Correspondence

Computation of Sensitivities for Noncommensurate Networks

Abstract—The adjoint network approach to automated network design as presented by Director and Rohrer is extended for use in gradient calculations for noncommensurate networks. The networks can contain distributed elements such as uniform transmission lines and *RC* lines. An example investigates gradient computations for a noncommensurate microwave network having 13 variable parameters.

In recent contributions [1], [2] Director and Rohrer discussed the concept of the adjoint network and its relevance to automated design of networks in the frequency and time domains. Employing Tellegen's theorem [3] they demonstrated how the gradient vector for a least-squares type of response objective function with respect to all existing (and nonexisting, if desired) elements could be evaluated from only two complete analyses, one of the given network and one of its topologically equivalent adjoint network. In the frequency domain [2] they considered both reciprocal and nonreciprocal lumped, linear, and time-invariant elements. More recently [4], [5] it was shown how their approach could be implemented for least *p*th and minimax response objective functions [6], [7].

The purpose of this correspondence is to show how the adjoint network approach may be used to advantage in gradient calculations for a class of commensurate and noncommensurate networks in the frequency domain. The networks can contain the conventional lumped, linear, and time-invariant elements and distributed parameter elements such as uniform transmission lines and *RC* lines. The results can then be incorporated into an automatic optimization algorithm in which such functions as gain, insertion loss, reflection coefficient, or any other desired response function can be optimized to meet least-*p*th or minimax performance specifications.

Given a network for optimization its adjoint has to be found. For networks consisting of lumped elements and commonly used uniformly distributed elements falling within the broad class illustrated in Fig. 1, the derivation of the adjoint is very straightforward.

Consider, for example, the uniformly distributed line shown in Fig. 1(a). Using an equivalent circuit based on the impedance matrix for convenience and with the notation of Fig. 1(a),

$$V = ZI \tag{1}$$

where

$$V \triangleq \begin{bmatrix} V_p \\ V_q \end{bmatrix} \tag{2}$$

$$I \triangleq \begin{bmatrix} I_p \\ I_q \end{bmatrix} \tag{3}$$

$$Z \triangleq Z\begin{bmatrix} \coth\theta & \operatorname{csch}\theta \\ \operatorname{csch}\theta & \coth\theta \end{bmatrix}. \tag{4}$$

It is readily shown that for the adjoint element

$$\Psi = Z^T\Phi \tag{5}$$

where Ψ and Φ are the adjoint network variables corresponding to V and I, respectively. Since $Z^T = Z$ for the reciprocal example under consideration, the adjoint element is, as expected, identical to the original one.

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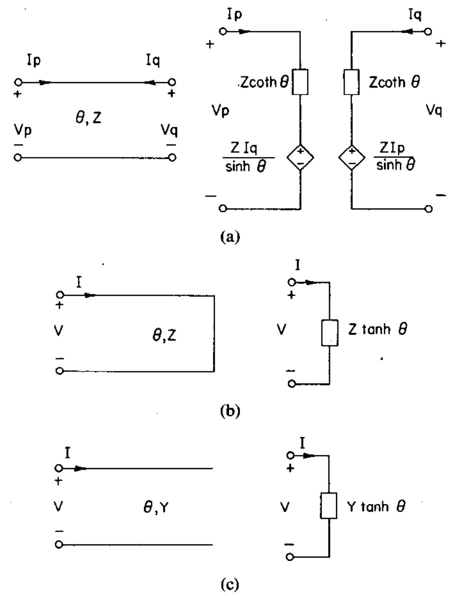

Fig. 1. Uniformly distributed elements with convenient representations. (a) Uniform line. (b) Short-circuited line. (c) Open-circuited line.

It may also be shown by invoking Tellegen's theorem that the expression leading to the sensitivity based on the impedance matrix is in general

$$I^T \Delta Z^T \Phi. \tag{6}$$

Using (4) and (6) we obtain

$$\begin{aligned} I^T\Delta Z^T\Phi &= I^T\left(\Delta Z\begin{bmatrix} \coth\theta & \operatorname{csch}\theta \\ \operatorname{csch}\theta & \coth\theta \end{bmatrix} - \frac{Z\Delta\theta}{\sinh\theta}\begin{bmatrix} \operatorname{csch}\theta & \coth\theta \\ \coth\theta & \operatorname{csch}\theta \end{bmatrix}\right)^T\Phi \\ &= \left(\frac{\Delta Z}{Z}ZI - \frac{\Delta\theta}{\sinh\theta}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}ZI\right)^T\Phi \\ &= \frac{\Delta Z}{Z}V^T\Phi - \frac{\Delta\theta}{\sinh\theta}V^T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\Phi. \end{aligned} \tag{7}$$

Equation (7) contains the essential information needed to generate the partial derivatives of a response function with respect to changes in the adjustable parameters in the distributed element.

Consider a lossless transmission line of length l and characteristic impedance Z. It is readily shown that with $\theta = j\beta l$ where (using the usual notation) $\beta = \omega/c$, (7) can be written as

$$\frac{\Delta Z}{Z}V^T\Phi - \frac{\beta\Delta l}{\sin\beta l}V^T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\Phi. \tag{8}$$

Consider a uniform *RC* line. Letting R be the total resistance and C the total capacitance of the section,

$$Z = \sqrt{\frac{R}{sC}} \tag{9}$$

$$\theta = \sqrt{sRC} \tag{10}$$

TABLE I

SENSITIVITY EXPRESSIONS FOR SOME LUMPED AND UNIFORMLY DISTRIBUTED LINE ELEMENTS

Element	Equation (for adjoint replace Ψ for V and Φ for I)	Sensitivity (component of $\mathbf{G}$)	Increment
Resistor	$V = RI$	$I\Phi$	ΔR
	$I = GV$	$-V\Psi$	ΔG
Inductor	$V = j\omega LI$	$j\omega I\Phi$	ΔL
	$I = \frac{1}{j\omega}\Gamma V$	$-\frac{1}{j\omega}V\Psi$	$\Delta\Gamma$
Capacitor	$V = \frac{1}{j\omega}SI$	$\frac{1}{j\omega}I\Phi$	ΔS
	$I = j\omega CV$	$-j\omega V\Psi$	ΔC
Short-circuited uniformly distributed line	$V = Z\tanh\theta\, I$	$\tanh\theta\, I\Phi$	ΔZ
		$Z\,\mathrm{sech}^2\,\theta\, I\Phi$	$\Delta\theta$
	$I = Y\coth\theta\, V$	$-\coth\theta\, V\Psi$	ΔY
		$Y\,\mathrm{csch}^2\,\theta\, V\Psi$	$\Delta\theta$
Open-circuited uniformly distributed line	$V = Z\coth\theta\, I$	$\coth\theta\, I\Phi$	ΔZ
		$-Z\,\mathrm{csch}^2\,\theta\, I\Phi$	$\Delta\theta$
	$I = Y\tanh\theta\, V$	$-\tanh\theta\, V\Psi$	ΔY
		$-Y\,\mathrm{sech}^2\,\theta\, V\Psi$	$\Delta\theta$
Uniformly distributed line	$\mathbf{V} = Z\begin{bmatrix}\coth\theta & \mathrm{csch}\,\theta\\ \mathrm{csch}\,\theta & \coth\theta\end{bmatrix}\mathbf{I}$	$\frac{1}{Z}\mathbf{V}^T\mathbf{\Phi}$	ΔZ
		$-\frac{1}{\sinh\theta}\mathbf{V}^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathbf{\Phi}$	$\Delta\theta$
	$\mathbf{I} = Y\begin{bmatrix}\coth\theta & -\mathrm{csch}\,\theta\\ -\mathrm{csch}\,\theta & \coth\theta\end{bmatrix}\mathbf{V}$	$-\frac{1}{Y}\mathbf{I}^T\mathbf{\Psi}$	ΔY
		$-\frac{1}{\sinh\theta}\mathbf{I}^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathbf{\Psi}$	$\Delta\theta$

where s is the complex frequency variable. Here

$$\Delta Z = \frac{\Delta R}{2\theta} - \frac{Z\Delta C}{2C} \tag{11}$$

$$\Delta\theta = \frac{\Delta R}{2Z} + \frac{\theta\Delta C}{2C}. \tag{12}$$

Equation (7) for a uniform RC line becomes

$$\frac{\Delta R}{2R}\left(\mathbf{V}^T\mathbf{\Phi} - \frac{\theta}{\sinh\theta}\mathbf{V}^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathbf{\Phi}\right) - \frac{\Delta C}{2C}\left(\mathbf{V}^T\mathbf{\Phi} + \frac{\theta}{\sinh\theta}\mathbf{V}^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathbf{\Phi}\right)$$

which simplifies somewhat to

$$\frac{\Delta R}{2R}\mathbf{V}^T\begin{bmatrix}1 & -\frac{\theta}{\sinh\theta}\\ -\frac{\theta}{\sinh\theta} & 1\end{bmatrix}\mathbf{\Phi} - \frac{\Delta C}{2C}\mathbf{V}^T\begin{bmatrix}1 & \frac{\theta}{\sinh\theta}\\ \frac{\theta}{\sinh\theta} & 1\end{bmatrix}\mathbf{\Phi}. \tag{13}$$

Tables I and II summarize the sensitivity components of a number of commonly used elements. They may all be derived in a manner similar to the one just outlined. A number of observations need to be made at this point. The sensitivities depend on various values of current and voltage associated with the original and adjoint networks and on some element values. Although it may be convenient to use impedance or admittance matrix equivalent circuits in deriving the sensitivities, any suitable method of network analysis can be used in practice as long as the sign convention of Fig. 1 is strictly adhered to. Furthermore, it may be verified that

$$V\Phi \equiv I\Psi \tag{14}$$

for the two-terminal elements and

$$\mathbf{V}^T\mathbf{\Phi} \equiv \mathbf{I}^T\mathbf{\Psi} \tag{15}$$

$$\mathbf{V}^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathbf{\Phi} \equiv \mathbf{I}^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathbf{\Psi} \tag{16}$$

for the two-port elements of Tables I and II, which allows some flexibility in the choice of sensitivity expressions.

Referring to Fig. 2, suppose we have to minimize

$$U = \sum_{\Omega_d}\frac{1}{p}|L(\omega_d) - \hat{L}(\omega_d)|^p \tag{17}$$

where L is the insertion loss between R_g and R_L, $\hat{L}$ is the desired insertion loss between R_g and R_L, Ω_d is a set of discrete frequencies ω_d, and p is a positive integer, that is to say, to approximate a specified insertion loss function in a least-pth sense over a set of frequencies in the range of interest.

$$L(\omega_d) = -20\log_{10}\left|\frac{I_L(j\omega_d)}{V_g(j\omega_d)}\right|(R_g + R_L) \tag{18}$$

TABLE II
SENSITIVITY EXPRESSIONS FOR SOME SPECIAL CASES OF UNIFORMLY DISTRIBUTED LINE ELEMENTS

Element	Equation (for adjoint replace Ψ for V and Φ for I)	Sensitivity (component of G)	Increment
Short-circuited lossless transmission line	$V = jZ \tan \beta l\, I$	$j \tan \beta l\, I\Phi$	ΔZ
		$jZ\beta \sec^2 \beta l\, I\Phi$	Δl
	$I = -jY \cot \beta l\, V$	$j \cot \beta l\, V\Psi$	ΔY
		$-jY\beta \csc^2 \beta l\, V\Psi$	Δl
Open-circuited lossless transmission line	$V = -jZ \cot \beta l\, I$	$-j \cot \beta l\, I\Phi$	ΔZ
		$jZ\beta \csc^2 \beta l\, I\Phi$	Δl
	$I = jY \tan \beta l\, V$	$-j \tan \beta l\, V\Psi$	ΔY
		$-jY\beta \sec^2 \beta l\, V\Psi$	Δl
Lossless transmission line	$V = -jZ\begin{bmatrix} \cot \beta l & \csc \beta l \\ \csc \beta l & \cot \beta l \end{bmatrix} I$	$\frac{1}{Z} V^T \Phi$	ΔZ
		$-\frac{\beta}{\sin \beta l} V^T \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \Phi$	Δl
	$I = -jY\begin{bmatrix} \cot \beta l & -\csc \beta l \\ -\csc \beta l & \cot \beta l \end{bmatrix} V$	$-\frac{1}{Y} I^T \Psi$	ΔY
		$-\frac{\beta}{\sin \beta l} I^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Psi$	Δl
Uniform RC line	as for uniformly distributed line in Table I with $Z = \sqrt{R/sC}$ and $\theta = \sqrt{sRC}$	$\frac{1}{2R} V^T \begin{bmatrix} 1 & -\frac{\theta}{\sinh \theta} \\ -\frac{\theta}{\sinh \theta} & 1 \end{bmatrix} \Phi$	ΔR
		$-\frac{1}{2C} V^T \begin{bmatrix} 1 & \frac{\theta}{\sinh \theta} \\ \frac{\theta}{\sinh \theta} & 1 \end{bmatrix} \Phi$	ΔC

so that

$$\nabla U = \sum_{\Omega_d} |L(\omega_d) - \hat{L}(\omega_d)|^{p-2}[L(\omega_d) - \hat{L}(\omega_d)] \cdot \left[-10(\log_{10} e) \frac{I_L(j\omega_d)\nabla I_L^*(j\omega_d) + I_L^*(j\omega_d)\nabla I_L(j\omega_d)}{I_L(j\omega_d)I_L^*(j\omega_d)} \right]$$
$$= -20(\log_{10} e) \sum_{\Omega_d} |L(\omega_d) - \hat{L}(\omega_d)|^{p-2}[L(\omega_d) - \hat{L}(\omega_d)] \cdot \mathrm{Re}\left\{ \frac{\nabla I_L(j\omega_d)}{I_L(j\omega_d)} \right\} \quad (19)$$

where R_g, R_L, and V_g are constant. One analysis of the network yields I_L, L, and U. If, for the adjoint network, $\Psi_g = 0$ then

$$\nabla I_L(j\omega_d) = \frac{1}{\Psi_L(j\omega_d)} G(j\omega_d) \quad (20)$$

where Ψ_L is an appropriate adjoint excitation and G contains sensitivity components as shown in Tables I and II. Hence the components of $\nabla I_L(j\omega_d)$ are obtained from analyses of both original and adjoint networks.

Fig. 3 shows a noncommensurate network having 13 variable parameters. An objective function of the form of (17) was chosen with $\hat{L} = 0$; $p = 10$; and $\Omega_d = \{0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ GHz. For the element values given in Fig. 3, $U = 3.04383 \times 10^8$.

Table III shows the components of ∇U estimated from 1- and 0.001-percent incremental changes in the parameters compared with those obtained using (19) and the appropriate expressions from Tables I and II. In the incremental case the changes in insertion loss were calculated via changes in input impedance due to changes in parameter values. For the adjoint network approach it was found convenient for analysis purposes to calculate the voltages and currents in the original network associated with $I_L = 1$ (see Fig. 2) and the voltages and currents in the adjoint network associated with $\Phi_g = 1$. By reciprocity, of course, the independent sources needed to produce these currents, namely, V_g and Ψ_L, are equal.

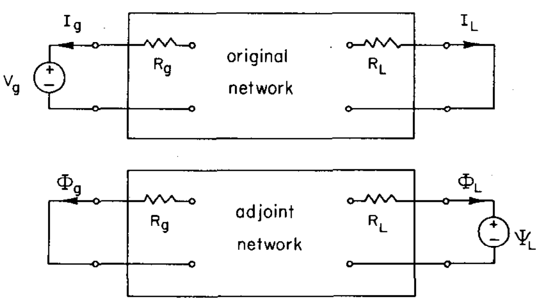

Fig. 2. Network for insertion loss design.

One of the attractive features of the adjoint network approach to computer-aided design is the relative ease with which gradients with respect to nonexistent or zero-valued lumped elements can be obtained. Thus elements may be grown from a short circuit or open circuit if the

TABLE III

Comparison of Gradient Components for the 13 Variable Noncommensurate Network Obtained by Incremental and Adjoint Network Methods

Element			Gradient Components		
Type	Parameter	Value	1-Percent Increment	0.001-Percent Increment	Adjoint Network
Parallel capacitor	C	2 pF	1.416×10^7	1.3756×10^7	1.3755×10^7
Parallel short-circuited line	Y	0.0125 mho	-7.324×10^9	-7.3952×10^9	-7.3952×10^9
	l	6 cm	1.884×10^7	1.8510×10^7	1.8509×10^7
Transmission line	Z	25 Ω	-1.166×10^8	-1.2276×10^8	-1.2276×10^8
	l	8 cm	-1.275×10^7	-1.3300×10^7	-1.3300×10^7
Parallel inductor	Γ	0.1 (nH)$^{-1}$	-3.178×10^9	-3.2166×10^9	-3.2167×10^9
Parallel capacitor	C	3 pF	6.563×10^7	6.5130×10^7	6.5130×10^7
Series open-circuited line	Z	40 Ω	3.439×10^7	3.3764×10^7	3.3763×10^7
	l	5 cm	-4.408×10^8	-4.5832×10^8	-4.5834×10^8
Parallel capacitor	C	4 pF	3.296×10^8	3.2384×10^8	3.2384×10^8
Transmission line	Z	50 Ω	-4.191×10^6	-4.2482×10^6	-4.2483×10^6
	l	1 cm	1.393×10^8	1.3932×10^8	1.3932×10^8
Series inductor	L	3 nH	3.442×10^6	3.3200×10^6	3.3199×10^6

gradient, which is a function of voltages and currents only, indicates an increase in element value.

Unfortunately, as Tables I and II show, things get more complicated with distributed elements. Consider a uniform line (Fig. 1(a)). For $\theta=0$ but $Z\neq 0$, the sensitivity with respect to Z is

$$\frac{1}{Z} V^T \Phi = 0 \tag{21}$$

but the sensitivity with respect to θ is

$$-\frac{1}{\sinh\theta} V^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Phi \to ZI\Phi - \frac{V\Psi}{Z} \tag{22}$$

where

$$\begin{aligned} I &= I_p \\ \Phi &= \Phi_p = -\Phi_q \\ V &= V_p = V_q \\ \Psi &= \Psi_p = \Psi_q. \end{aligned}$$

Equation (22) is readily derived by means of an *ABCD* matrix description of the line. Observe that $ZI\Phi$ is the sensitivity of a zero-length short-circuited line and $-V\Psi/Z$ is the corresponding expression for an open-circuited line. Clearly, also

$$ZI\Phi - \frac{V\Psi}{Z} \to \begin{cases} ZI\Phi & Z \to \infty \\ \dfrac{-V\Psi}{Z} & Z \to 0. \end{cases} \tag{23}$$

Furthermore, depending on the type of distributed element under consideration, as $\theta \to 0$ we can obtain appropriate lumped equivalent circuits. So, it does not seem obvious then what type of element is to be grown, whether lumped or distributed, from a knowledge only of currents and voltages. In practice, the physical constraints on circuit configuration may predetermine the allowable element types.

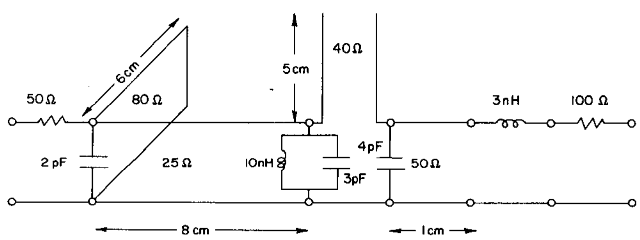

Fig. 3. Noncommensurate network having 13 variables terminated in 50 and 100 Ω.

Most published work to date on the computer-aided optimization of noncommensurate networks, in particular at microwave frequencies, indicates that direct-search methods of the pattern-search type have been widely employed [7]–[10]. Following the results presented in this correspondence such networks may now be readily designed using efficient gradient methods of minimization. Furthermore, the computational inefficiency and uncertainty inherent in the numerical estimation of partial derivatives by perturbation can be circumvented.

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An Algorithm for Generating Constituent Matrices

Abstract—A simple procedure for computer evaluation of the constituent matrices is introduced. The power of this procedure lies in the ease of generating the constituent matrices where the characteristic polynomial has a number of multiple-order zeros.

Wehrhahn [1] and Karni [2]–[4] accomplish the evaluation of the constituent matrices from Frame's algorithm. Their methods involve division of two polynomials and the generation and recombination of a great many terms in order to obtain the needed partial fraction expansion to give the constituent matrices.

The method described in this correspondence will accomplish the same result by an algorithm that involves addition, subtraction, and division of only two polynomials, the division known to be exact.

Frame's algorithm [5] will generate a matric function whose partial fraction expansion will give the needed constituent matrices

$$\frac{F(s)}{d(s)} = \frac{Us^{n-1} + F_1 s^{n-2} + \cdots + F_{n-1}}{S^n + d_1 s^{n-1} + \cdots + d_n} = \frac{F(s)}{(s-s_1)^j (s-s_2)^k \cdots (s-s_n)^q} = \frac{F(s)}{(s-s_1)^j q_1(s)}. \tag{1}$$

Note that capital letters represent matrices and small letters represent scalars. Expanding (1) in partial fraction form

$$\frac{F(s)}{(s-s_1)^j q_1(s)} = \frac{Z_1}{(s-s_1)} + \frac{Z_2}{(s-s_1)^2} + \cdots + \frac{Z_j}{(s-s_1)^j} + \Sigma \frac{Z_r}{(s-s_r)} \tag{2}$$

where $\Sigma\,(Z_r/(s-s_r))$ represents the rest of the partial fraction expansion due to the zeros of $q_1(s)$.

Evaluating

$$Z_j = \left.\frac{F(s)}{q_1(s)}\right|_{s=s_1}$$

and subtracting from both sides of (2), we obtain

$$\frac{F(s) - Z_j q_1(s)}{(s-s_1)^j q_1(s)} = \frac{Z_1}{(s-s_1)} + \frac{Z_2}{(s-s_2)} + \cdots + \frac{Z_{j-1}}{(s-s_1)^{j-1}} + \Sigma \frac{Z_r}{(s-s_r)}. \tag{3}$$

Since the subtraction of $Z_j q_1(s)$ from $F(s)$ makes the numerator in (3) exactly divisible by $(s-s_1)$, we obtain

$$\frac{D_1(s)}{q_1(s)(s-s_1)^{j-1}} = \frac{Us^{n-2} + D_1 s^{n-3} + \cdots + D_{n-2}}{(s-s_1)^{j-1} q_1(s)} = \frac{Z_1}{(s-s_1)} + \frac{Z_2}{(s-s_1)^2} + \cdots + \frac{Z_{j-1}}{(s-s_1)^{j-1}} + \Sigma \frac{Z_r}{(s-s_r)}. \tag{4}$$

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Thus

$$Z_{j-1} = \left.\frac{D_1(s)}{q_1(s)}\right|_{s=s_1}.$$

Continuing the procedure $j-1$ times, all the Z_j terms can be found.

Returning to (1), rewrite it in the form

$$\frac{F(s)}{d(s)} = \frac{F(s)}{(s-s_k)^p q_k(s)}. \tag{5}$$

All of the Z_p associated with the s_k pole in (5) can be found by the same procedure.

Generalizing the procedure, let

$$\frac{F(s)}{d(s)} = \frac{F(s)}{(s-s_1)^j q_i(s)}. \tag{6}$$

Defining

$$Z^j(s) = F(s) \quad \text{and} \quad Z_j = \left.\frac{F(s)}{q_i(s)}\right|_{s=s_i}$$

then

$$Z_{j-1} = \left.\frac{F(s) - Z_j q_i(s)}{(s-s_i) q_i(s)}\right|_{s=s_i} = \left.\frac{Z^{j-1}(s)}{q_i(s)}\right|_{s=s_i}$$

where

$$Z^{j-1}(s) = \frac{F(s) - Z_j q_i(s)}{(s-s_i)}$$

and

$$Z_{j-2} = \left.\frac{Z^{j-1}(s) - Z_{j-1} q_i(s)}{(s-s_i) q_i(s)}\right|_{s=s_i} = \left.\frac{Z^{j-2}(s)}{q_i(s)}\right|_{s=s_i}.$$

Thus, starting with $n=0$,

$$Z_{j-n} = \left.\frac{Z^{j-n}(s)}{q_i(s)}\right|_{s=s_i} \tag{7}$$

and

$$Z^{j-n} = \frac{Z^{j-n+1}(s) - Z_{j-n+1} q_i(s)}{(s-s_i)}, \qquad n \neq 0$$

where

$$Z^j = F(s), \qquad n = 0. \tag{8}$$

As n goes from 0 to $j-1$, all of the Z_j associated with the s_i pole will be generated. Rewriting $F(s)/d(s)$ as in (5) and using the generalized procedure will generate all the Z_p associated with the pole s_k.

However, to save storage in the computer, instead of returning to (5), just start the generalized procedure over again on the expression

$$\frac{F^*(s)}{(s-s_k)^p q_k(s)} = \frac{Z^1(s)}{(s-s_k)^p q_k(s)} = \frac{Z_1}{(s-s_i)} + \frac{Z_1^*}{(s-s_k)} + \frac{Z_2^*}{(s-k)^2} + \cdots + \frac{Z_p^*}{(s-s_k)^p} + \Sigma \frac{K_r^*}{(s-k)^2}$$

where $Z^1(s)$ is the function generated by (7) when $n=j-1$. This allows the evaluation of all the constituent matrices without the problem of storing the results of Frame's algorithm.

Example

Given

$$P = \begin{bmatrix} -13 & -21 & -11 \\ 5 & 8 & 5 \\ 6 & 11 & 4 \end{bmatrix}$$