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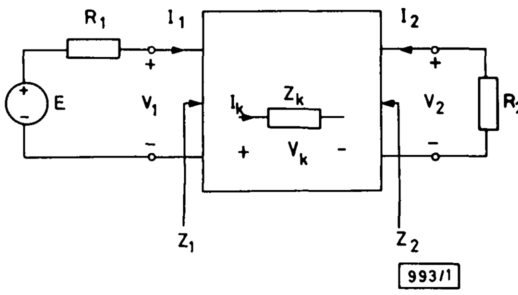
PROOF AND EXTENSION OF GENERAL SENSITIVITY FORMULAS FOR LOSSLESS TWO-PORTS

Indexing term: Circuit theory and design

An elegant and simple proof of an important result in sensitivity analysis of lossless two-ports stated by Orchard, Temes and Cataltepe is presented using Tellegen's theorem. Our derivation is extended to the computation of group delay.

A general formula for sensitivities of lossless two-ports, requiring only one analysis of the original circuit, has been presented by Orchard, Temes and Cataltepe.[1] It can be extended to yield results of significant value. The proof of the key formula, their eqn. 4, was indicated by the authors to be involved and lengthy. Different, elegant approaches to simple proofs of the theory, however, have been found. One of them, based on Tellegen's theorem,[2] is presented here. An extension of the formula to the computation of group delay of a general lossless two-port is included here.



By Tellegen's theorem, for the two-port shown in Fig. 1, we have

$$-\frac{\partial V_1(s)}{\partial \phi} I_1(-s) - \frac{\partial I_1(s)}{\partial \phi} V_1(-s) - \frac{\partial V_2(s)}{\partial \phi} I_2(-s)$$

$$-\frac{\partial I_2(s)}{\partial \phi} V_2(-s) + \sum_k \left[\frac{\partial V_k(s)}{\partial \phi} I_k(-s) + \frac{\partial I_k(s)}{\partial \phi} V_k(-s)\right] = 0 \quad (1)$$

where s denotes the complex frequency, ϕ is an arbitrary variable in the two-port and the summation is taken over all the internal branches. For the input port we have

$$V_1(s) = E - I_1(s)R_1 \quad (2)$$

and, for constant E and R_1,

$$\frac{\partial V_1(s)}{\partial \phi} = -R_1 \frac{\partial I_1(s)}{\partial \phi} \quad (3)$$

Therefore, for the first two terms of eqn. 1,

$$\frac{\partial V_1(s)}{\partial \phi} I_1(-s) + \frac{\partial I_1(s)}{\partial \phi} V_1(-s) = \frac{\partial I_1(s)}{\partial \phi}[E - 2I_1(-s)R_1]$$

$$= E\rho_1(-s)\frac{\partial I_1(s)}{\partial \phi} \quad (4)$$

where the input reflection coefficient is defined as

$$\rho_1(s) \triangleq \frac{Z_1(s) - R_1}{Z_1(s) + R_1} = 1 - \frac{2R_1}{Z_1(s) + R_1} = 1 - \frac{2R_1 I_1(s)}{E} \quad (5)$$

For the output port we have

$$V_2(s) = -R_2 I_2(s) \quad (6)$$

so, for constant R_2,

$$\frac{\partial V_2(s)}{\partial \phi} = -R_2 \frac{\partial I_2(s)}{\partial \phi} \quad (7)$$

Hence, for the third and fourth terms of eqn. 1,

$$\frac{\partial V_2(s)}{\partial \phi} I_2(-s) + \frac{\partial I_2(s)}{\partial \phi} V_2(-s) = 2I_2(-s)\frac{\partial V_2(s)}{\partial \phi} \quad (8)$$

Now consider an internal branch for which

$$V_k(s) = Z_k(s)I_k(s) \quad (9)$$

and, invoking the lossless property $Z_k(-s) = -Z_k(s)$,

$$V_k(-s) = Z_k(-s)I_k(-s) = -Z_k(s)I_k(-s) \quad (10)$$

It follows for the kth branch of eqn. 1 that

$$\frac{\partial V_k(s)}{\partial \phi} I_k(-s) + \frac{\partial I_k(s)}{\partial \phi} V_k(-s)$$

$$= \left[Z_k(s)\frac{\partial I_k(s)}{\partial \phi} + \frac{\partial Z_k(s)}{\partial \phi} I_k(s)\right] I_k(-s)$$

$$+ \frac{\partial I_k(s)}{\partial \phi}[-Z_k(s)I_k(-s)] = \frac{\partial Z_k(s)}{\partial \phi} I_k(s)I_k(-s) \quad (11)$$

Substituting eqns. 4, 8 and 11 into eqn. 1, the Tellegen sum is reduced to

$$-E\rho_1(-s)\frac{\partial I_1(s)}{\partial \phi} - 2I_2(-s)\frac{\partial V_2(s)}{\partial \phi} +$$

$$\sum_k \left[\frac{\partial Z_k(s)}{\partial \phi} I_k(s)I_k(-s)\right] = 0$$

which can be rewritten as

$$\frac{\partial V_2(s)}{\partial \phi} = \frac{1}{2I_2(-s)}$$

$$\times \left\{-E\rho_1(-s)\frac{\partial I_1(s)}{\partial \phi} + \sum_k \frac{\partial Z_k(s)}{\partial \phi} I_k(s)I_k(-s)\right\} \quad (12)$$

Taking $\phi = Z_k(s)$ and using the formula from Bandler,[3]

$$E\frac{\partial I_1(s)}{\partial Z_k(s)} = -I_k^2(s) \quad (13)$$

the key formula of Orchard *et al.*[1] is proved to be

$$\frac{\partial \theta}{\partial Z_k} = -\frac{1}{V_2(s)}\frac{\partial V_2(s)}{\partial Z_k} = \frac{I_k(s)I_k(-s) + \rho_1(-s)I_k^2(s)}{-2V_2(s)I_2(-s)} \quad (14)$$

where θ is the transducer coefficient.

It is well known[4] that derivatives with respect to nonexistent elements can be computed—hence the usefulness of eqn. 14 in the prediction of the effects of small losses and parasitics by a first-order Taylor expansion evaluated at the nominal (lossless, ideal) design.

The dual to eqn. 14, namely eqn. 9 of Reference 1, is easily derived in a similar manner.

Here, we extend the formula to group delay computation as follows. Taking $s = j\omega$ and $\phi = \omega$ in eqn. 12, and using an extension of eqn. 13 from Reference 3 as

$$E\frac{\partial I_1}{\partial \omega} = -\sum \frac{\partial Z_k}{\partial \omega} I_k^2 \quad (15)$$

the group delay $T_G(\omega)$ is given by

$$T_G(\omega) = -\mathrm{Im}\left\{\frac{1}{V_2}\frac{\partial V_2}{\partial \omega}\right\}$$

$$= \frac{1}{2P_2}\,\mathrm{Im}\left\{\sum_k \left(I_k^* + \rho_1^* I_k\right)\frac{\partial Z_k}{\partial \omega} I_k\right\} \qquad (16)$$

where P_2 is the power in R_2 and * stands for the complex conjugate. The dual formula can be readily derived as

$$T_G(\omega) = \frac{1}{2P_2}\,\mathrm{Im}\left\{\sum_k (V_k^* - \rho_1^* V_k)\frac{\partial Y_k}{\partial \omega} V_k\right\} \qquad (17)$$

We have checked the results obtained from eqn. 17 with those using the standard adjoint network approach[3,5] for the filter used by Orchard *et al.*[1] as an example. The results agree exactly. In the range $0 \le \omega \le 1$, for example, we have, in seconds, 30·1941, 9·7952, 6·1723, 5·1344, 4·9551, 5·2210, 5·8766, 7·2968, 10·5619 and 25·0132, corresponding to frequencies uniformly spaced 0·1 rad/s apart from 0·1 to 1·0 rad/s.

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