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OPTIMUM APPROXIMATION OF HIGH-ORDER SYSTEMS BY LOW-ORDER MODELS USING RECENT GRADIENT METHODS

J.W. BANDLER, N.D. MARKETTOS and N.K. SINHA
Department of Electrical Engineering and
Communications Research Laboratory
McMaster University
Hamilton, Ontario, Canada

Abstract A comparison is made between three minimization techniques, the highly-regarded Fletcher-Powell method, the new Fletcher method and the new Jacobson-Oksman method by obtaining second-order models for a seventh-order system so that the step-response of the model approximates that of the system in the least-squares sense.

INTRODUCTION

A number of different methods have been proposed recently [1-5] for determining low-order models for high-order complex systems, in order to simplify the preliminary design and optimization of such systems. With the exception of the methods of Anderson [2] and Sinha and Pille [5], however, all of them approximate the response of the system in a qualitative manner in addition to requiring that the exact transfer function or vector differential equation for the high-order system must be specified in order that the reduced model may be obtained. In practice, this is seldom possible, and one has often to identify the system. Thus, a more realistic approach to the derivation of the simplified model should be based on using directly the measured input-output data for the system. Furthermore, it may be desirable to obtain an optimum model of a given order minimizing a specified criterion of error between the response of the model and that of the high-order system to a given input function, usually a unit step. Some suitable criteria for obtaining such a model are the minimization of (i) the sum of the squares of the error between sampled values of the two responses [2,5], (ii) the sum of the absolute values of the sampled errors, (iii) the sum of the pth power of the sampled errors, (iv) the maximum value of the errors between the responses, and (v) the maximum value of the perpendicular distance between the responses [6].

In a recent paper [7], Sinha and Bereznai have used the pattern search method of Hooke and Jeeves [8] for the determination of optimum low-order models using various criteria, and have calculated a number of such models for a given seventh-order system.

Due to the poor convergence properties of the pattern search algorithm, it was decided to investigate the application of efficient gradient methods to this problem. As well as the highly-regarded Fletcher-Powell method [9], two new gradient methods, one by Fletcher [10] and the other by Jacobson and Oksman [11] are available. This paper, therefore, attempts a comparative study of these three gradient methods by using each of them to obtain optimum second-order models, based on the least-squares criterion, for the same seventh-order system that was approximated by Sinha and Bereznai [7].

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The statement of the problem will be followed by a brief review of the two new gradient methods. The results of computation and a critical comparison between the three methods will then be presented.

It may be added that as far as the authors are aware, this is the first comparative study of these three gradient methods.

STATEMENT OF THE PROBLEM

Given the following transfer function of a seventh-order system, representing the control-system of the pitch rate of a supersonic transport aircraft [12]

$$G(s) = \frac{375000(s+0.08333)}{s^7+83.64s^6+4097s^5+70342s^4+853703s^3+2814271s^2+3310875s+281250} \tag{1}$$

determine the transfer function, H(s), of a second-order model such that the response of the model to a unit step approximates that of the system in the least-squares sense. In other words, if $g_1(t)$ is the response of G(s) to a unit step and $h_1(t)$ is the response of the model H(s) to a unit step then the parameters of the model must be selected such that the objective function

$$J_1 = \int_0^\infty [g_1(t) - h_1(t)]^2 dt \tag{2}$$

is minimized.

As it is desired to solve the problem using a digital computer, the objective function (2) can be conveniently replaced by the following discrete version

$$J_2 = \sum_{k=1}^{N} [g_1(kT) - h_1(kT)]^2 \tag{3}$$

where N is sufficiently large to include the portion of response which should be approximated, and T a suitable sampling interval. As the system takes a considerable time to reach the steady-state value, and since it is more desirable to emphasize the initial portion of the transient response, it was decided to select the parameters of the second-order model so that its steady-state response was constrained to be equal to the steady-state response of the system. The application of this constraint made it possible to reduce the value of N without increasing the error for large values of t.

The resulting second-order models were of two types, those with a finite zero, and those without a finite zero. These may be written, respectively, as

$$H_1(s) = \frac{Ea_0}{s^2+a_1s+a_0} \tag{4}$$

and

$$H_2(s) = \frac{b_1s+Ea_0}{s^2+a_1s+a_0} \tag{5}$$

where E is the steady-state response of the system to a unit step input. The model represented by equation (4) has two variable parameters, while that represented by equation (5) has three variable parameters.

To give a basic description of the new Fletcher method it is necessary to briefly review the Fletcher-Powell method first. Suppose that it is desired to minimize a function $F(\underset{\sim}{x})$ of an n-dimensional vector $\underset{\sim}{x}$, given by

$$\underset{\sim}{x} = [x_1 \quad x_2 \cdots x_n]^T \tag{6}$$

where the superscript T stands for transpose.

Let $\underset{\sim}{g}(\underset{\sim}{x})$ be the gradient of $F(\underset{\sim}{x})$ with respect to $\underset{\sim}{x}$ and let $\underset{\sim}{G}$ denote the corresponding Hessian matrix. The inverse Hessian $\underset{\sim}{G}^{-1}$ will be approximated by the matrix $\underset{\sim}{H}$. Define

$$\underset{\sim}{\delta} = \Delta\underset{\sim}{x} \tag{7}$$

as the increment in $\underset{\sim}{x}$, which is the correction made on the parameter vector $\underset{\sim}{x}$ in order to decrease $F(\underset{\sim}{x})$.

The main feature of the Fletcher-Powell method is that the increment

$$\underset{\sim}{\delta} = \alpha\underset{\sim}{s} \tag{8}$$

is taken along the direction

$$\underset{\sim}{s} = -\underset{\sim}{H}\underset{\sim}{g} \tag{9}$$

where α in (8) is that value of λ which minimizes $F(\underset{\sim}{x} + \lambda\underset{\sim}{s})$ along the direction of $\underset{\sim}{s}$. In practice, α is determined by a linear search. The matrix $\underset{\sim}{H}$ is updated at each iteration using the formula

$$\underset{\sim}{H}^{i+1} = \underset{\sim}{H}^{i} + \frac{\underset{\sim}{\delta}^{i}\underset{\sim}{\delta}^{i^T}}{\underset{\sim}{\delta}^{i^T}\underset{\sim}{\chi}^{i}} - \frac{\underset{\sim}{H}^{i}\underset{\sim}{\chi}^{i}\underset{\sim}{\chi}^{i^T}\underset{\sim}{H}^{i}}{\underset{\sim}{\chi}^{i^T}\underset{\sim}{H}^{i}\underset{\sim}{\chi}^{i}} \tag{10}$$

where

$$\underset{\sim}{\chi}^{i} = \underset{\sim}{g}^{i+1} - \underset{\sim}{g}^{i} \tag{11}$$

and the superscript i denotes the value at the ith iteration.

The updating formula (10) has the property that if $\underset{\sim}{H}^{i}$ is positive definite then $\underset{\sim}{H}^{i+1}$ must also be positive definite. Since $\underset{\sim}{H}$ is initially the unit matrix, $\underset{\sim}{H}^{i}$ is positive definite for all i.

The method has the property of quadratic convergence, that is, for a quadratic function the minimum can be located in at most n iterations. However, it depends on accurate location of the minimum along each direction of search. This is done by cubic interpolation which, although it is the most efficient method of search, requires several function and gradient evaluations.

The new Fletcher method dispenses with the linear search. The property of quadratic convergence, which depends on linear search, is replaced by a property which requires, for quadratic functions, that the eigenvalues of $\underset{\sim}{H}$ tend monotonically to those of $\underset{\sim}{G}^{-1}$. The decrease in F must be sufficiently large to guarantee ultimate convergence. This is taken care of by the following test. The change, ΔF in F on an iteration would be expected by Taylor's series expansion to be approximately $\underset{\sim}{g}^{T}\underset{\sim}{\delta}$ for a small $\underset{\sim}{\delta}$, but much less than $\underset{\sim}{g}^{T}\underset{\sim}{\delta}$ in absolute value when the position of the minimum along a line is overestimated. The change in F relative to $\underset{\sim}{g}^{T}\underset{\sim}{\delta}$ cannot become arbitrarily small if

$$\frac{\Delta F}{\underset{\sim}{g}^T \underset{\sim}{\delta}} \geq \mu \tag{12}$$

where $0<\mu<<1$, a preassigned small quantity set at 0.0001. If corrections are determined by

$$\underset{\sim}{\delta} = -\lambda \underset{\sim}{H} \underset{\sim}{g} \tag{13}$$

then trying values of $\lambda=1,w,w^2,w^3,\ldots$ for $w=0.1$ will eventually produce a $\underset{\sim}{\delta}$ that satisfies equation (12).

Although the conditions imposed so far are simple, failure can occur because $\underset{\sim}{H}$, updated using formula (10), can become singular. To overcome this problem the following updating formula is derived

$$\underset{\sim}{H}^{i+1} = \underset{\sim}{H}^i - \frac{\underset{\sim}{\delta}^i \underset{\sim}{\gamma}^{iT} \underset{\sim}{H}^i}{\underset{\sim}{\delta}^{iT} \underset{\sim}{\gamma}^i} - \frac{\underset{\sim}{H}^i \underset{\sim}{\gamma}^i \underset{\sim}{\delta}^{iT}}{\underset{\sim}{\delta}^{iT} \underset{\sim}{\gamma}^i} + \left(1 + \frac{\underset{\sim}{\gamma}^{iT} \underset{\sim}{H}^i \underset{\sim}{\gamma}^i}{\underset{\sim}{\delta}^{iT} \underset{\sim}{\gamma}^i}\right) \frac{\underset{\sim}{\delta}^i \underset{\sim}{\delta}^{iT}}{\underset{\sim}{\delta}^{iT} \underset{\sim}{\gamma}^i} \tag{14}$$

Denote $\underset{\sim}{H}^{i+1}$ in (14) as $\underset{\sim}{H}_1^{i+1}$ and $\underset{\sim}{H}^{i+1}$ in (10) as $\underset{\sim}{H}_2^{i+1}$. The formula in (14) has the property that the eigenvalues of $\underset{\sim}{H}$ tend monotonically to those of $\underset{\sim}{C}^{-1}$. Note, however, that $\underset{\sim}{H}_1^{i+1}$ does not necessarily replace $\underset{\sim}{H}_2^{i+1}$, as one of the two is chosen for updating on the basis of the following test. If

$$\underset{\sim}{\delta}^{iT} \underset{\sim}{\gamma}^i \geq \underset{\sim}{\gamma}^{iT} \underset{\sim}{H}^i \underset{\sim}{\gamma}^i \tag{15}$$

then $\underset{\sim}{H}_1^{i+1}$ is used; otherwise $\underset{\sim}{H}_2^{i+1}$ is used.

The algorithm is terminated when $||\underset{\sim}{g}|| < \varepsilon$, a tolerance level, which was set at 1.0×10^{-6}.

THE JACOBSON-OKSMAN METHOD

This method is based on homogeneous rather than quadratic functions. A consequence of this is that convergence is obtained in n+2 steps for a homogeneous function of the form

$$F(\underset{\sim}{x}) = \frac{1}{\theta}(\underset{\sim}{x}-\check{\underset{\sim}{x}})^T \underset{\sim}{g}(\underset{\sim}{x}) + F(\check{\underset{\sim}{x}}) \tag{16}$$

where $\underset{\sim}{x}$ is an n-dimensional parameter vector as before, $\underset{\sim}{g}(\underset{\sim}{x})$ is the gradient of $F(\underset{\sim}{x})$ as defined previously, θ is the degree of homogeneity and $\check{\underset{\sim}{x}}$ is the location of the minimum of $F(\underset{\sim}{x})$. On the other hand, a quadratic objective function may be expressed as

$$F(\underset{\sim}{x}) = \tfrac{1}{2}(\underset{\sim}{x}-\check{\underset{\sim}{x}})^T \underset{\sim}{Q}(\underset{\sim}{x}-\check{\underset{\sim}{x}}) + F(\check{\underset{\sim}{x}}) \tag{17}$$

where Q is a constant positive definite matrix. Thus we see that equation (17) is a special case of the homogeneous equation (16) with $\theta = 2$.

The basis of the method will now be discussed. Equation (16) may be rearranged as

$$\check{\underset{\sim}{x}}^T \underset{\sim}{g}(\underset{\sim}{x}) + \theta\, F(\underset{\sim}{x}) - w = \underset{\sim}{x}^T \underset{\sim}{g}(\underset{\sim}{x}) \tag{18}$$

where

$$w = \theta\, F(\check{\underset{\sim}{x}}) \; . \tag{19}$$

Let

$$v \triangleq \underset{\sim}{x}^T \underset{\sim}{g}(\underset{\sim}{x})$$

$$\underset{\sim}{y} \triangleq [\underset{\sim}{g}^T(\underset{\sim}{x}) \quad F(\underset{\sim}{x}) \quad -1]^T \tag{20}$$

$$\underset{\sim}{\alpha} \triangleq [\underset{\sim}{\check{x}}^T \quad \theta \quad w]^T$$

For some point $\underset{\sim}{x}_{i+1}$, where in the interest of clarity, subscripts now denote iteration numbers, equation (18) becomes

$$v_{i+1} = \underset{\sim}{\alpha}^T \underset{\sim}{y}_{i+1} = \underset{\sim}{y}_{i+1}^T \underset{\sim}{\alpha} \tag{21}$$

where $\underset{\sim}{\alpha}$ and $\underset{\sim}{y}$ are (n+2)-vectors. The vector $\underset{\sim}{\alpha}$ contains the unknowns $\underset{\sim}{\check{x}}$ and w and must be determined. If we evaluate v and $\underset{\sim}{y}$ at n+2 distinct points, $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots, \underset{\sim}{x}_{n+2}$, so that the resultant $\underset{\sim}{y}_i$ are linearly independent, we have

$$\begin{bmatrix} \underset{\sim}{y}_1^T \\ \underset{\sim}{y}_2^T \\ \vdots \\ \underset{\sim}{y}_{n+2}^T \end{bmatrix} \underset{\sim}{\alpha} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n+2} \end{bmatrix} \tag{22}$$

or, in matrix form

$$\underset{\sim}{Y}\, \underset{\sim}{\alpha} = \underset{\sim}{v} \tag{23}$$

Since the $\underset{\sim}{y}_i$ are linearly independent, the matrix $\underset{\sim}{Y}$ is non-singular, giving

$$\underset{\sim}{\alpha} = \underset{\sim}{Y}^{-1}\, \underset{\sim}{v} \tag{24}$$

To avoid matrix inversion, a recursive formula is used as new $\underset{\sim}{y}_i$ and v_i are evaluated. This is done as follows:

Let $\underset{\sim}{P}_0^{-1} = \underset{\sim}{I}$ an (n+2)x(n+2) identity matrix

and

$\underset{\sim}{v}_0 = \underset{\sim}{\alpha}_0$, where $\underset{\sim}{\alpha}_0$ is given.

At each iteration we successively replace corresponding rows and elements of $\underset{\sim}{P}_0^{-1}$ and $\underset{\sim}{v}_0$ with calculated values of $\underset{\sim}{y}_i$ and v_i, respectively, using

$$\underset{\sim}{P}_{i+1}^{-1} = \underset{\sim}{P}_i^{-1} + \underset{\sim}{e}_j\, (\underset{\sim}{y}_{i+1}^T - \underset{\sim}{e}_j^T\, \underset{\sim}{P}_i^{-1}) \tag{25}$$

$$\underset{\sim}{v}_{i+1} = \underset{\sim}{v}_i + \underset{\sim}{e}_j (v_{i+1} - \underset{\sim}{e}_j^T\, \underset{\sim}{v}_i) \tag{26}$$

where $\underset{\sim}{e}_j$ is a unit vector having unity as the jth element and zeros elsewhere, with j = i+1.

Using Householder's formula

$$\underset{\sim}{P}_{i+1} = \underset{\sim}{P}_i - \frac{\underset{\sim}{P}_i\, \underset{\sim}{e}_j\, (\underset{\sim}{y}_{i+1}^T\, \underset{\sim}{P}_i - \underset{\sim}{e}_j^T)}{\underset{\sim}{y}_{i+1}^T\, \underset{\sim}{P}_i\, \underset{\sim}{e}_j} \tag{27}$$

Successive estimates of the vector $\underset{\sim}{\alpha}$ are given by

$$\underset{\sim}{a}_{i+1} = \underset{\sim}{a}_i + \frac{\underset{\sim}{P}_i \quad \underset{\sim}{e}_j (v_{i+1} - \underset{\sim}{y}_{i+1}^T \underset{\sim}{a}_i)}{\underset{\sim}{y}_{i+1}^T \underset{\sim}{P}_i \underset{\sim}{e}_j} \tag{28}$$

Given the linear independence of $\underset{\sim}{y}_i$ it can be shown that for a homogeneous function

$$\underset{\sim}{P}_{n+2} = \underset{\sim}{X}^{-1} \tag{29}$$

and

$$\underset{\sim}{a}_{n+2} = \underset{\sim}{a} \tag{30}$$

Thus, the algorithm finds the minimum, the degree of homogeneity and the value of the minimum after n+2 iterations.

APPLICATION TO THE PROBLEM OF OPTIMUM APPROXIMATION

The gradient methods discussed in the previous two sections depend considerably upon the availability of the first partial derivatives of the objective function with respect to the parameters of the model. Fortunately, it is fairly easy to derive analytical expressions for the gradient for a second-order model. Hence, the methods could be applied directly.

Since the efficiency of an optimization method is the speed with which the algorithm proceeds to the optimum, a fair indication of performance is the number of times the function and its gradient have to be evaluated. This is a valid assumption for the problem under consideration since the time taken for one evaluation of the function and its gradient is much more than the "housekeeping"-logic and other simple operations in the algorithm. Moreover, in the following discussion, a function evaluation will imply the complete evaluation of the function and its gradient, since only gradient methods are being considered.

RESULTS

First the two-parameter problem resulting from equation (4) was tried. The computer used was a CDC 6400 and typical C.P. times for the two-parameter problem were about 4 seconds. Three different starting points were considered for each of the three methods and in every case the algorithms ultimately converged to the same optimum parameters a_0=3.195912, a_1=2.281056, with the optimum value of the objective function 7.50758×10^{-4} and the components of the gradient less than 1.0×10^{-9}. Figure 1 shows the corresponding response.* Table 1 compares the number of function evaluations required for each method for the objective function to reach the value 7.50759×10^{-4}. This value is 1.0×10^{-9} higher than the optimum ultimately obtained. From Table 1 it can be seen that both the new Fletcher method and the Jacobson-Oksman method show an improvement over the Fletcher-Powell method with the Jacobson-Oksman method being slightly better than the new Fletcher method.

For the three-parameter problem represented by equation (5) typical computer C.P. times were about 6 seconds. Again three different starting points were tried with each method and in each case the algorithms converged to the optimum a_0=1.997397, a_1=1.660663, b_1=4.370715 x 10^{-2} with the optimum value of the objective function 1.582215 x 10^{-4} and the gradient components less than 1.0×10^{-9}. Figure 2 shows the corresponding response.* Table 2

*Figures do not show corresponding responses, but responses of models with steady state values equal to the system response at 8 sec.

compares the number of function evaluations required to reach the objective function value 1.582225×10^{-4} which is 1.0×10^{-9} higher than the optimum.

Starting points	a_0=3.0, a_1=2.0	a_0=0.5, a_1=0.5	a_0=1.0, a_1=1.0
Jacobson-Oksman	21	19	14
New Fletcher	21	22	19
Fletcher-Powell	29	49	32

Table 1: Number of function evaluations required to reach the objective function value 7.50759×10^{-4} for the 2-parameter problem.

Here again it is seen that the Jacobson-Oksman and the new Fletcher methods are superior to the Fletcher-Powell method and of the two methods the Jacobson-Oksman is slightly better. However, there was one case, not shown, when the Jacobson-Oksman method failed. This was due to a very large step that made the objective function too large for the computer to handle with single precision. In situations like this one is tempted to limit the step size, however it was decided not to interfere with the algorithm in any way, as this might slow down the method at the beginning.

Starting points	a_0=1.0, a_1=1.0, b_1=1.0	a_0=0.5, a_1=2.0, b_1=4.0	a_0=1.0, a_1=0.5, b_1=0.1
Jacobson-Oksman	39	39	29
New Fletcher	27	76	35
Fletcher-Powell	60	274	58

Table 2: Number of function evaluations required to reach the objective function value 1.582225×10^{-4} for the 3-parameter problem.

CONCLUSIONS

From the example considered, it may be concluded that all the three gradient methods can be successfully used for the determination of optimum low-order models for a high-order system. However, the two new methods are much superior to the older but highly-regarded Fletcher-Powell method. Between the two new methods, there is not much to choose, although the Jacobson-Oksman method was slightly faster for most of the cases considered.

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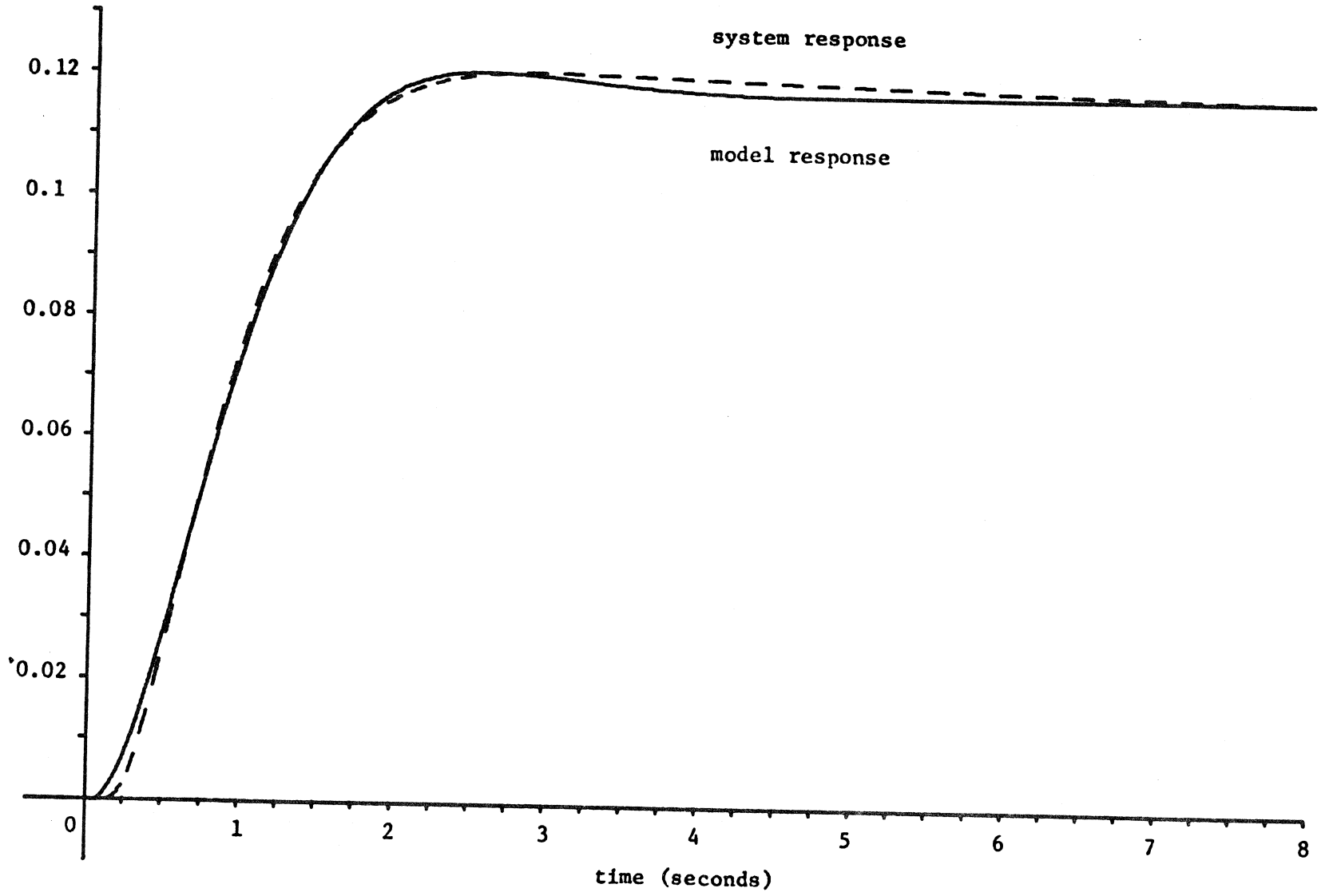

Figure 1. Response of the 2-parameter model.

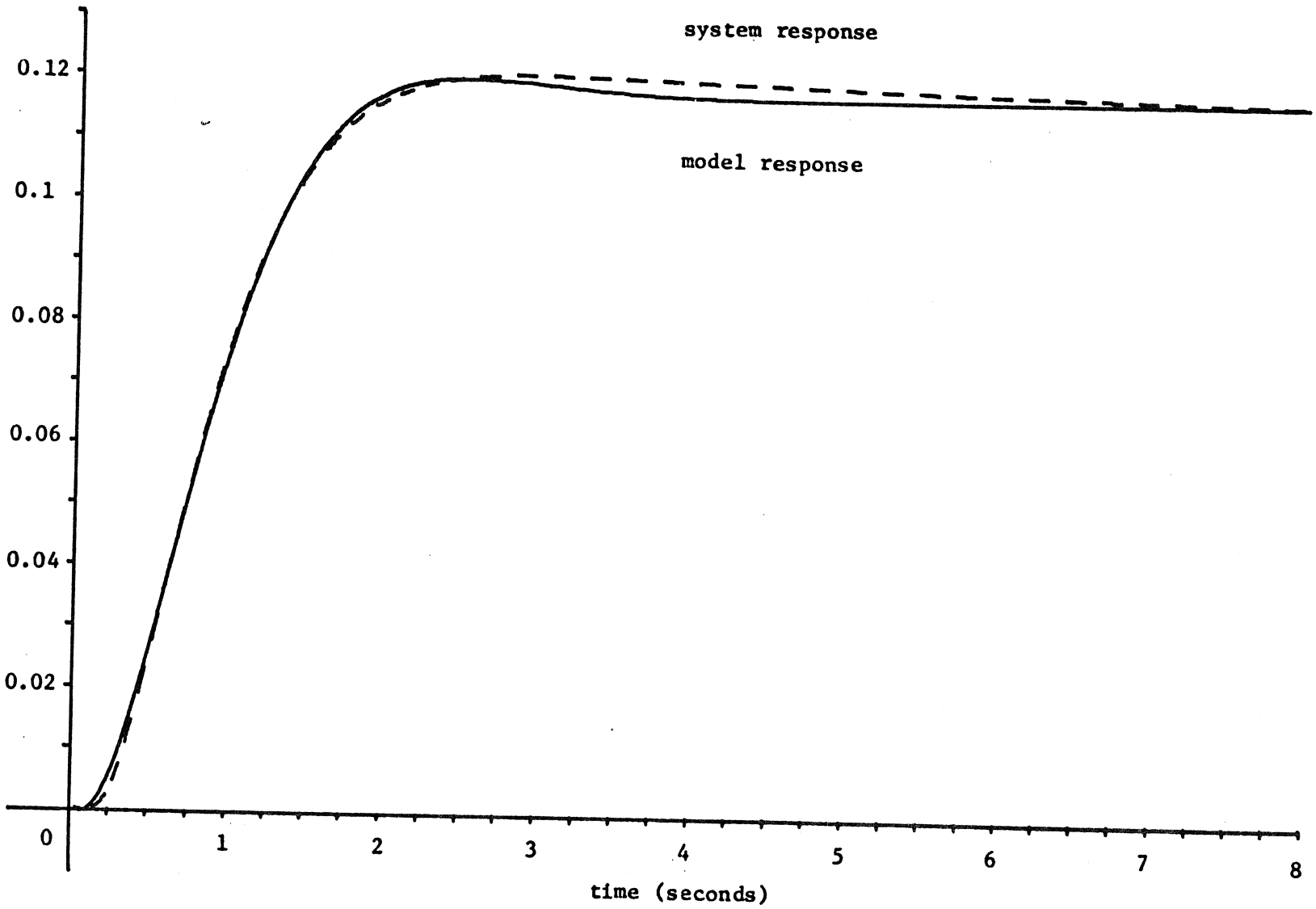

Figure 2. Response of the 3-parameter model.