modulator could have a lower $V_{\lambda/2}$ than the corresponding 0° z cut (i.e. $R > 1$), but this would ultimately depend on the ratio of the electro-optic coefficients.

The piezoelectric behaviour of the 45° y' cut transverse modulators has also been examined. Since there is a component of electric field along the z axis, stronger resonances than those observed in the 45° y cut modulators should be present.[5] Calculations show that the effective piezoelectric constants $d_{36}/\sqrt{2}$, $d_{36}/2\sqrt{2}$ and $(d_{14}/\sqrt{2})-(d_{36}/2\sqrt{2})$ give rise to strains along the crystal width, thickness and length, and also to shear strains in the thickness–length plane. Consequently, several modes of vibrations should be present.

We have tested a compensated 45° y' cut ADP modulator of crystal dimensions $4 \times 5 \times 36$ mm; the electric field being in the 4 mm direction. This modulator has a $V_{\lambda/2}$ of approximately 400 V and a capacitance of 37·0 pF. Only weak resonances in the 60–220 KHz range were observed, but the case used was designed to reduce mechanical resonances by clamping the crystal and by oil immersion.[6]

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DIRECT METHOD FOR EVALUATING SCATTERING-MATRIX SENSITIVITIES*

Indexing terms: Sensitivity analysis, Matrix algebra

A method is presented for evaluating first- and second-order sensitivities of, in general, a nonreciprocal network described by a scattering matrix. It avoids the analysis of an adjoint network, using only the results of an analysis of the given network.

Methods of evaluating first- and second-order sensitivities and gradients with respect to network parameters using wave variables have been recently proposed.[1,2] Thus far, the concept of the adjoint network[3,4] has been employed, requiring, in general, no more than two network analyses, for example, for evaluating first-order sensitivities. It is the purpose of this letter to show how one might avoid the use of the adjoint network in the computation of the desired first- and second-order sensitivities by using the results of only one analysis of the given network.

It has been found convenient to use the scattering-matrix analysis technique previously proposed by Monaco and Tiberio[5] Let

$$\begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_1 & 0 \ldots 0 \\ 0 & \boldsymbol{S}_2 \ldots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \ldots \boldsymbol{S}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix} \quad . \quad . \quad . \quad . \quad (1)$$

represent the relation between all the port incident and reflected waves in a multiport network consisting of n uncoupled component multiports. Thus $\boldsymbol{S}_i$ would be the scattering matrix of the ith component multiport with $\boldsymbol{a}_i$ representing the incident waves and $\boldsymbol{b}_i$ the reflected waves.

Eqn. 1 may be rewritten as

$$\begin{bmatrix} \boldsymbol{b}_E \\ \boldsymbol{b}_I \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{EE} & \boldsymbol{S}_{EI} \\ \boldsymbol{S}_{IE} & \boldsymbol{S}_{II} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} = \boldsymbol{\Sigma} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} \quad . \quad . \quad . \quad (2)$$

where appropriate rows and columns have been interchanged so that $\boldsymbol{a}_E$ and $\boldsymbol{b}_E$ denote vectors associated with external ports and $\boldsymbol{a}_I$ and $\boldsymbol{b}_I$ denote vectors associated with the remaining or internal ports.

Each internal port is connected to another internal adjacent port. The waves at these ports are related by

$$\boldsymbol{b}_I = \boldsymbol{\Gamma} \boldsymbol{a}_I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$\boldsymbol{\Gamma}$ will contain elements of value 0 or 1, for example, if adjacent ports have the same normalisation numbers. From eqns. 2 and 3

$$\boldsymbol{b}_I = \boldsymbol{\Gamma} \boldsymbol{a}_I = \boldsymbol{S}_{IE} \boldsymbol{a}_E + \boldsymbol{S}_{II} \boldsymbol{a}_I \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

so that

$$\boldsymbol{a}_I = (\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1} \boldsymbol{S}_{IE} \boldsymbol{a}_E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\boldsymbol{b}_E = \boldsymbol{S}_{EE} \boldsymbol{a}_E + \boldsymbol{S}_{EI} \boldsymbol{a}_I = \{\boldsymbol{S}_{EE} + \boldsymbol{S}_{EI}(\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1} \boldsymbol{S}_{IE}\} \boldsymbol{a}_E \quad (6)$$

The overall scattering matrix of the network is then

$$\boldsymbol{S} = \boldsymbol{S}_{EE} + \boldsymbol{S}_{EI}(\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1} \boldsymbol{S}_{IE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

First-order sensitivities: Let ϕ be a variable parameter in the given network. Suppose it is desired to evaluate $\partial \boldsymbol{b}_E/\partial\phi$. Applying $\partial/\partial\phi$ to eqn. 2

$$\frac{\partial}{\partial\phi} \begin{bmatrix} \boldsymbol{b}_E \\ \boldsymbol{b}_I \end{bmatrix} = \frac{\partial\boldsymbol{\Sigma}}{\partial\phi} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} + \boldsymbol{\Sigma} \frac{\partial}{\partial\phi} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} \quad . \quad . \quad . \quad (8)$$

Assuming that the terminations are matched, $\partial \boldsymbol{a}_E/\partial\phi = \boldsymbol{0}$. Differentiating eqn. 4 and solving for $\partial \boldsymbol{a}_I/\partial\phi$, we can obtain

$$\frac{\partial}{\partial\phi} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \dfrac{\partial \boldsymbol{a}_I}{\partial\phi} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1} \end{bmatrix} \left(\frac{\partial}{\partial\phi} \begin{bmatrix} \boldsymbol{S}_{EE} & \boldsymbol{S}_{EI} \\ \boldsymbol{S}_{IE} & \boldsymbol{S}_{II} \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} \quad (9)$$

Substituting eqn. 9 into eqn. 8

$$\frac{\partial}{\partial\phi} \begin{bmatrix} \boldsymbol{b}_E \\ \boldsymbol{b}_I \end{bmatrix} = \frac{\partial\boldsymbol{\Sigma}}{\partial\phi} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} + \boldsymbol{\Sigma} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1} \end{bmatrix} \frac{\partial\boldsymbol{\Sigma}}{\partial\phi} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} \quad (10)$$

Replacing $\boldsymbol{a}_I$ by the expression in eqn. 5

$$\frac{\partial \boldsymbol{b}_E}{\partial\phi} = [\boldsymbol{1} \quad \boldsymbol{S}_{EI}(\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1}] \frac{\partial\boldsymbol{\Sigma}}{\partial\phi} \begin{bmatrix} \boldsymbol{1} \\ (\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1} \boldsymbol{S}_{IE} \end{bmatrix} \boldsymbol{a}_E \quad (11)$$

where $\boldsymbol{1}$ is the unit matrix of appropriate dimensions. $\partial \boldsymbol{S}/\partial\phi$ can be immediately identified in eqn. 11.

The evaluation of eqn. 11 should be fairly easy once the network has been analysed. As indicated in eqn. 7, $(\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1}$ and $(\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1} \boldsymbol{S}_{IE}$ are obtained during this analysis. $\boldsymbol{S}_{EI}$ will usually be sparse and, furthermore, $\boldsymbol{S}_{EI} = \boldsymbol{S}_{IE}^T$ for reciprocal networks, in which case

$$\boldsymbol{S}_{EI}(\boldsymbol{\Gamma} - \boldsymbol{S}_{II})^{-1}$$

need not be separately evaluated. Unless ϕ is frequency, $\partial\boldsymbol{\Sigma}/\partial\phi$ may also be quite sparse. To evaluate, for example, $\partial S_{jk}/\partial\phi$ one takes $\boldsymbol{a}_E = \boldsymbol{0}$ except for a_k which is set at 1. The jth row of eqn. 11 is then evaluated, giving $\partial S_{jk}/\partial\phi$.

Second-order sensitivities: Let ϕ and ψ be variable parameters in the given network. Applying $\partial^2/\partial\psi\,\partial\phi$ to eqn. 2,

$$\frac{\partial^2}{\partial\psi\,\partial\phi} \begin{bmatrix} \boldsymbol{b}_E \\ \boldsymbol{b}_I \end{bmatrix} = \frac{\partial\boldsymbol{\Sigma}}{\partial\psi} \frac{\partial}{\partial\phi} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} + \frac{\partial\boldsymbol{\Sigma}}{\partial\phi} \frac{\partial}{\partial\psi} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix}$$

$$+ \frac{\partial^2 \boldsymbol{\Sigma}}{\partial\psi\,\partial\phi} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} + \boldsymbol{\Sigma} \frac{\partial^2}{\partial\psi\,\partial\phi} \begin{bmatrix} \boldsymbol{a}_E \\ \boldsymbol{a}_I \end{bmatrix} \quad . \quad . \quad (12)$$

* This work was carried out under grants A7239 and A5277 from the National Research Council of Canada

Under matched terminations, both $\partial a_E/\partial\phi$ and $\partial a_E/\partial\psi$ are zero. Differentiating eqn. 4 twice and solving for $\partial^2 a_I/\partial\psi\,\partial\phi$, we obtain

$$\frac{\partial^2}{\partial\psi\,\partial\phi}\begin{bmatrix}a_E\\a_I\end{bmatrix}=\begin{bmatrix}0\\\dfrac{\partial^2 a_I}{\partial\psi\,\partial\phi}\end{bmatrix}=\begin{bmatrix}0&0\\0&(\Gamma-S_{II})^{-1}\end{bmatrix}$$

$$\times\left(\frac{\partial\Sigma}{\partial\psi}\frac{\partial}{\partial\phi}\begin{bmatrix}a_E\\a_I\end{bmatrix}+\frac{\partial\Sigma}{\partial\phi}\frac{\partial}{\partial\psi}\begin{bmatrix}a_E\\a_I\end{bmatrix}+\frac{\partial^2\Sigma}{\partial\psi\,\partial\phi}\begin{bmatrix}a_E\\a_I\end{bmatrix}\right) \qquad (13)$$

Substituting eqn. 13 into eqn. 12, we obtain, after some manipulation,

$$\frac{\partial^2 b_E}{\partial\psi\,\partial\phi}=[1\quad S_{Eb}(\Gamma-S_{II})^{-1}]$$

$$\times\left(\frac{\partial\Sigma}{\partial\psi}\begin{bmatrix}0\\\dfrac{\partial a_I}{\partial\phi}\end{bmatrix}+\frac{\partial\Sigma}{\partial\phi}\begin{bmatrix}0\\\dfrac{\partial a_I}{\partial\psi}\end{bmatrix}+\frac{\partial^2\Sigma}{\partial\psi\,\partial\phi}\begin{bmatrix}a_E\\a_I\end{bmatrix}\right) \qquad (14)$$

Using eqn. 9,

$$\frac{\partial^2 b_E}{\partial\psi\,\partial\phi}=[1\quad S_{EI}(\Gamma-S_{II})^{-1}]\left(\frac{\partial\Sigma}{\partial\psi}\begin{bmatrix}0&0\\0&(\Gamma-S_{II})^{-1}\end{bmatrix}\frac{\partial\Sigma}{\partial\phi}\right.$$

$$\left.+\frac{\partial\Sigma}{\partial\phi}\begin{bmatrix}0&0\\0&(\Gamma-S_{II})^{-1}\end{bmatrix}\frac{\partial\Sigma}{\partial\psi}+\frac{\partial^2\Sigma}{\partial\psi\,\partial\phi}\right)\begin{bmatrix}a_E\\a_I\end{bmatrix} \qquad (15)$$

As before, eqn. 5 can be used to remove a_I, resulting in

$$\frac{\partial^2 b_E}{\partial\psi\,\partial\phi}=[1\quad S_{EI}(\Gamma-S_{II})^{-1}]\left(\frac{\partial\Sigma}{\partial\psi}\begin{bmatrix}0&0\\0&(\Gamma-S_{II})^{-1}\end{bmatrix}\frac{\partial\Sigma}{\partial\phi}\right.$$

$$\left.+\frac{\partial\Sigma}{\partial\phi}\begin{bmatrix}0&0\\0&(\Gamma-S_{II})^{-1}\end{bmatrix}\frac{\partial\Sigma}{\partial\psi}+\frac{\partial^2\Sigma}{\partial\psi\,\partial\phi}\right)$$

$$\times\begin{bmatrix}1\\(\Gamma-S_{II})^{-1}S_{IE}\end{bmatrix}a_E \qquad (16)$$

$\partial^2 S/\partial\psi\,\partial\phi$ can be immeditely identified in eqn. 16.

Similar comments to those made for computation of first-order sensitivities can be made for second-order sensitivities.

It is a fairly straightforward matter to show how the results presented here relate to the adjoint network approach which the authors presented previously.[2] The present method has the advantage that only one matrix inversion is required in the evaluation of first- and second-order sensitivities.

The authors are pleased to acknowledge the co-operation of M. Sablatash of the University of Toronto.

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3rd November 1970

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SOME POSSIBILITIES OF INDIRECT DETERMINATION OF NETWORK-FUNCTION SENSITIVITY TO THE IDEAL VOLTAGE-AMPLIFIER AMPLIFICATION

Indexing terms: Linear network analysis, Computer-aided circuit analysis, Sensitivity analysis

The sensitivity of network functions of a circuit containing an ideal voltage amplifier to the amplification-factor variation of this element is expressed with the help of various auxiliary network functions determined in the original circuit or in the circuit from which the ideal voltage amplifier was removed. The procedure is suitable for practical sensitivity measurements.

A recent letter[1] shows a simple procedure for the sensitivity calculation of a linear network function to the amplification factor of an ideal voltage amplifier (voltage-controlled voltage source). The mentioned procedure is suitable for digital computers. In this letter, some possibilities of an illustrative or even practical interpretation of the indirect determination of network-function sensitivities will be presented. The results can be applied to calculations and also to experimental sensitivity measurements.

Let us consider a 4-port network containing an ideal voltage amplifier (i.v.a.) shown in Fig. 1. The system under consideration is assumed to be analysed by the node-voltage method with the external port voltages as shown in Fig. 1.

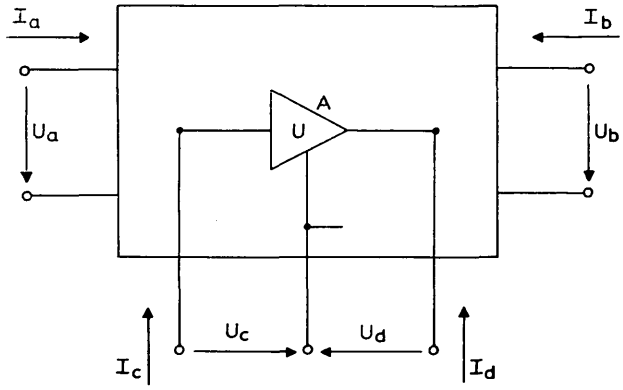

Fig. 1 *4-port network containing ideal voltage amplifier*

In the following text the ports will be referred to as *a*, *b*, *c* and *d*.

If port *a* is the input port ($I_c = I_d = 0$) and *b* is the output port, the following network functions can be considered for the resultant 2-port network:

(i) voltage-transfer ratio $K_{ab} = U_b/U_a$, $I_b = 0$ (the first letter in the function subscript means the port where the system is supplied; the second letter means the output port)

(ii) current-transfer ratio $H_{ab(b)} = -I_b/I_a$, $U_b = 0$ (the letter in parentheses means the port to be short-circuited)

(iii) input impedance (with output port open-circuited)
$Z_a = U_a/I_a \qquad I_b = 0$

(iv) output impedance (with input port short-circuited)
$Z_{b(a)} = U_b/I_b \qquad U_a = 0$

(v) transfer impedance (with output port open-circuited)
$Z_{ab} = U_b/I_a \qquad I_b = 0$

(vi) transfer admittance (with output port short-circuited)
$Y_{ab(b)} = -I_b/U_a \qquad U_b = 0$

Any of the given network functions is dependent on the amplification factor A of the ideal voltage amplifier. This dependence can be generally written as follows:

$$F(A)=\frac{N}{V}=\frac{\tilde{M}-A\tilde{m}}{\tilde{W}-A\tilde{w}} \qquad (1)$$