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ON REALISTIC MINIMAX MODELLING OF

HIGH-ORDER SYSTEMS

J.W. Bandler and T.V. Srinivasan
Department of Electrical Engineering
McMaster University
Hamilton, Ontario, Canada.

This paper investigates the feasibility of automated design of lower-order models for high-order systems where the order of the models can be increased efficiently according to desired performance criteria. The modelling can be done for a variety of objectives with or without constraints, so that a realistic on-line or off-line design can be achieved to satisfy a set of arbitrary transient and steady-state response specifications. Suitable examples are chosen to illustrate the modelling procedure.

Introduction

A number of methods are now available for determining low-order models for high-order complex systems for a variety of performance criteria and objectives. Least-squares and minimax system models have been derived for high-order systems recently[1-3] using direct search[4] and gradient[5-7] optimization techniques. These models were derived on the basis of measured input-output data of the system and the steady-state of the model was fixed at a certain value.

Minimax objectives have been considered throughout this paper for purposes of modelling, though any other objective could also suit the purpose. It is now possible to tackle constrained minimax problems rather efficiently[8-9], and a generalized objective function can be formulated, which can accomodate the steady-state error between system and model responses, which makes the whole modelling procedure rather flexible and meaningful. Thus, arbitrary transient and steady-state response specifications can be imposed on the model for a desired performance criterion.

The whole modelling procedure can be automated, so that it is possible to move from optimal low-order models to higher-order models without degradation in the objective function value. By this procedure, modelling may be continued and the order of the model increased, until the error criterion meets the desired objectives of the user. This can be done on-line or off-line, though automated modelling would be quite important in on-line operations.

Once a set of parameters for a model has been obtained by optimization, it may be important to investigate the solution for optimality, and recently a program has been developed[10], which is capable of testing a solution for minimax optimality conditions. If the solution is optimal, the user may decide to increase the order of the model for improvement in the objective function, while if it is not optimal, the user may decide to use another optimization technique to improve the results.

Second- and third-order models have been derived for a nuclear-reactor system described by a ninth-order non-linear differential equation with and without steady-state constraint specifications, and the solutions have been verified to satisfy the necessary optimality conditions. It is proposed to apply constraints on the model parameters so as to guarantee that the pole-zero locations of the model in the s-domain may not contribute to instability of the model.

Minimax System Modelling

The modelling problem considered assumes that input-output data of the system is known, and that the error criterion considered is minimax (or Chebyshev). It is required to find a transfer function of a given order such that its response is an approximation to the response of the high-order system in the minimax sense. The problem may be tackled by efficient direct[7] or indirect[11] minimax algorithms.

In general the transfer function of a given order n may be written as

$$H_{m,n}(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$= \frac{\sum_{i=0}^{m} b_i s^i}{s^n + \sum_{j=1}^{n} a_{n-j} s^{n-j}} \quad (1)$$

where $m \leq n$ for physical systems. For this work the input is a unit step and the criterion chosen

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is to minimize the maximum error between the system and the model responses over a specified time-interval [0,T], where the vector of variable model parameters is given by

$$\underset{\sim}{\phi} = [a_0\ a_1\ \ldots\ a_{n-1}\ b_0\ b_1\ \ldots\ b_m]^T \qquad (2)$$

In this paper

t_i	is an ith time instant in [0,T]
c_i^s	is the response of the system at t_i
$c_i^m(\underset{\sim}{\phi})$	is the response of the approximating model at t_i
$e_i(\underset{\sim}{\phi})=c_i^m(\underset{\sim}{\phi})-c_i^s$	is the error between the system and model responses at t_i
c_∞^s	is the steady-state value of the system
c_∞^m	is the steady-state value of the model

The usual approximation problem that has been considered in the past[2-3] assumes that c_∞^m is fixed at a convenient value (usually c_∞^s or c_i^s at t_i=T), so that the objective is to minimize

$$U(\underset{\sim}{\phi}) = \max_{t_i \in [0,T]} |e_i(\underset{\sim}{\phi})| \qquad (3)$$

It may, however, be unacceptable to fix c_∞^m at a certain value, in which case a realistic tradeoff between transient and steady-state errors can be achieved.

A Generalized Objective Function

It has been recently proposed[8] that a constrained minimization problem can be formulated as an unconstrained minimax problem. This approach has been used by Bandler and Srinivasan[9] to consider the constrained minimax problem as a nonlinear programming problem, and then reformulating it into an unconstrained minimax problem. It is now possible to apply this method to system modelling so that a generalized objective function can be defined to take into account both the transient and steady-state response errors. The following additional notation is required.

$S_{u\infty}$	is the upper bound of the system specifications at steady-state
$S_{\ell\infty}$	is the lower bound of the system specifications at steady-state
$e_{u\infty}=c_\infty^m-S_{u\infty}$	is the error between upper system specifications and model steady-state value
$e_{\ell\infty}=c_\infty^m-S_{\ell\infty}$	is the error between lower system specifications and model steady-state value

The problem may now be formulated into two forms as follows.

Formulation 1

$$V(\underset{\sim}{\phi},\alpha,\alpha_{\ell\infty},\alpha_{u\infty}) = \max_{t_i \in [0,T]} [z, z-\alpha(z-|e_i(\underset{\sim}{\phi})|), z-\alpha_{\ell\infty}e_{\ell\infty}, z+\alpha_{u\infty}e_{u\infty}] \qquad (4)$$

where α, $\alpha_{\ell\infty}$ and $\alpha_{u\infty}$ are positive. For sufficiently large values of α, $\alpha_{\ell\infty}$ and $\alpha_{u\infty}$ one can obtain, in principle, the exact optimal solution for the original problem by minimizing this reformulated objective function. If c_∞^m is fixed such that $e_{\ell\infty}$ and $-e_{u\infty}$ are positive, the objective function (4) reduces essentially to $U(\underset{\sim}{\phi})$.

Formulation 2

Minimize

$$W(\underset{\sim}{\phi},w_{\ell\infty},w_{u\infty}) = \max_{t_i \in [0,T]} [|e_i(\underset{\sim}{\phi})|, -w_{\ell\infty}e_{\ell\infty}, w_{u\infty}e_{u\infty}] \qquad (5)$$

where

$$w_{\ell\infty} \begin{cases} = 0 \text{ for } -e_{\ell\infty} < 0 \\ > 0 \text{ for } -e_{\ell\infty} \geq 0 \end{cases}$$

$$w_{u\infty} \begin{cases} = 0 \text{ for } e_{u\infty} < 0 \\ > 0 \text{ for } e_{u\infty} \geq 0 \end{cases}$$

If c_∞^m is fixed within satisfied specifications the above objective function reduces to $U(\underset{\sim}{\phi})$.

Comments

In cases where suitable constraints - including parameter constraints - are imposed, the above procedure may be used to incorporate this in the objective function[9]. In many cases it is convenient to choose $S_{\ell\infty}=S_{u\infty}=c_\infty^s$.

Automated Lower-order Models

One of the major problems that is encountered during modelling is to decide whether a certain lower-order model is acceptable or not. If the model is too simple so that computing time for optimizing model parameters is small, the approximation to the original system may be very bad, while if the model is complex, then the very need for system modelling is lost. If one were to strike a reasonable compromise between the speed with which the model is optimized, and the accuracy of the approximation, it would not be unreasonable to devise a scheme whereby one could increase the complexity of the model in an automated fashion after a certain number of iterations or computer time. It is, however, important to keep in mind the desirability of making this increase in complexity as smooth as possible, so that the objective function value is not degraded. Thus, either the number of parameters could be increased for a model with a certain order, or the order of the model itself can be increased.

Let $H^*_{m,n}$ denote an optimized model of the form (1). Three possibilities occur as follows.

Increase in Parameters Only

$$H^*_{m,n}(s) \rightarrow H_{m+p,n}(s)$$

Here $b_{m+p}, b_{m+p-1}, \ldots, b_{m+1}$ are assumed to be initially zero so that $H_{m+p,n} = H^*_{m,n}$ in the first iteration.

Increase in Order

$$H^*_{m,n}(s) \rightarrow H_{m+q,n+q}(s)$$

Here q poles of $H_{m+q,n+q}(s)$ are assumed to cancel with q zeros initially, so that $H_{m+q,n+q} = H^*_{m,n}$ in the first iteration. In this case, initial guesses for q poles (or zeros) are necessary.

Increase in Order and Parameters

$$H^*_{m,n}(s) \rightarrow H_{m+p+q,n+q}(s)$$

Here $b_{m+q+p}, \ldots, b_{m+q+1}$ are assumed to be zero initially and that there is a cancellation of q zeros and q poles at start, so that $H_{m+p+q,n+q} = H^*_{m,n}$ in the first iteration.

A careful choice of initial parameters can make the increase in model complexity smooth so that the whole modelling procedure can be automated on a small digital computer on-line.

Optimality Conditions

Once a certain lower-order model has been optimized using an optimization technique, it may be necessary to investigate the parameter solution for minimax optimality conditions[12]. Though the necessary optimality conditions seem straightforward to verify it is both tedious and difficult in practice. Bandler and Srinivasan[10] have recently proposed a user-oriented computer program which makes the testing of a solution for optimality practicable.

It is now possible to investigate the solutions after a certain number of iterations of the modelling algorithm, or when a certain convergence criterion is reached, so that one may decide whether to carry on with further optimization, to increase the order of the model, or to terminate altogether.

Results

For the examples considered, two second-order models and a third-order model were chosen.

$$H_{02}(s) = \frac{b_0}{s^2+a_1s+a_0}$$

$$H_{12}(s) = \frac{B_1s+B_0}{s^2+A_1s+A_0}$$

$$H_{23}(s) = \frac{x_6s^2+x_5s+x_4}{(s+x_3)(s^2+x_2s+x_1)}$$

The transition between the three models can be made smooth by making the following substitutions at the start of the new model.

$$H^*_{02} \rightarrow H_{12} : A_0=a^*_0, A_1=a^*_1, B_0=b^*_0, B_1=0$$

$$H^*_{02} \rightarrow H_{23} : x_1=a^*_0, x_2=a^*_1, x_3 = \text{positive value}$$
$$x_4=x_3b^*_0, x_5=b^*_0, x_6=0$$

$$H^*_{12} \rightarrow H_{23} : x_1=A^*_0, x_2=A^*_1, x_3= \text{positive value}$$
$$x_4=B^*_0x_3, x_5=B^*_1x_3+B^*_0, x_6=B^*_1$$

Two cases were considered for both examples.

c^m_∞ fixed

In this case,

$$w_{\ell\infty} = w_{u\infty} = 0$$

and

$$U(\phi) = \max_{t_i \in [0,T]} |e_i(\phi)|$$

c^m_∞ varied

In this case,

$$w_{\ell\infty} = w_{u\infty} = w_\infty$$

and

$$U(\phi) = \max_{t_i \in [0,T]} [|e_i(\phi)|, -w_\infty e_{\ell\infty}, w_\infty e_{u\infty}]$$

Example 1

A 9th-order nuclear reactor system[13] was chosen, where a step input is considered so that the power level of the reactor system changes from 90% to 100% of the full power. T was equal to 10 seconds.

The results, shown in Table I, indicate that the increase in order of the model did not produce any large improvement in $\check{U}$, the minimum value of U, and in this case a model increase is quite wasteful from the computing viewpoint. On the other hand, an improvement in the transient error at a slight expense on the steady-state error is obtained.

Table I. Results for Example 1

Case	Model	1000 $\check{U}$	1000 max$[-e_{\ell\infty}, e_{u\infty}]$
c_∞^m fixed at c_∞^s	H_{02}	2.9234	0
	H_{12}	2.7018	0
	H_{23}	2.4040	0
c_∞^m varied, $w_\infty = 1$, $S_{\ell\infty}=S_{u\infty}=c_\infty^s$	H_{23}	1.2167	1.2166

Example 2

The system considered was a 7th-order control system for the pitch rate of a supersonic aircraft[2,3]. T was equal to 8 seconds though the responses shown in Figs. 1 to 3 were taken up to 20 seconds. c_∞^s was equal to 0.11111. The results are summarized in Table II.

Table II. Results for Example 2

Case	Model	1000 $\check{U}$	1000 max $[-e_{\ell\infty}, e_{u\infty}]$	Fig.
c_∞^m fixed at c_i^s for $t_i = T$	H_{02}	3.7635		1
	H_{12}	2.4872		2
	H_{23} (6 ripple)	1.0207		3
	H_{23} (5 ripple)	1.2140		-
c_∞^m varied, $w_\infty = 1$, $S_{\ell\infty}=S_{u\infty}=c_\infty^s$	H_{02}	4.1656	4.1656	1
	H_{12}	4.1582	4.1582	2
	H_{23}	1.0201	0.91785	3
c_∞^m varied, $w_\infty = 10^6$, $S_{\ell\infty}=0.11061$, $S_{u\infty}=0.11161$	H_{02}	7.7657	7.6945 $\times 10^{-6}$	-
	H_{12}	7.8624	0.	-
	H_{23}	1.0201	9.8483 $\times 10^{-7}$	-

The results indicate that when c_∞^m is fixed increasing the order of the model does improve the transient errors, and it has been shown that for the third-order model both the 5-ripple and 6-ripple solutions satisfy the necessary minimax optimality conditions[3]. It is interesting to note that in all the cases considered, the third-order model gives the best result corresponding to the same transient error and three different steady-state errors. Some of the optimal parameters when c_∞^m is fixed tend to have nearly zero real parts which may make the model oscillatory. Using appropriate parameter constraints (as indicated in an earlier section) satisfactory results can be obtained which would guarantee a minimum damping of the model for a step input.

Conclusions

Lower-order modelling of high-order systems can be automated rather easily, and, with the availability of efficient optimization techniques, on-line system modelling and control is entirely feasible. The proposed ideas can be effectively used to get desired optimal models in the minimax sense within user-specified computing times and error allowances.

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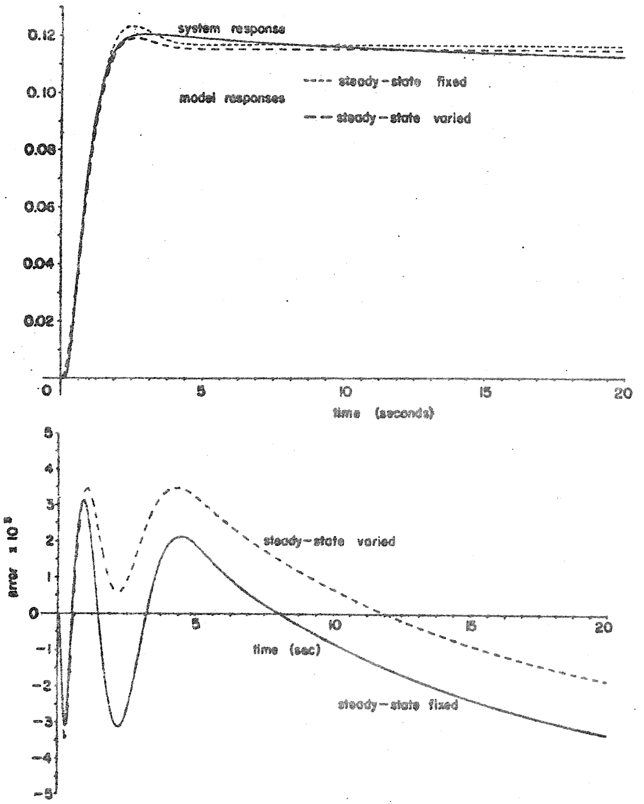

FIG. 1

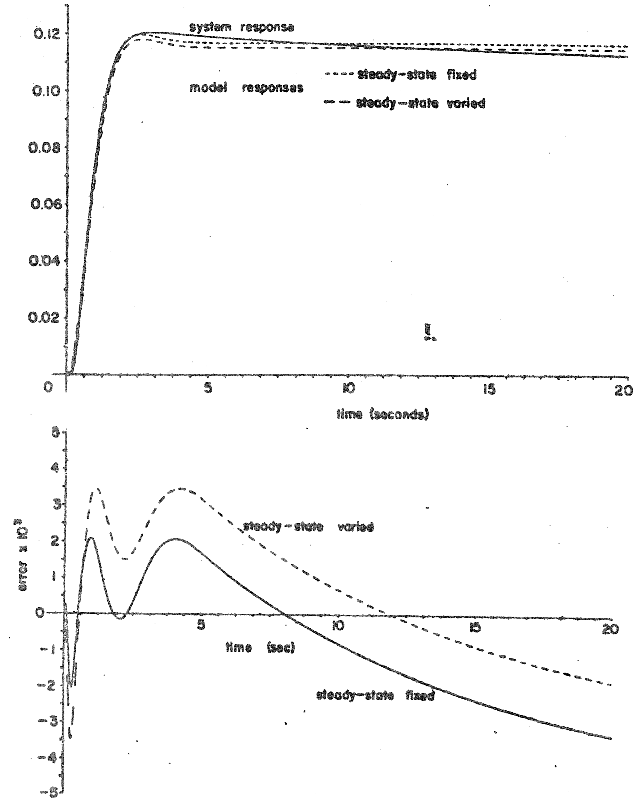

FIG. 2

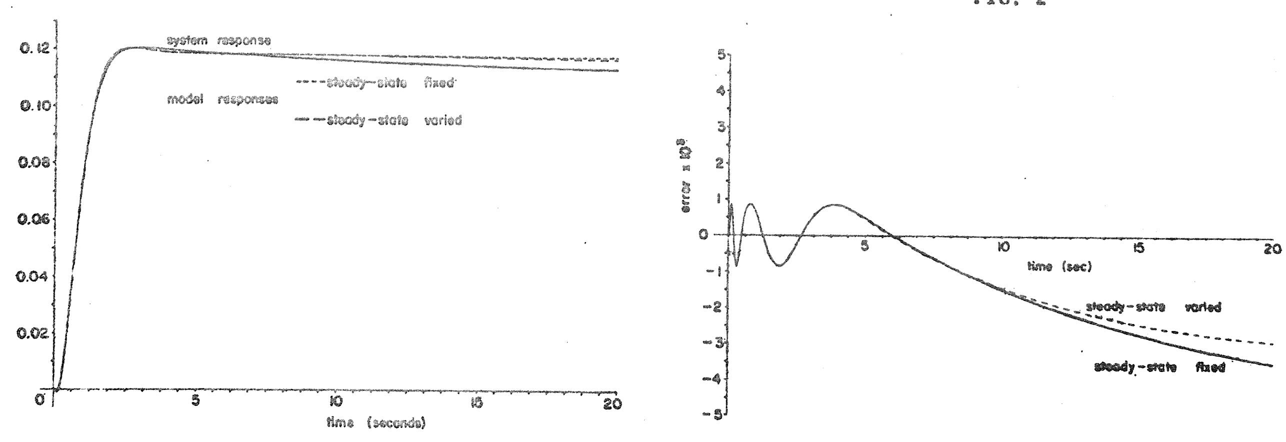

FIG. 3