\mathrm{Re}\,\{\hat{V}_l V_l^*(V_n - V_l)\}$
B_{ln}	susceptance between load and slack buses	$-2\,\mathrm{Im}\,\{\hat{V}_l V_l^*(V_n - V_l)\}$
G_{gn}	conductance between generator and slack buses	$2\hat{V}_{g1}\,\mathrm{Re}\,\{V_g^*(V_n - V_g)\}$
B_{gn}	susceptance between generator and slack buses	$-2\hat{V}_{g1}\,\mathrm{Im}\,\{V_g^* V_n\}$

and

$$\frac{\mathrm{d}f}{\mathrm{d}B_{12}} = -2\,\mathrm{Im}\,\{\hat{V}_1 V_1^*(V_2 - V_1)\} = -0{\cdot}0502$$

where $G_{mm'}$ and $B_{mm'}$ denote, respectively, the conductance and susceptance of line mm' connecting buses m and m', $m' = 0$ denotes the ground.

Example 6

For the second system, we have

$$\frac{\mathrm{d}f}{\mathrm{d}P_1} = 2\hat{V}_{11} = 0{\cdot}0603$$

$$\frac{\mathrm{d}f}{\mathrm{d}|V_1|} = 2\hat{V}_{12} = -0{\cdot}0577$$

$$\frac{\mathrm{d}f}{\mathrm{d}V_{21}} = 2\hat{V}_{21} = 0{\cdot}5346$$

$$\frac{\mathrm{d}f}{\mathrm{d}B_{10}} = 0{\cdot}0$$

$$\frac{\mathrm{d}f}{\mathrm{d}G_{12}} = 2\hat{V}_{11}\,\mathrm{Re}\,\{V_1^*(V_2 - V_1)\} = 0.0044$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}B_{12}} = -2\hat{V}_{11}\,\mathrm{Im}\,\{V_1^* V_2\} = -0{\cdot}0108$$

The gradients obtained can be easily checked by small perturbations about the base case values.

SENSITIVITY OF COMPLEX FUNCTIONS

In the previous sections, we have derived the required sensitivity expressions and gradients for a general real function. The relationships between different modes of formulation have been investigated and expressions relating the RHS and solution vector of corresponding adjoint systems have been derived.

The sensitivities of a general complex function can be obtained using the previous formulae derived simply by considering the real and imaginary parts separately. In this case, only the RHS of the adjoint system of equations has to be changed. In other words, only one forward and one backward substitution are required for each real function, provided that the LU factors of the formed matrix of coefficients are stored and that the base case point remains unchanged.

In this section, we show how the compact complex formulation can be exploited to formulate the adjoint system corresponding to a general complex function and to derive the required sensitivities. The relationships between different modes of formulation are again investigated for the complex function case.

For a complex function f, we may write, using (3)

$$\delta f = [\hat{\boldsymbol{\mu}}^{\mathrm{T}} \ \hat{\hat{\boldsymbol{\mu}}}^{\mathrm{T}}] \begin{bmatrix} \delta \mathbf{V}_{\mathrm{M}} \\ \delta \mathbf{V}_{\mathrm{M}}^{*} \end{bmatrix} + \delta f_{p} \tag{132}$$

where we have defined

$$\hat{\boldsymbol{\mu}} \triangleq \frac{\partial f}{\partial \mathbf{V}_{\mathrm{M}}} \tag{133}$$

and

$$\hat{\hat{\boldsymbol{\mu}}} \triangleq \frac{\partial f}{\partial \mathbf{V}_{\mathrm{M}}^{*}} \tag{134}$$

δf_p being the change in f due to changes in other variables in terms of which f may be explicitly expressed. Hence, from (69)

$$\delta f = [\hat{\boldsymbol{\mu}}^{\mathrm{T}} \ \hat{\hat{\boldsymbol{\mu}}}^{\mathrm{T}}] \begin{bmatrix} \mathbf{K} & \bar{\mathbf{K}} \\ \bar{\mathbf{K}}^{*} & \mathbf{K}^{*} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d} \\ \mathbf{d}^{*} \end{bmatrix} + \delta f_{p} \tag{135}$$

or

$$\delta f = [\hat{\mathbf{V}}^{\mathrm{T}} \ \hat{\hat{\mathbf{V}}}^{\mathrm{T}}] \begin{bmatrix} \mathbf{d} \\ \mathbf{d}^{*} \end{bmatrix} + \delta f_{p} \tag{136}$$

where

$$\begin{bmatrix} \mathbf{K}^{\mathrm{T}} & \bar{\mathbf{K}}^{*\mathrm{T}} \\ \bar{\mathbf{K}}^{\mathrm{T}} & \mathbf{K}^{*\mathrm{T}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{V}} \\ \hat{\hat{\mathbf{V}}} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\mu}} \\ \hat{\hat{\boldsymbol{\mu}}} \end{bmatrix} \tag{137}$$

which represents the adjoint system of equations to be solved. The first-order change of the complex function f can be evaluated by solving (137) and substituting into (136).

The relationships between the adjoint solutions of different modes of formulation are derived as follows. Let

$$f = f_1 + j f_2 \tag{138}$$

hence

$$\delta f = \delta f_1 + j \delta f_2 \tag{139}$$

and let $\hat{\mathbf{V}}_{\mathrm{r}}^{1}$ and $\hat{\mathbf{V}}_{\mathrm{s}}^{1}$ be the solution vector of the adjoint system (81) using Cartesian co-ordinates for the real function f_1. Similarly, let $\hat{\mathbf{V}}_{\mathrm{r}}^{2}$ and $\hat{\mathbf{V}}_{\mathrm{s}}^{2}$ be the solution vector of (81) for the real function f_2. Hence, using (80) and (136), one may write

$$\hat{\mathbf{V}}^{\mathrm{T}} \mathbf{d} + \hat{\hat{\mathbf{V}}}^{\mathrm{T}} \mathbf{d}^{*} = (\hat{\mathbf{V}}_{\mathrm{r}}^{1\mathrm{T}} \mathbf{d}_1 - \hat{\mathbf{V}}_{\mathrm{s}}^{1\mathrm{T}} \mathbf{d}_2) + j(\hat{\mathbf{V}}_{\mathrm{r}}^{2\mathrm{T}} \mathbf{d}_1 - \hat{\mathbf{V}}_{\mathrm{s}}^{2\mathrm{T}} \mathbf{d}_2) \tag{140}$$

hence, from (44),

$$\hat{\mathbf{V}} = (\hat{\mathbf{V}}_{\mathrm{r}}^1 - \hat{\mathbf{V}}_{\mathrm{s}}^2)/2 + j(\hat{\mathbf{V}}_{\mathrm{s}}^1 + \hat{\mathbf{V}}_{\mathrm{r}}^2)/2 \tag{141}$$

and

$$\hat{\hat{\mathbf{V}}} = (\hat{\mathbf{V}}_{\mathrm{r}}^1 + \hat{\mathbf{V}}_{\mathrm{s}}^2)/2 + j(-\hat{\mathbf{V}}_{\mathrm{s}}^1 + \hat{\mathbf{V}}_{\mathrm{r}}^2)/2 \tag{142}$$

Equations (141) and (142) relate the solutions of the adjoint system (81) for both f_1 and f_2 to the solution of (137) for the complex function f.

Similarly, let $\hat{\mathbf{V}}_\delta^1$ and $\hat{\mathbf{V}}_v^1$ be the solution vector of the adjoint system (91) using polar co-ordinates for the real function f_1. Also, let $\hat{\mathbf{V}}_\delta^2$ and $\hat{\mathbf{V}}_v^2$ be the solution vector of (91) for the real function f_2. Hence, using (90) and (136), one may write

$$\hat{\mathbf{V}}^{\mathrm{T}}\mathbf{d} + \hat{\hat{\mathbf{V}}}^{\mathrm{T}}\mathbf{d}^* = (\hat{\mathbf{V}}_\delta^{1\mathrm{T}}\mathbf{d}_1 - \hat{\mathbf{V}}_v^{1\mathrm{T}}\mathbf{d}_2) + j(\hat{\mathbf{V}}_\delta^{2\mathrm{T}}\mathbf{d}_1 - \hat{\mathbf{V}}_v^{2\mathrm{T}}\mathbf{d}_2) \tag{143}$$

hence, from (44)

$$\hat{\mathbf{V}} = (\hat{\mathbf{V}}_\delta^1 - \hat{\mathbf{V}}_v^2)/2 + j(\hat{\mathbf{V}}_v^1 + \hat{\mathbf{V}}_\delta^2)/2 \tag{144}$$

and

$$\hat{\hat{\mathbf{V}}} = (\hat{\mathbf{V}}_\delta^1 + \hat{\mathbf{V}}_v^2)/2 + j(-\hat{\mathbf{V}}_v^1 + \hat{\mathbf{V}}_\delta^2)/2 \tag{145}$$

Equations (144) and (145) relate the solutions of the adjoint system (91) for both f_1 and f_2 to the solution of (137) for the complex function f.

For gradient calculations, we proceed as before and use the partitioned forms (98), (100) and

$$\hat{\hat{\mathbf{V}}} = \begin{bmatrix} \hat{\hat{\mathbf{V}}}_{\mathrm{L}} \\ \hat{\hat{\mathbf{V}}}_{\mathrm{G}} \\ \hat{\hat{V}}_n \end{bmatrix} \tag{146}$$

and we write (74) in the form

$$\delta f = \hat{\mathbf{V}}_{\mathrm{L}}^{\mathrm{T}}\mathbf{d}_{\mathrm{L}} + \hat{\mathbf{V}}_{\mathrm{G}}^{\mathrm{T}}\mathbf{d}_{\mathrm{G}} + \hat{V}_n d_n + \left(\frac{\partial f}{\partial \boldsymbol{\rho}}\right)^{\mathrm{T}} \delta\boldsymbol{\rho} + \hat{\hat{\mathbf{V}}}_{\mathrm{L}}^{\mathrm{T}}\mathbf{d}_{\mathrm{L}}^* + \hat{\hat{\mathbf{V}}}_{\mathrm{G}}^{\mathrm{T}}\mathbf{d}_{\mathrm{G}}^* + \hat{\hat{V}}_n d_n^* + \left(\frac{\partial f}{\partial \boldsymbol{\rho}^*}\right)^{\mathrm{T}} \delta\boldsymbol{\rho} \tag{147}$$

The first, second and third terms of (147) are given by (103), (105) and (106), respectively. The fifth term of (147) is given, using (99), by

$$\hat{\hat{\mathbf{V}}}_{\mathrm{L}}^{\mathrm{T}}\mathbf{d}_{\mathrm{L}}^* = \sum_{l=1}^{n_{\mathrm{L}}} (\hat{\hat{V}}_l \delta S_l) + \sum_{l=1}^{n_{\mathrm{L}}} \sum_{\substack{m=1 \\ m\neq l}}^{n} \hat{\hat{V}}_l V_l (V_m^* - V_l^*)\delta y_{lm}^* - \sum_{l=1}^{n_{\mathrm{L}}} \hat{\hat{V}}_l V_l V_l^* \delta y_{l0}^* \tag{148}$$

Also, the sixth term of (147) is given, using (33) by

$$\hat{\hat{\mathbf{V}}}_{\mathrm{G}}^{\mathrm{T}}\mathbf{d}_{\mathrm{G}}^* = \sum_{g=n_{\mathrm{L}}+1}^{n-1} \hat{\hat{V}}_g(\delta P_g + j\delta|V_g|) + \sum_{g=n_{\mathrm{L}}+1}^{n-1} \sum_{\substack{m=1 \\ m\neq g}}^{n} \hat{\hat{V}}_g \operatorname{Re}\{V_g^*(V_m - V_g)\delta y_{gm}\} - \sum_{g=n_{\mathrm{L}}+1}^{n-1} \hat{\hat{V}}_g \operatorname{Re}\{V_g^* V_g \delta y_{g0}\} \tag{149}$$

and the seventh term of (147) is given, using (19) by

$$\hat{\hat{V}}_n d_n^* = \hat{\hat{V}}_n \delta V_n \tag{150}$$

Equations (103), (105), (106), (148), (149) and (150) provide useful information for gradient evaluation of the complex function f w.r.t. the control variables of interest. Under the assumption that $\boldsymbol{\rho}$ does not contain such control variables, the derivatives of the complex function f are obtained as follows.

Load bus control variables

From (103) and (148), the derivatives of f w.r.t. the demand S_l and S_l^* at load bus l is given by

$$\frac{\mathrm{d}f}{\mathrm{d}S_l} = \hat{\hat{V}}_l \tag{151}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}S_l^*} = \hat{V}_l \tag{152}$$

Generator bus control variables

From (105) and (149), the derivatives of f w.r.t. the generator control variables are given by

$$\frac{\mathrm{d}f}{\mathrm{d}\tilde{S}_g} = \hat{\hat{V}}_g \tag{153}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}\tilde{S}_g^*} = \hat{V}_g \tag{154}$$

where $\tilde{S}_g$ is given by (22).

Slack bus control variables

From (106) and (150), the derivatives of f w.r.t. the slack bus voltage V_n and V_n^* are given by

$$\frac{\mathrm{d}f}{\mathrm{d}V_n} = \hat{\hat{V}}_n \tag{155}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}V_n^*} = \hat{V}_n \tag{156}$$

Line control variables

The derivatives of f w.r.t. line control variables y_{ij} can be obtained from (103), (105), (148) and (149) as follows. For $y_{ll'}$ between load buses l and l', we have from (103) and (148)

$$\frac{\mathrm{d}f}{\mathrm{d}y_{ll'}} = (\hat{V}_l V_l^* - \hat{V}_{l'} V_{l'}^*)(V_{l'} - V_l) \tag{157}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{ll'}^*} = (\hat{\hat{V}}_l V_l - \hat{\hat{V}}_{l'} V_{l'})(V_{l'}^* - V_l^*) \tag{158}$$

For y_{l0} between load bus l and ground, we have from (103) and (148)

$$\frac{\mathrm{d}f}{\mathrm{d}y_{l0}} = -\hat{V}_l V_l^* V_l \tag{159}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{l0}^*} = -\hat{\hat{V}}_l V_l V_l^* \tag{160}$$

For $y_{gg'}$ between generator buses g and g', we have from (105) and (149)

$$\frac{\mathrm{d}f}{\mathrm{d}y_{gg'}}=\tfrac{1}{2}[(\hat{V}_g+\hat{\hat{V}}_g)V_g^*-(\hat{V}_{g'}+\hat{\hat{V}}_{g'})V_g^*](V_{g'}-V_g) \tag{161}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{gg'}^*}=\tfrac{1}{2}[(\hat{V}_g+\hat{\hat{V}}_g)V_g-(\hat{V}_{g'}+\hat{\hat{V}}_{g'})V_{g'}](V_{g'}^*-V_g^*) \tag{162}$$

For y_{g0} between generator bus g and ground, we have from (105) and (149)

$$\frac{\mathrm{d}f}{\mathrm{d}y_{g0}}=\frac{\mathrm{d}f}{\mathrm{d}y_{g0}^*}=-\tfrac{1}{2}(\hat{V}_g+\hat{\hat{V}}_g)V_g^*V_g \tag{163}$$

For y_{lg} between load bus l and generator bus g, we have from (103), (105), (148) and (149)

$$\frac{\mathrm{d}f}{\mathrm{d}y_{lg}}=[\tfrac{1}{2}(\hat{V}_g+\hat{\hat{V}}_g)V_g^*-\hat{V}_lV_l^*](V_l-V_g) \tag{164}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{lg}^*}=[\tfrac{1}{2}(\hat{V}_g+\hat{\hat{V}}_g)V_g-\hat{\hat{V}}_lV_l](V_l^*-V_g^*) \tag{165}$$

For y_{ln} between load bus l and the slack bus n, we have from (103) and (148)

$$\frac{\mathrm{d}f}{\mathrm{d}y_{ln}}=\hat{V}_lV_l^*(V_n-V_l) \tag{166}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{ln}^*}=\hat{\hat{V}}_lV_l(V_n^*-V_l^*) \tag{167}$$

Finally, for y_{gn} between generator bus g and the slack bus n, we have from (105) and (149)

$$\frac{\mathrm{d}f}{\mathrm{d}y_{gn}}=\tfrac{1}{2}(\hat{V}_g+\hat{\hat{V}}_g)V_g^*(V_n-V_g) \tag{168}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}y_{gn}^*}=\tfrac{1}{2}(\hat{V}_g+\hat{\hat{V}}_g)V_g(V_n^*-V_g^*) \tag{169}$$

Remarks

If $\boldsymbol{\rho}$ of (147) contains any of the above control variables, the partial derivatives of f w.r.t. appropriate control variables must be added to the expressions (151)–(169).

Equations (151)–(169) compactly define the required formal derivatives of the complex function f w.r.t. complex control variables. The gradients of f w.r.t. real and imaginary parts of the control variables are obtained using

$$\frac{\mathrm{d}f}{\mathrm{d}u_{k1}}=\frac{\mathrm{d}f}{\mathrm{d}u_k}+\frac{\mathrm{d}f}{\mathrm{d}u_k^*} \tag{170}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}u_{k2}}=j\left(\frac{\mathrm{d}f}{\mathrm{d}u_k}-\frac{\mathrm{d}f}{\mathrm{d}u_k^*}\right) \tag{171}$$

where u_k is given by (131).

Expressions of forms (170) and (171) can be directly obtained from (151)–(169).

Example 7

Now, we consider the first 2-bus system and the complex function

$$f=V_1=V_{11}+jV_{12}$$

Using Cartesian co-ordinates, the adjoint system solutions for V_{11} and V_{12} are given, respectively, by

$$\hat{\mathbf{V}}_r^1=\begin{bmatrix}0{\cdot}0883\\2{\cdot}3144\end{bmatrix}$$

$$\hat{\mathbf{V}}_s^1=\begin{bmatrix}0{\cdot}1161\\0{\cdot}2041\end{bmatrix}$$

$$\mathbf{V}_r^2=\begin{bmatrix}0{\cdot}0428\\0{\cdot}1117\end{bmatrix}$$

and

$$\mathbf{V}_s^2=\begin{bmatrix}-0{\cdot}0187\\0{\cdot}7352\end{bmatrix}$$

hence, from (141) and (142)

$$\hat{\mathbf{V}}=\begin{bmatrix}0{\cdot}0535+j\,0{\cdot}0794\\0{\cdot}7896+j\,0{\cdot}1579\end{bmatrix}$$

and

$$\hat{\hat{\mathbf{V}}}=\begin{bmatrix}0{\cdot}0348-j\,0{\cdot}0366\\1{\cdot}5248-j\,0{\cdot}0462\end{bmatrix}$$

The derivatives of f w.r.t. control variables are calculated, using the derived expressions, as follows. For S_1

$$\frac{\mathrm{d}f}{\mathrm{d}S_1}=\hat{\hat{V}}_1=0{\cdot}0348-j\,0{\cdot}0366$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}S_1^*}=\hat{V}_1=0{\cdot}0535+j\,0{\cdot}0794$$

hence, from (170) and (171)

$$\frac{\mathrm{d}f}{\mathrm{d}P_1}=0{\cdot}0883-j\,0{\cdot}0428$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}Q_1}=0{\cdot}1161-j\,0{\cdot}0187$$

For V_2,

$$\frac{df}{dV_2} = \hat{\hat{V}}_2 = 1 \cdot 5248 - j\,0 \cdot 0462$$

and

$$\frac{df}{dV_2^*} = \hat{V}_2 = 0 \cdot 7896 + j\,0 \cdot 1579$$

hence, from (170)

$$\frac{df}{dV_{21}} = 2 \cdot 3144 + j\,0 \cdot 1117$$

For y_{10},

$$\frac{df}{dy_{10}} = -|V_1|^2 \hat{V}_1 = -0 \cdot 0311 - j\,0 \cdot 0462$$

and

$$\frac{df}{dy_{10}^*} = -|V_1|^2 \hat{\hat{V}}_1 = -0 \cdot 0203 + j\,0 \cdot 0213$$

hence, from (170) and (171)

$$\frac{df}{dG_{10}} = -0 \cdot 0514 - j\,0 \cdot 0249$$

and

$$\frac{df}{dB_{10}} = 0 \cdot 0676 - j\,0 \cdot 0109$$

For y_{12},

$$\frac{df}{dy_{12}} = \hat{V}_1 V_1^*(V_2 - V_1) = -0 \cdot 0080 + j\,0 \cdot 0231$$

and

$$\frac{df}{dy_{12}^*} = \hat{\hat{V}}_1 V_1(V_2^* - V_1^*) = -0 \cdot 0022 - j\,0 \cdot 0127$$

hence, from (170) and (171)

$$\frac{df}{dG_{12}} = -0 \cdot 0102 + j\,0 \cdot 0104$$

and

$$\frac{df}{dB_{12}} = -0 \cdot 0358 - j\,0 \cdot 0059$$

APPLICATIONS TO A 6-BUS SAMPLE POWER SYSTEM

In this section, we present some of the numerical results obtained for a 6-bus power system[13] using the sensitivity formulae derived in the paper.

The system consists of three specified load buses ($l=1, 2, 3$), two generator buses ($g=4, 5$), the slack bus ($n=6$) and eight transmission lines ($t=7, \ldots, 14$). The single line diagram for this system is shown in Figure 3. The line and bus data are shown, respectively, in Tables II and III. All values shown are in per units. The application of the adjoint network approach results in the load flow solution shown in Table IV.

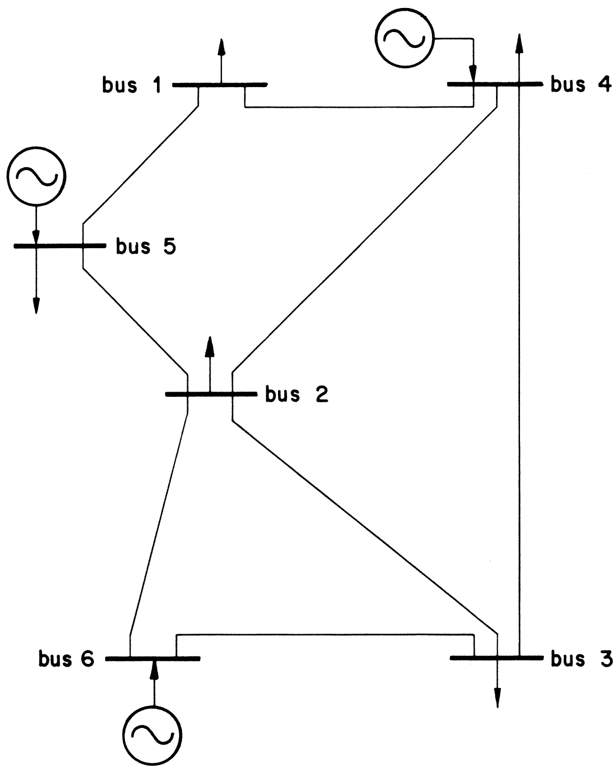

Figure 3. Six-bus sample power system

Examples of sensitivities of bus states, namely $|V_1|$, Q_4, δ_1 and δ_4 w.r.t. system bus and line control variables are shown in Tables V–VIII. The estimated effects of the line and circuit outages on the different states, based on first-order changes, are also shown.

Observe that the sensitivities w.r.t. non-existing elements, e.g. the shunt parameters in Tables V–VIII can be evaluated as well.

Although the sensitivities of a general function can be evaluated using the same adjoint matrix of coefficients at the load flow solution and by defining the RHS of the adjoint equations corresponding to the function considered, these sensitivities can also be obtained, directly, using the results of Tables V–VIII. For example, consider the function

$$f=|I_{14}|^2=|V_1-V_4|^2|Y_{14}|^2 \tag{172}$$

which may denote the loading of line 1, 4. The sensitivity of this function w.r.t. a control variable u_k is given by

$$\frac{\mathrm{d}f}{\mathrm{d}u_k}=\frac{\partial f}{\partial u_k}+2|V_1-V_4||Y_{14}|^2\frac{\partial|V_1-V_4|}{\partial u_k} \tag{173}$$

Table II. Line data for 6-bus power system

Line No.	Terminal buses	Resistance R_t (pu)	Reactance X_t (pu)	Number of lines
1	1, 4	0·05	0·20	1
2	1, 5	0·025	0·10	2
3	2, 3	0·10	0·40	1
4	2, 4	0·10	0·40	1
5	2, 5	0·05	0·20	1
6	2, 6	0·01875	0·075	4
7	3, 4	0·15	0·60	1
8	3, 6	0·0375	0·15	2

Table III. Bus data for 6-bus power system

Bus index, m	Bus type	P_m (pu)	Q_m (pu)	$\lvert V_m\rvert \angle\delta_m$ (pu)
1	load	−2·40	0	— $\angle -$
2	load	−2·40	0	— $\angle -$
3	load	−1·60	−0·40	— $\angle -$
4	generator	−0·30	—	1·02 $\angle -$
5	generator	1·25	—	1·04 $\angle -$
6	slack	—	—	1·04 $\angle 0$

Table IV. Load flow solution of 6-bus power system

Load buses	Generator buses	Slack buses
$V_1=0{\cdot}9787\angle -0{\cdot}6602$	$Q_4=0{\cdot}7866$, $\delta_4=-0{\cdot}5566$	$P_6=6{\cdot}1298$, $Q_6=1{\cdot}3546$
$V_2=0{\cdot}9633\angle -0{\cdot}2978$	$Q_5=0{\cdot}9780$, $\delta_5=-0{\cdot}4740$	
$V_3=0{\cdot}9032\angle -0{\cdot}3036$		

which, when substituting values at the load flow solution and noting that $|V_4|$ is constant, reduces to

$$\frac{\mathrm{d}f}{\mathrm{d}u_k}=\frac{\partial f}{\partial u_k}-1{\cdot}6871\frac{\partial|V_1|}{\partial u_k}-4{\cdot}8588\frac{\partial\delta_1}{\partial u_k}+4{\cdot}8588\frac{\partial\delta_4}{\partial u_k}$$

Now, let u_k denote the conductance of line 2, 4. Hence, from Tables V, VII and VIII, we get

$$\frac{\mathrm{d}f}{\mathrm{d}G_{24}}=-0{\cdot}0324$$

Table V. Six-bus system: sensitivities of $|V_1|$

Line quantities

Line	Total derivatives: Conductance	Total derivatives: Susceptance	Contingency effect: Outage of one line	Contingency effect: Outage of circuit
1, 4	−0·006326	−0·005283	0·017421	0·017421
1, 5	−0·011838	−0·008884	0·027880	0·055760
2, 3	0·000027	−0·000012	0·000044	0·000044
2, 4	−0·000207	−0·000597	−0·001282	0·001282
2, 5	0·000163	0·000294	−0·001192	−0·001192
2, 6	−0·000002	0·000039	−0·000123	−0·000494
3, 4	−0·000265	−0·000443	0·000591	0·000591
3, 6	−0·000017	−0·000120	0·000362	0·000724

Load bus quantities—total derivatives

Bus	Real power	Reactive power	Shunt conductance	Shunt susceptance
1	0·029522	0·070273	−0·028275	−0·067306
2	−0·000131	−0·000005	0·000122	0·000005
3	0·000378	0·000169	−0·000308	−0·000138

Generator bus quantities—total derivatives

Bus	Voltage magnitude	Real power	Shunt conductance	Shunt susceptance
4	0·357365	0·002243	−0·002334	0·0
5	0·732004	−0·001804	0·001951	0·0

Similarly, if u_k denotes the susceptance of line 2, 4, we get

$$\frac{df}{dB_{24}} = -0{\cdot}0932$$

The effect of line 2, 4 outage on the function considered can be estimated using the relation

$$\delta f = -\frac{df}{dG_{24}} G_{24} - \frac{df}{dB_{24}} B_{24} \tag{174}$$

where we have set the changes in line conductance and susceptance, respectively, to $-G_{24}$ and $-B_{24}$. Substituting the values of G_{24} (=0·5882) and B_{24} (=−2·3529) in (174), we get

$$\delta f = 0{\cdot}019 - 0{\cdot}219 = 0{\cdot}200$$

which is identical to the result presented in the Tellegen's theorem approach of[13] where the function $f = |I_{14}|^2$ was considered, directly, in the adjoint simulation without state transformations.

CONCLUSIONS

A unified study for the class of adjoint network approaches to power system sensitivity analysis which exploits the Jacobian matrix of the load flow solution has been presented. Generalized sensitivity expressions which are easily derived, compactly described and effectively used for calculating first-order changes and gradients of functions of interest have been obtained. These generalized sensitivity expressions are common to all modes of formulation, e.g. polar and Cartesian.

Table VI. Six-bus system: sensitivities of Q_4

Line quantities				
	Total derivatives		Contingency effect	
Line	Conductance	Susceptance	Outage of one line	Outage of circuit
1, 4	−0·056140	−0·044515	0·143437	0·143437
1, 5	0·065943	0·060168	−0·205565	−0·411130
2, 3	0·000236	0·004289	−0·009954	−0·009954
2, 4	0·256340	0·022413	0·098051	0·098051
2, 5	−0·015503	0·028010	−0·150048	−0·150048
2, 6	0·046139	0·039093	−0·086459	−0·345835
3, 4	0·243148	−0·031249	0·144371	0·144371
3, 6	0·062174	0·056610	−0·128837	−0·257674
Load bus quantities—total derivatives				
Bus	Real power	Reactive power	Shunt conductance	Shunt susceptance
1	−0·457852	−0·358531	0·438519	0·343391
2	−0·115872	−0·168723	0·107512	0·156551
3	−0·127525	−0·258052	0·104029	0·210506
Generator bus quantities—total derivatives				
Bus	Voltage magnitude	Real power	Shunt conductance	Shunt susceptance
4	7·51274	−0·550625	0·572870	0·0
5	−4·66462	−0·219233	0·237122	0·0

A first step towards deriving these generalized sensitivity expressions has been performed where we have used a special complex notation to compactly describe the transformations relating different ways of formulating power network equations to a standard complex form. This special notation and the derived transformations have been used to effectively derive the required sensitivity expressions only by matrix manipulations.

The use of these generalized sensitivity expressions requires only the solution of an adjoint system of linear equations, the matrix of coefficients of which is simply the transpose of the Jacobian matrix of the load flow solution in any mode of formulation. These generalized sensitivity expressions are applicable to both real and complex modes of performance functions as well as the control variables defined in a particular study.

Table VII. Six-bus system: sensitivities of δ_1

Line quantities

Line	Total derivatives		Contingency effect	
	Conductance	Susceptance	Outage of one line	Outage of circuit
1, 4	0·001197	−0·010358	0·050152	0·050152
1, 5	−0·004594	−0·016180	0·070737	0·141473
2, 3	−0·001609	0·000178	−0·001366	−0·001366
2, 4	−0·010354	−0·031650	0·068981	0·068981
2, 5	−0·011653	−0·025839	0·107885	0·107885
2, 6	−0·005283	−0·025867	0·077008	0·308030
3, 4	−0·020029	−0·036084	0·054530	0·054530
3, 6	−0·002723	−0·019449	0·058881	0·117762

Load bus quantities—total derivatives

Bus	Real power	Reactive power	Shunt conductance	Shunt susceptance
1	0·309969	−0·002339	−0·296880	0·002240
2	0·085296	0·026631	−0·079143	−0·024709
3	0·061420	0·027332	−0·050104	−0·022297

Generator bus quantities—total derivatives

Bus	Voltage magnitude	Real power	Shunt conductance	Shunt susceptance
4	0·192793	0·208858	−0·217296	0·0
5	0·271949	0·223549	−0·241790	0·0

Table VIII. Six-bus system: sensitivities of δ_4

Line quantities

Line	Total derivatives		Contingency effect	
	Conductance	Susceptance	Outage of one-line	Outage of circuit
1, 4	−0·006119	0·005953	−0·035213	−0·035213
1, 5	−0·004959	−0·011033	0·046087	0·092174
2, 3	−0·000725	−0·000212	0·000073	0·000073
2, 4	−0·017094	−0·051045	0·110050	0·110050
2, 5	−0·006343	−0·016282	0·069157	0·069157
2, 6	−0·005360	−0·024608	0·072997	0·291989
3, 4	−0·028650	−0·050482	0·067952	0·067952
3, 6	−0·003276	−0·023336	0·017661	0·035321

Table VIII (*cont.*)

Load bus quantities—total derivatives

Bus	Real power	Reactive power	Shunt conductance	Shunt susceptance
1	0·222333	0·005176	−0·212945	−0·004957
2	0·081031	0·026460	−0·075185	−0·024551
3	0·073688	0·032826	−0·060111	−0·026778

Generator bus quantities—total derivatives

Bus	Voltage magnitude	Real power	Shunt conductance	Shunt susceptance
4	−0·047087	0·281747	−0·293130	0·0
5	0·272518	0·164929	−0·178387	0·0

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REFERENCES

1. B. Stott, 'Review of load-flow calculation methods', *Proc. IEEE*, **62**, 916–929 (1974).
2. J. W. Bandler and M. A. El-Kady, 'A new method for computerized solution of power flow equations', *IEEE Trans. Power Apparatus and Systems*, **PAS-101**, 1–10 (1982).
3. J. Peschon, D. S. Piercy, W. F. Tinney and O. J. Tveit, 'Sensitivity in power systems', *IEEE Trans. Power Apparatus and Systems*, **PAS-87**, 1687–1696 (1968).
4. J. W. Bandler and M. A. El-Kady, 'Power network sensitivity analysis and formulation simplified', *IEEE Trans. Automatic Control*, **AC-26**, 773–775 (1981).
5. J. W. Bandler and M. A. El-Kady, 'Sensitivity analyses and reduced gradient evaluation for optimization of power systems', *Proc. European Conf. Circuit Theory and Design*, The Hague, Netherlands, 69–74 (1981).
6. H. W. Dommel and W. F. Tinney, 'Optimal power flow solutions', *IEEE Trans. Power Apparatus and Systems*, **PAS-87**, 1866–1876 (1968).
7. A. M. Sasson and H. M. Merrill, 'Some applications of optimization techniques to power system problems', *Proc. IEEE*, **62**, 969–972 (1974).
8. S. W. Director and R. L. Sullivan, 'A tableau approach to power system analysis and design', *Proc. IEEE Int. Symp. Circuits and Systems*, New York, 605–609 (1978).
9. J. W. Bandler and M. A. El-Kady, 'A complex Lagrangian approach with applications to power network sensitivity analysis', *IEEE Trans. Circuits and Systems*, **CAS-29**, 1–6 (1982).
10. J. W. Bandler and M. A. El-Kady, 'The method of complex Lagrange multipliers with applications', *Proc. IEEE Int. Symp. Circuits and Systems*, Chicago, IL, 773–777 (1981).
11. M. A. El-Kady, 'A unified approach to generalized network sensitivities with applications to power system analysis and planning', *Ph.D. Thesis*, McMaster University, Hamilton, Canada, 1980.
12. B. A. Fuks, *Introduction to the Theory of Analytic Functions of Several Complex Variables*, American Mathematical Society, Providence, Rhode Island, 1963.
13. J. W. Bandler and M. A. El-Kady, 'A unified approach to power system sensitivity analysis and planning, part II: special class of adjoint systems', *Proc. IEEE Int. Symp. Circuits and Systems* Houston, TX 688–692 (1980).