LEAST PTH OPTIMIZATION OF RECURSIVE DIGITAL FILTERS

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Least pth optimization of recursive digital filters using large values of p, typically 10,000 is presented. The Fletcher method is used in conjunction with the Bandler-Charalambous method. Local optimality and stability checking of the solution along with an option for increasing the order complexity of the filter is implemented.

Introduction

This paper describes the application of the Bandler-Charalambous[1] method involving least pth approximation using extremely large values of p, typically 1,000 to 1,000,000, for choosing the coefficients of a recursive digital filter to meet arbitrary specifications of the magnitude characteristics. The local optimality of the least pth solution is checked by perturbation. The order complexity of the filter can be increased through growing of filter sections to meet the prescribed specifications. A pole inversion technique[2] to meet the stability requirements is also implemented. A comparison is made between the Fletcher-Powell method[3] and the more recent Fletcher method[4] in conjunction with the application of least pth optimization to a recursive digital filter design example. An example where effectively negative values of p were used is also presented.

The Problem

Suppose that upper and lower bounds on the magnitude characteristics of a recursive digital filter are prescribed at a discrete set of frequencies $f_1,\ldots,f_m$. These correspond to a discrete set of values of the variable z evaluated on the unit circle in the z-domain

$$z_i = e^{j\psi_i\pi}\ , \quad i = 1,\ldots,m$$

where

$$\psi_i = \frac{2f_i}{f_s}\ , \quad i = 1,\ldots,m$$

and f_s is the sampling frequency. The transfer function of a recursive digital filter is chosen to be of the cascade form, namely,

$$H(z) = A \prod_{k=1}^{K} \frac{1 + a_k z^{-1} + b_k z^{-2}}{1 + c_k z^{-1} + d_k z^{-2}}$$

All the poles of the transfer function should lie within the unit circle in the z-domain in order that the filter be stable.

The inversion of a pole of the transfer function with respect to the unit circle in the z-domain is equivalent to multiplying the transfer function by a particular all pass function, implying that the inversion of a pole of the transfer function with respect to the unit circle does not affect the shape of the magnitude characteristics[2]. Thus, all the poles that do not lie within the unit circle in the z-domain can be inverted with respect to the unit circle to insure the stability of the filter.

The problem is to find the n-dimensional parameter vector $\underset{\sim}{\phi} = [a_1\ b_1\ c_1\ d_1\ a_2\ b_2\ c_2\ d_2\ \ldots\ A]^T$, where $n=4K+1$, to minimize an appropriately chosen objective function comprising real error functions related to the upper and lower specified bounds[1].

The following notation is used[1]

- $S_u(\psi)$ upper specified function (desired response bound)
- $S_u'(\psi,\xi)$ artificial upper specified function
- $S_\ell(\psi)$ lower specified function (desired response bound)
- $S_\ell'(\psi,\xi)$ artificial lower specified function
- $w_u(\psi)$ upper positive weighting function
- $w_\ell(\psi)$ lower positive weighting function
- ξ margin of errors with respect to the artificial and desired specifications
- q p sgn(maximum artificial error)

Description of the Program

A general computer program package CADRDF was developed utilizing the aforementioned ideas. The user should specify the initial number of second-order filter sections, the maximum acceptable number of filter sections, the required precision, an option for local optimality checking either by perturbation of the parameters or by increasing ξ and then restarting the optimization process. The user should also specify the upper and lower bounds on the magnitude response, the discrete set of frequency points and the required optimization algorithm (Fletcher method[4] or Fletcher-Powell method[3]).

Starting with the initial number of filter sections, the coefficients of the digital filter are evaluated by minimizing an objective function using the chosen optimization algorithm. It is to be noted that the value of p is increased successively and the optimization is carried out for each subsequent value of p, until the absolute value of the relative change in maximum error becomes less than some small quantity (taken to be 0.001). To insure the stability of the filter a stability checking is provided whereby all the poles that do not lie within the unit circle in the z-domain are inverted with respect to the unit circle.

Local optimality of the solution $\check{\phi}$ is checked by either perturbing $\check{\phi}$ or increasing ξ, restarting the optimization process using the highest attained value of p and comparing the solutions before and after perturbation. It is to be noted that a minimax optimum will not be affected by increasing ξ.

If the specifications are not satisfied and the

This work was supported by the National Research Council of Canada under grants A7239 and C154.

maximum specified number of second-order filter sections has not been exceeded, a second-order filter section is grown by increasing the number of the independent parameters n by 4, assigning a starting value of zero for each of the grown filter coefficients a_k, b_k, c_k and d_k, then repeating the whole design process.

Examples

Example 1

Consider the design of a low pass digital filter of the cascade form whose ideal magnitude response is specified by

$$\text{Ideal magnitude response} = \begin{cases} 1 \text{ for } \psi \epsilon W_p \\ 0 \text{ for } \psi \epsilon W_s \end{cases}$$

where $W_p = [0. , 0.09]$ is the passband and $W_s = [0.11 , 1.0]$ is the stopband.

Let

$$S_u(\psi) = S_\ell(\psi) = 1 \text{ for } \psi \epsilon W_p$$

$$S_u(\psi) = 0 \text{ for } \psi \epsilon W_s$$

$$w_u(\psi) = w_\ell(\psi) = 1 \text{ for } \psi \epsilon W_p \cup W_s$$

All the functions of ψ will be evaluated at a finite discrete set of values of ψ taken from the closed intervals W_p and W_s as follows:

ψ=0.0,0.08(0.01) ; $S_u(\psi_i)=S_\ell(\psi_i)=1$; i=1,...,9

ψ=0.0801,0.09(0.00045); $S_u(\psi_i)=S_\ell(\psi_i)=1$; i=10,...,32

ψ=0.11,0.2(0.01) ; $S_u(\psi_i)=0$; i=33,...,42

ψ=0.3,1.(0.1) ; $S_u(\psi_i)=0$; i=43,...,50

Using a second-order filter section of the cascade form $\phi=[a_1\ b_1\ c_1\ d_1\ A]^T$. A starting point $\phi_0=[0\ 0\ 0\ -0.25\ 0.1]^T$ was taken. Test quantities for the Fletcher and Fletcher-Powell methods were 10^{-6}, and $\xi=0$.

The results are shown in Table 1. Growing another filter section and restarting the optimization process gave the results shown in Table 2. The magnitude response is depicted in Figs. 1 and 2, and the pole-zero configuration is shown in Fig. 3. It is to be noted that 201 equidistant values of ψ were used for response evaluation and plotting in each frequency band.

Example 2

Consider the design of a low pass recursive digital filter of the cascade form, for a 10 kHz sampling rate, whose upper and lower magnitude response bounds are specified by

f = 0,900(100) ; $S_u(f) = 1.1$; $S_\ell(f) = 0.9$

f = 1200 ; $S_u(f) = 0.1$

f = 1500,5000(500) ; $S_u(f) = 0.1$

The specifications can be prescribed as

ψ=0.,0.18(0.02); $S_u(\psi_i)=1.1$; $S_\ell(\psi_i)=0.9$; i=1,...,10

ψ=0.24 ; $S_u(\psi_i)=0.1$; i=11

ψ=0.3,1.(0.1) ; $S_u(\psi_i)=0.1$; i=12,...,19

Using a second-order recursive digital filter section, the same starting point as that used by Suk and Mitra[5], namely, $\phi_0 = [0\ 1\ -1\ 0.5\ 0.1]^T$ was taken. Weighting factors, test quantities and ξ were as in Example 1.

Optimization using the Fletcher method gave the results shown in Table 3. Growing another filter section from the one section locally optimum solution and restarting the optimization process gave the results shown in Table 4.

The magnitude response is depicted in Figs. 4 and 5, and the pole-zero configuration is shown in Fig. 6.

The sign of q was positive so long as the magnitude response did not lie within the specified bounds, but a change of sign of q to negative, occurred, when the specifications were met, but that did not stop the optimization process as it went on to produce a locally optimum solution for the case when the specifications were met. As the magnitude response was initially specified at a discrete set of frequency points, the magnitude response of the final solution was considered only at that discrete set of frequency points as shown in Figs. 4 and 5. Increasing the value of ξ did not affect the least 10000th solution.

Conclusions

The application of the Bandler-Charalambous method using extremely large values of p, typically 10,000, to recursive digital filter design problems seems to yield reasonably well-conditioned objective functions. Effectively negative values of p can be used to obtain the coefficients of a recursive digital filter that meets or exceeds the prescribed specifications. The use of the Fletcher method in conjunction with least pth optimization seems to be more efficient than that of the Fletcher-Powell method.

Acknowledgements

The authors thank Mrs. J.R. Popović, whose computer program on least pth approximation was incorporated into CADRDF. C. Charalambous is thanked for helpful suggestions.

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q = p	Maximum Error	Objective Function	Number of Function Evaluations	
			Fletcher method	Fletcher-Powell method
2	0.446136	0.905656	106	245
10	0.278126	0.308162	33	173
100	0.248415	0.251337	27	71
1000	0.246809	0.247100	37	60
10000	0.246713	0.246741	29	56

Table 1. Results for Example 1 using one section. The final solution is

$$\underset{\sim}{\check{\phi}}=[-1.816464 \quad 0.999999 \quad -1.837254 \quad 0.896054 \quad 0.241338]^T$$

q = p	Maximum Error	Objective Function	Number of Function Evaluations Using Fletcher method
2	0.094612	0.183426	130
10	0.046434	0.055707	80
100	0.043999	0.044486	58
1000	0.043639	0.043688	49
10000	0.043610	0.043614	52

Table 2. Results for Example 1 using two sections. The final solution is

$$\underset{\sim}{\check{\phi}}=[-1.870741 \quad 0.999999 \quad -1.874095 \quad 0.953439 \quad -1.520276 \quad 1. \quad -1.752557 \quad 0.787996 \quad 0.043369]^T$$

q = p	Maximum Error	Objective Function	Number of Function Evaluations using Fletcher method
2	0.192456	0.244387	44
10	0.109918	0.121980	60
100	0.101718	0.102881	27
1000	0.101302	0.101417	43
10000	0.101271	0.101282	24

Table 3. Results for Example 2 using one section. The final solution is

$$\underset{\sim}{\check{\phi}}=[-1.166418 \quad 1.000001 \quad -1.545223 \quad 0.764801 \quad 0.210399]^T$$

q = -p	Maximum Error	Objective Function	Number of Function Evaluations using Fletcher method
-2	-0.062826	-0.016925	182
-10	-0.068650	-0.059397	85
-100	-0.073797	-0.072980	94
-1000	-0.074321	-0.074240	58
-10000	-0.074369	-0.074361	71

Table 4. Results for Example 2 using two sections. The final solution is

$$\underset{\sim}{\check{\phi}}=[-1.404118 \quad 0.999999 \quad -1.585808 \quad 0.888486 \quad -0.227133 \quad 0.999999 \quad -1.456438 \quad 0.584344 \quad 0.035707]^T$$

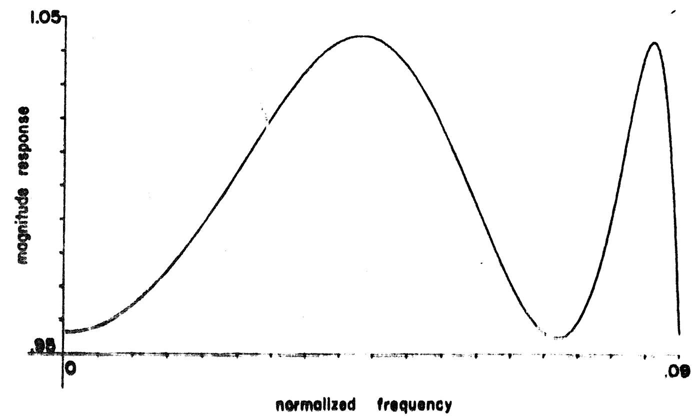

Fig.1. Magnitude characteristic of the pass band of the low pass filter of example 1.

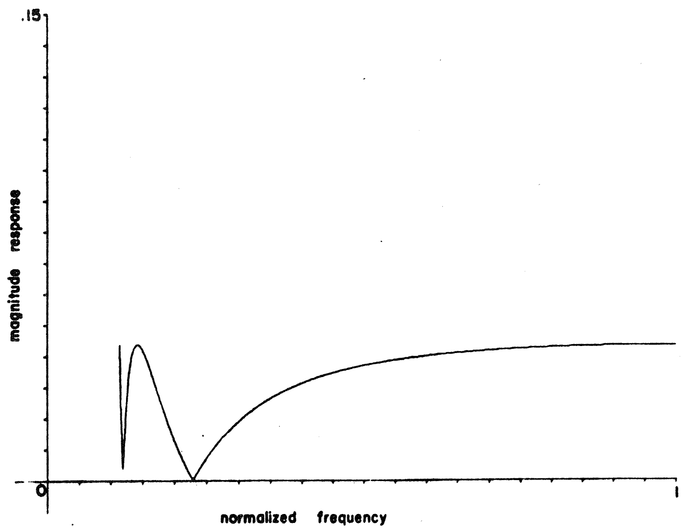

Fig.2. Magnitude characteristic of the stop band of the low pass filter of example 1.

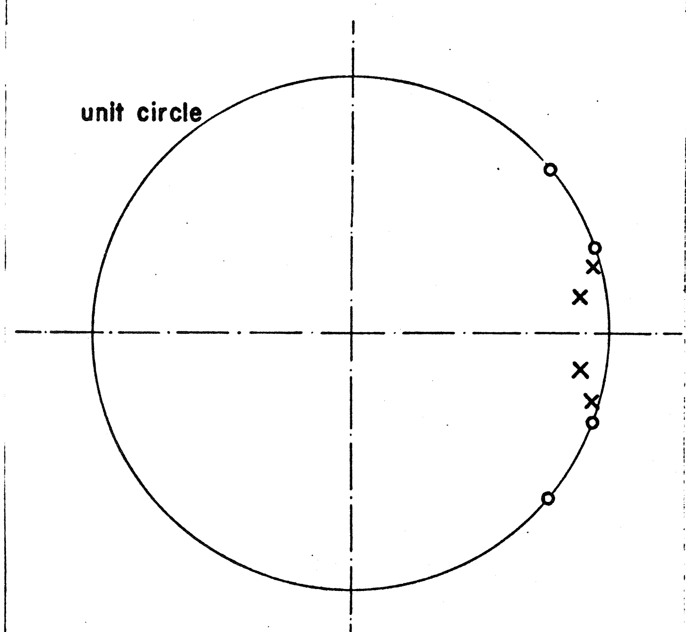

Fig.3. Pole-zero configuration for the low pass filter of example 1.

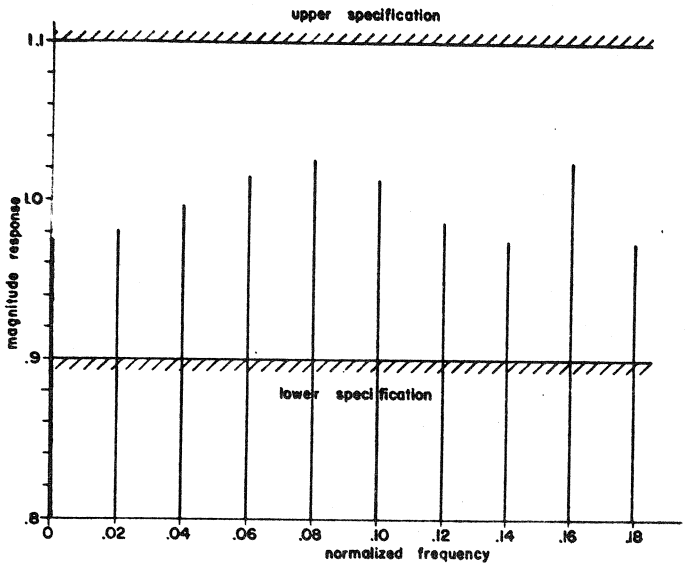

Fig.4. Magnitude characteristic of the pass band of the low pass filter of example 2.

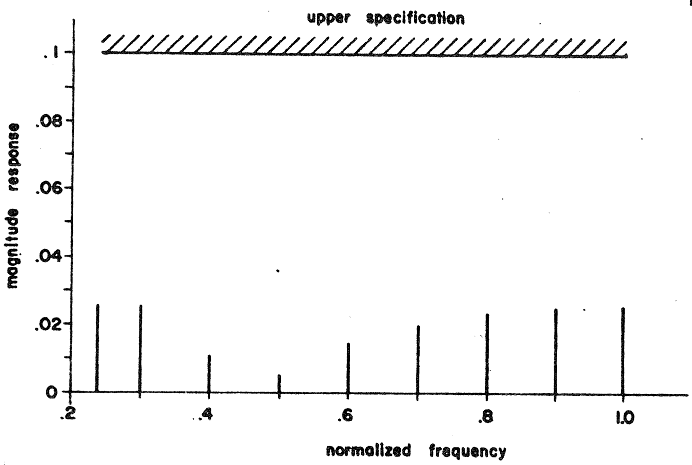

Fig.5. Magnitude characteristic of the stop band of the low pass filter of example 2.

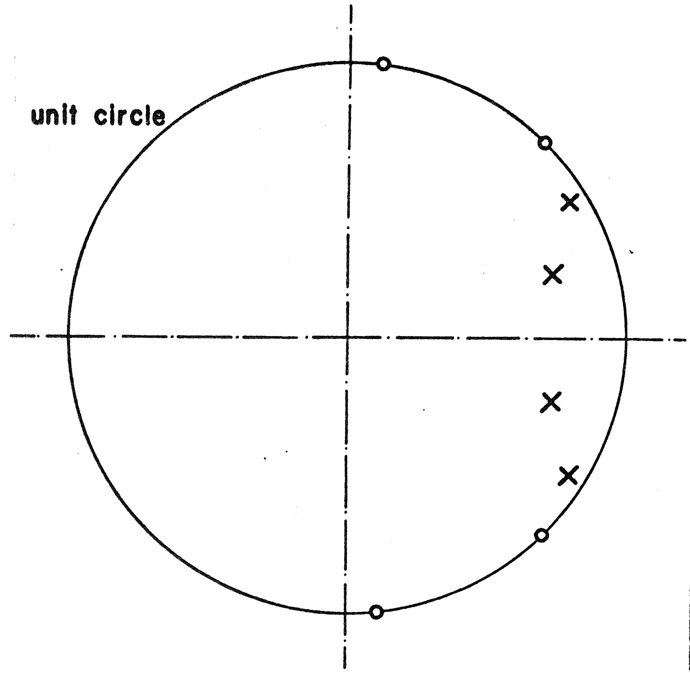

Fig.6. Pole-zero configuration for the low pass filter of example 2.

Biographies

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