PROCEEDINGS
SIXTH HAWAII INTERNATIONAL CONFERENCE ON SYSTEM SCIENCES

Edited By
Art Lew

Conference Held
January 9, 10, 11, 1973
University of Hawaii
Honolulu, Hawaii

Sponsored By:
Department of Electrical Engineering and
Department of Information and Computer Sciences
University of Hawaii

Supported By:
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Circuit Theory
Systems, Man and Cybernetics
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Society for Industrial and Applied Mathematics

CONSTRAINED MINIMAX OPTIMIZATION BY GRAZOR SEARCH

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Abstract

A practical approach to solving constrained minimax problems is presented. A method has been proposed recently in which a constrained minimization problem has been formulated as an unconstrained minimax problem. The essense of this approach can be used to treat the constrained minimax problem as a nonlinear programming problem, and then reformulating it into an unconstrained minimax problem. A recently proposed optimization algorithm called grazor search can suitably be used to solve the reformulated unconstrained minimax problem. The proposed method can handle any set of constraints-parameter constraints in particular.

1. INTRODUCTION

The problem of unconstrained minimax approximation is an important one, and has been considered by a number of authors[1-3]. These methods are, however, not very efficient both in terms of rate of convergence towards the solution, and the computation time required to reach the vicinty of the optimum. A new method called grazor search has been recently proposed[4], and has been found to be both efficient and reliable in effectively handling the solution of unconstrained minimax problems.

2. GRAZOR SEARCH METHOD[4,5]

Suppose we have the problem of minimizing

$$U(\underset{\sim}{\phi}) = \max_{1 \le i \le n} y_i(\underset{\sim}{\phi}) \qquad (1)$$

where $\underset{\sim}{\phi}$ denotes the k independent parameters and y_i are real, nonlinear, differentiable functions in general. Let $\hat{y}_\ell(\underset{\sim}{\phi})$, $\ell=1, \ldots, n_r$ be the largest local discrete maxima (ripples) of $y_i(\underset{\sim}{\phi})$, $1 \le i \le n$, in descending magnitude. The grazor search method consists of solving a linear program at the point $\underset{\sim}{\phi}^j$

$$\text{maximize} \quad \alpha_{k_r+1}(\underset{\sim}{\phi}^j) \ge 0 \qquad (2)$$

subject to

$$- \underset{\sim}{\nabla}^T \hat{y}_i(\underset{\sim}{\phi}^j) \sum_{\ell=1}^{k_r} \alpha_\ell^j \underset{\sim}{\nabla} \hat{y}_\ell(\underset{\sim}{\phi}^j) \le - \alpha_{k_r+1}^j \quad i=1,\ldots,k_r \qquad (3)$$

$$\alpha_i^j \ge 0 \qquad i=1,\ldots,k_r \qquad (4)$$

$$\sum_{\ell=1}^{k_r} \alpha_\ell^j = 1 \qquad (5)$$

where $\hat{y}_\ell(\underset{\sim}{\phi}^j)$, $\ell=1,\ldots,k_r$ are the highest ripples under consideration ($k_r \le n_r$). We next define

$$\Delta\underset{\sim}{\phi}^j = - \sum_{\ell=1}^{k_r} \alpha_\ell^j \underset{\sim}{\nabla} \hat{y}_\ell(\underset{\sim}{\phi}^j) \qquad (6)$$

which is normalised to

$$\Delta\underset{\sim}{\phi}_n^j = \Delta\underset{\sim}{\phi}^j / \|\Delta\underset{\sim}{\phi}^j\| \tag{7}$$

Starting at $\underset{\sim}{\phi}^j$, one or more steps are taken in the direction of $\Delta\underset{\sim}{\phi}_n^j$ until an improved point is obtained for a step equal to $\Delta\underset{\sim}{\phi}^o$. Next, a method based on golden section search to find the γ^{j*} corresponding to the constrained minimum value of $U(\underset{\sim}{\phi}^j + \gamma^j \Delta\underset{\sim}{\phi}^o)$ is used. The jth iteration ends by setting

$$\underset{\sim}{\phi}^{j+1} = \underset{\sim}{\phi}^j + \gamma^{j*} \Delta\underset{\sim}{\phi}^o \tag{8}$$

This method is guaranteed to converge under certain conditions.

3. CONSTRAINED MINIMAX PROBLEM

Consider the problem of minimizing (1) subject to

$$g_j(\underset{\sim}{\phi}) \geq 0 \quad j=1,2,\ldots,m \tag{9}$$

where g_j are non-linear functions of the parameters in general. This problem reduces to minimizing z subject to (9) and

$$z - y_i(\underset{\sim}{\phi}) \geq 0 \quad i=1,2,\ldots,n \tag{10}$$

The above problem can be reformulated as an unconstrained minimax problem by two methods, one using a recently proposed method due to Bandler and Charalambous[6], and the other using weighting functions.

Formulation 1

The problem can be reformulated as minimizing

$$V(\underset{\sim}{\phi},\underset{\sim}{\alpha}) = \max_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} [z, z-\alpha_1(z-y_i(\underset{\sim}{\phi})), z-\alpha_{j+1}g_j(\underset{\sim}{\phi})] \tag{11}$$

where $$\underset{\sim}{\alpha} \triangleq [\alpha_1\ \alpha_2\ \ldots\ \alpha_{m+1}]^T \tag{12}$$

$$\alpha_j > 0 \quad j=1,2,\ldots,m+1 \tag{13}$$

For a large enough value of $\underset{\sim}{\alpha}$ one can obtain, in principle, the exact optimal solution for the original problem by minimizing this reformulated objective function.

When implementing this scheme one can, for the problem defined earlier, slightly modify the formulation in order to save on computational effort, so that the minimization function chosen is

$$V'(\underset{\sim}{\phi},\underset{\sim}{\alpha}) = \max_{\substack{1\leq \ell\leq n_r \\ 1\leq j\leq m}} [z, z-\alpha_1(z-\hat{y}_\ell(\underset{\sim}{\phi})), z-\alpha_{j+1}g_j(\underset{\sim}{\phi})] \tag{14}$$

Formulation 2

Minimize

$$W(\underset{\sim}{\phi},\underset{\sim}{w}) = \max_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} [y_i(\underset{\sim}{\phi}), -w_j g_j(\underset{\sim}{\phi})] \tag{15}$$

where $$\underset{\sim}{w} \triangleq [w_1\ w_2\ \ldots\ w_m]^T \tag{16}$$

$$w_j > 0 \quad j=1,2,\ldots,m \tag{17}$$

For purposes of practical implementation, as long as $U(\underset{\sim}{\phi}) > 0$ and one wishes to apply nonzero weights only to violated constraints of (9), the minimization function may be chosen as

$$W'(\underset{\sim}{\phi},\underset{\sim}{w}') = \max_{\substack{1\leq \ell\leq n_r \\ 1\leq j\leq m}} [\hat{y}_\ell, -w'_j g_j(\underset{\sim}{\phi})] \tag{18}$$

where $$\underset{\sim}{w}' \triangleq [w'_1\ w'_2\ \ldots\ w'_m]^T \tag{19}$$

$$\begin{aligned} w'_j &> 0 \quad \text{for } g_j(\underset{\sim}{\phi}) < 0 \\ w'_j &= 0 \quad \text{for } g_j(\underset{\sim}{\phi}) \geq 0 \end{aligned} \qquad j=1,2,\ldots,m \tag{20}$$

The advantage of this formulation is apparent when U>0 implies that certain specifications are violated and U<0 implies that they are satisfied. In this case, comparison with violated and satisfied constraints seems appropriate.

By proper choice of the elements of $\underset{\sim}{\alpha}$, $\underset{\sim}{w}$ or $\underset{\sim}{w}'$, the reformulated functions V, V', W or W' can be minimized by using a suitable algorithm.

In case of parameter constraints, upper and lower specifications can be considered as follows

$$g_{2\ell-1}(\underset{\sim}{\phi}) = \phi_i - \phi_{i\ell} \geq 0 \qquad \ell=1,2,\ldots k \qquad (21)$$

$$g_{2\ell}(\underset{\sim}{\phi}) = -(\phi_i - \phi_{iu}) \geq 0$$

$$g_j(\underset{\sim}{\phi}) \geq 0 \qquad (22)$$

$$j=2k+1, 2k+2\ldots, m$$

4. RESULTS

The proposed approach was applied to the design of a 5-section cascaded lossless transmission-line filter with unit impedance terminations. The problem has been previously considered[7] for lengths fixed at a quarter-wavelength of ℓ_q=2.5 cm corresponding to 3 GHz, and for a required attenuation of 0.4 db in the passband (0-1 GHz). Optimal values have been derived for characteristic impedance values when a stopband frequency of 3 GHz was chosen. The functions chosen for the problem were

$$y_i(\underset{\sim}{\phi}) = |\rho_i| - r \quad f_i \epsilon 0\text{-}1 \text{ GHz}$$

$$y_i(\underset{\sim}{\phi}) = 1 - |\rho_i| \quad f_i = 3 \text{ GHz} \qquad (23)$$

where $\rho_i = \rho(f_i)$ is an ith reflection coefficient magnitude at a discrete frequency f_i, and r corresponds to an attenuation of 0.4 db, respectively. Twenty-one uniformly-spaced points were chosen in the passband.

The lengths ℓ_i were initially fixed at ℓ_q and the impedances Z_i were varied. The impedance constraints imposed were $0.5 \leq Z_i \leq 2.0$, i=1,2,...,5 while the minimization function was chosen to be W'. Further, n=22, m=10 and w_j=1000 (for unsatisfied constraints) or 0 (for satisfied constraints) for j=1,2,...,m. Table I shows the results of optimizing the impedances, and it is observed that some of the impedances of the constrained solution lie on constraint boundaries.

Parameters	Unconstrained optimal solution	Constrained solution (i)	Constrained solution (ii)
Z_1	3.151	0.5683	1.760
Z_2	0.4416	2.000	0.5000
Z_3	4.419	0.5000	2.000
Z_4	0.4416	2.000	0.5000
Z_5	3.151	0.5683	1.760
U	3.951×10^{-5}	3.255×10^{-3}	3.255×10^{-3}
W'	2.419×10^{3}	3.255×10^{-3}	3.255×10^{-3}

Table I 5-section filter design problem

Moreover, there are two distinct solutions, for which the impedances are reciprocals of each other.

As a further step, it was desired to improve the performance, if possible, by allowing both the lengths and impedances to vary, and imposing the following constraints: $0 \leq \ell_{in} \leq 2$, $0.4416 \leq Z_i \leq 4.419$ (corresponding to lower and upper limits of the unconstrained optimum of Table I) for i=1,2,...,5, $0 \leq \sum_{j=1}^{5} \ell_{jn} \leq 5$, where $\ell_{in} = \ell_i/\ell_q$. The function to be minimized was chosen as V, while n=22, m=22, α_j=10 for j=1,2,...,m+1. The starting parameters corresponded to optimal values when the lengths are fixed at ℓ_q and the above mentioned impedance constraints are imposed. It was observed that no improvement could be achieved from the starting point and that the starting point satisfies the necessary conditions for a minimax optimum[8].

5. CONCLUSIONS

The experience with the approach seems to indicate that the method is very useful in tackling constrained minimax problems effectively, now that a reliable unconstrained minimax algorithm is available. The method has a number of applications, including high-order system modelling and control system designs, where constraints may have to be imposed on the pole-zero locations of the system responses.

6. ACKNOWLEDGEMENTS

This work was supported by the National Research Council of Canada under grants A7239 and C154. Discussions with C. Charalambous of McMaster University are gratefully acknowledged.

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