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## THE TOLERANCE-TUNING PROBLEM: A NONLINEAR PROGRAMMING APPROACH

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**Abstract** A theory of optimal worst case design embodying centering, tolerancing and tuning with circuit applications is presented. Some simplified problems and special cases are discussed. Projections and slack variables are used to explain some of the concepts.

### INTRODUCTION

Much attention has been drawn recently to the component tolerance assignment and the design centering problem [1] - [4]. An approach whereby the nominal point as well as the tolerances are simultaneously optimized to meet minimum cost requirement in the fabrication of the design has been proposed [1], [2], [4]. On the other hand, component tuning, a very important related subject, is usually dealt with separately on a case-by-case study, or merely as an alignment procedure after the product is built [5].

A theory of optimal worst-case design embodying all the centering, tolerancing and tuning problems in a unified formulation at the design stage is presented here. Our approach incorporates the nominal circuit parameter values, the corresponding tolerances and tuning variables simultaneously into an optimization procedure designed to obtain the best values for all of them in an effort to reduce cost, or make an otherwise impractically toleranced design more attractive. Intuitively, our aim is to produce the best nominal point to permit the largest tolerances and the smallest tuning ranges (preferably zero) such that we can guarantee in advance that, in the worst case, the circuit will meet all the constraints and specifications.

### THE GENERAL FORMULATION

A design consists of design data of the nominal point  $\phi^0 \triangleq [\phi_1^0 \phi_2^0 \dots \phi_k^0]^T$ , the tolerances  $\epsilon \triangleq [\epsilon_1 \epsilon_2 \dots \epsilon_k]^T$  and the tuning vector  $t \triangleq [t_1 t_2 \dots t_k]^T$ , where  $k$  is the number of network parameters. Note that not all the components of  $\phi^0$ ,  $\epsilon$  and  $t$  need to be variables. Let  $I_\phi \triangleq \{1, 2, \dots, k\}$  be the index set for these parameters. The problem is formulated as the nonlinear programming problem:

$$\begin{aligned} & \text{minimize } C(\phi, \epsilon, t) \\ \text{subject to } & \phi \in R_c \end{aligned} \quad (1)$$

where

$$\phi = \phi^0 + \epsilon \mu + t \rho \quad (2)$$

and constraints on  $\phi^0$ ,  $\epsilon$ ,  $t \geq 0$ , for all  $\mu \in R_\mu$  and some  $\rho \in R_\rho$ , where

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$$E \triangleq \begin{bmatrix} \epsilon_1 & & & & & \\ & \epsilon_2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \epsilon_k & \end{bmatrix}, \quad T \triangleq \begin{bmatrix} t_1 & & & & & \\ & t_2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & t_k & \end{bmatrix}.$$

$R_\mu$  and  $R_\rho$  are sets of real number multipliers that will determine the outcome of a circuit and the amount of tuning required to bring the point into  $R_c$ .  $R_\mu$  and  $R_\rho$  are determined from realistic situations of the tolerance spread and tuning range. For example,

$$R_\mu = \{\mu_i | -1 \leq \mu_i \leq -a_i, a_i \leq \mu_i \leq 1, i \in I_\phi\} \quad (3)$$

where  $0 < a_i < 1, i \in I_\phi$ . The most commonly used continuous range is obtained by setting  $a_i$  to zero. A commercial stock will probably have the better toleranced components taken out, thus  $0 < a_i < 1$ . If  $a_i = 1$ ,  $R_\mu$  is identical to the set of vertices of the tolerance region. Some of the common examples for  $R_\rho$  will be,

$$R_\rho = \{\rho_i | -1 \leq \rho_i \leq 1, i \in I_\phi\} \quad (4)$$

or, in the case of one-way tuning or irreversible trimming,  $R_\rho = \{\rho_i | 0 < \rho_i \leq 1, i \in I_\phi\}$ , or  $R_\rho = \{\rho_i | -1 \leq \rho_i \leq 0, i \in I_\phi\}$ . Unless otherwise indicated, we will consider  $-1 \leq \mu_i \leq 1, -1 \leq \rho_i \leq 1, i \in I_\phi$ .

The constraint region  $R_c$  is given by  $R_c \triangleq \{\phi | g(\phi) \geq \rho\}$  and the tolerance region  $R_\epsilon \triangleq \{\phi | \phi^0 - \epsilon \leq \phi \leq \phi^0 + \epsilon\}$ . Thus, a tuning region  $R_t(\mu) \triangleq \{\phi | \phi^0 + \epsilon\mu - \epsilon \leq \phi \leq \phi^0 + \epsilon\mu + \epsilon\}$ . It is required that

$$R_t(\mu) \cap R_c \neq \emptyset, \quad (5)$$

where  $\emptyset$  is the empty set. See Fig. 1 for an illustration.

#### Tunable Constraint Region

$R_c$  is defined by a set of specifications and constraints. For tunable constraint regions the problem is subject to

$$\phi \in R_c(\psi) \quad (6)$$

where  $\psi$  represents other independent variables, and

$$\phi = \phi^0 + \epsilon\mu + T_\epsilon(\psi) \quad (7)$$

Fig. 2 depicts three different regions of  $R_c$ . Overlapping is, of course, allowable.

#### REDUCED PROBLEM 1

Consider the separation of the components into effectively tuned and effectively toleranced parameters. Let

$$I_\epsilon \triangleq \{i | \epsilon_i > t_i, i \in I_\phi\} \quad (8)$$

$$I_t \triangleq \{i | t_i \geq \epsilon_i, i \in I_\phi\} \quad (9)$$

$$\epsilon'_i \triangleq \epsilon_i - t_i, i \in I_\epsilon \quad (10)$$

and

$$t'_i \triangleq t_i - \epsilon_i, i \in I_t. \quad (11)$$

It is obvious that  $I_t$  and  $I_\epsilon$  are disjoint and  $I_t \cup I_\epsilon = I_\phi$ . Now, we may consider the problem subject to

$$\phi \in R_c$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \epsilon'_i \mu_i & \text{for } i \in I_\epsilon \\ t'_i \rho_i & \text{for } i \in I_t \end{cases} \quad (12)$$

for all  $-1 \leq \mu_i \leq 1, i \in I_\epsilon$  and for some  $-1 \leq \rho_i \leq 1, i \in I_t$ .

### Theorem 1

A feasible solution to the reduced problem 1 is a feasible solution to the original problem.

### Proof

Given  $\phi^0, \epsilon, t$  we will show that

$$(1) \quad \epsilon_i \mu_i + t_i \rho_i = \epsilon'_i \mu_i \quad i \in I_\epsilon$$

$$(2) \quad \epsilon_i \mu_i + t_i \rho_i = t'_i \rho_i \quad i \in I_t$$

under the restrictions on  $\mu_i, \rho_i$  and  $\rho'_i$ .

(1) Since  $\rho_i$  can be freely chosen from  $-1 \leq \rho_i \leq 1$ , we can let  $\rho_i = -\mu_i$  giving

$$(\epsilon_i - t_i)\mu_i = \epsilon'_i \mu_i.$$

(2) For any  $-1 \leq \rho'_i \leq 1$  and all  $-1 \leq \mu_i \leq 1$ , we can choose

$$-1 \leq \rho_i = \frac{(t_i - \epsilon_i)\rho'_i - \epsilon_i \mu_i}{t_i} \leq 1.$$

Thus, any point with components represented by (12) of the reduced problem can be represented by (2) of the original problem.

We state here without proof the following theorem.

### Theorem 2

A feasible solution to the original problem implies a feasible solution to the reduced problem if  $R_c$  is one-dimensionally convex.

A GEOMETRIC INTERPRETATION

Let

$$R_{\epsilon t} \triangleq \{ \phi \mid \phi_i^o - \epsilon_i' \leq \phi_i \leq \phi_i^o + \epsilon_i', i \in I_\epsilon \} \quad (13)$$

and

$$R_{t\epsilon} \triangleq \{ \phi \mid \phi_i^o - t_i' \leq \phi_i \leq \phi_i^o + t_i', i \in I_t \}. \quad (14)$$

Consider the following regions

$$R_{\epsilon t p} \triangleq \{ \phi_p \mid \phi_p = R\phi, \phi \in R_{\epsilon t} \}, \quad (15)$$

$$R_{ct\epsilon} \triangleq R_c \cap R_{t\epsilon}$$

and

$$R_{ct\epsilon p} \triangleq \{ \phi_p \mid \phi_p = R\phi, \phi \in R_{ct\epsilon} \} \quad (16)$$

where

$$R \triangleq \begin{bmatrix} p_1 & & & & \\ & p_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & p_k \end{bmatrix}$$

and

$$p_i = \begin{cases} 0 & \text{for } i \in I_t \\ 1 & \text{for } i \in I_\epsilon \end{cases}$$

The constraints of the problem are now interpreted by the requirement that

$$R_{\epsilon t p} \subseteq R_{ct\epsilon p}. \quad (17)$$

Fig. 3 depicts the definition of the different regions. Any point whose components are given by (12) is in the intersection of  $R_{\epsilon t}$  and  $R_{t\epsilon}$ , may be tuned into  $R_{ct\epsilon}$  by changing the value of  $\rho_i$ ,  $i \in I_t$  if the projection of  $R_{\epsilon t}$  onto the subspace spanned by the components subject to tolerances includes the projection of that point.

The reduced problem may be stated as: solve a pure tolerance problem (i.e., no tuning) in the subspace spanned by the toleranced variables with  $R_{\epsilon t p}$  as the tolerance region and  $R_{ct\epsilon p}$  as the constraint region.

Special Cases

- (1)  $I_\epsilon = \emptyset$ , the pure tuning problem.  
In this case,  $R_{\epsilon t}$  is the entire space and  $R$  is a zero matrix. The problem has a solution if

$$R_{ct\epsilon} \neq \emptyset. \quad (18)$$

- (2)  $I_t = \emptyset$ , the pure tolerance problem.  
 In this case,  $R_{ct\epsilon}$  is the entire space and  $R_t$  is a unit matrix. This problem has been treated thoroughly. It is required that,

$$R_{ct} \subseteq R_c \quad (19)$$

Fig. 4 illustrates a case where  $R_{ct\epsilon} \not\subseteq R_{ct\epsilon}$ . An outcome at  $\phi^0$  cannot be tuned to  $R_c$  within the effective tuning range. However, there exists a solution to the original formulation by tuning both components.  $R_c$  is not one-dimensionally convex in this case.

#### Extension for Tunable Constraint Region

Three types of components can be identified when the constraint region is considered to be tunable. They are:

- (a) Fixed components with tolerances
  - (b) Components tuned by the manufacturer
  - (c) Components tunable by the customer
- In this case,

$$\phi \in R_c(\psi)$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \epsilon_i^! \mu_i & \text{for } i \in I_\epsilon \\ t_i^! \rho_i^! & \text{for } i \in I_{tm} \\ t_i^! \rho_i^!(\psi) & \text{for } i \in I_{tc} \end{cases} \quad (20)$$

where  $I_{tm}$  identifies components (b) and  $I_{tc}$  identifies components (c).

Setting the value  $\psi$  to a particular value will control the setting of the value of  $\rho_i^!$ ,  $i \in I_{tc}$  such that  $\phi$  will be in that particular constraint region  $R_c(\psi)$ .

#### REDUCED PROBLEM 2

We consider  $-1 \leq \mu_i \leq 1$  replaced by  $\mu_i \in \{-1, 1\}$ ,  $i \in I_\epsilon$ .

Theorem 3 A feasible solution to reduced problem 2 implies a feasible solution to reduced problem 1 if  $R_{ct\epsilon}$  is one-dimensionally convex.

This is a pure tolerance problem in the subspace. The proof is given in [1].

#### Implementation for Frequency Domain Problem

Let  $\phi_p^r$  be a vertex of  $R_{ct\epsilon}$ . The constraints can be expressed explicitly as

$$g(\phi) = g(\phi_p^r + \sum_{i \in I_t} (\phi_i^0 + t_i^! \rho_i^r) e_i) \geq 0 \quad (21)$$

where  $e_i$  is the  $i$ th unit vector. The slack variables  $\rho_i^r$  are also to be constrained by  $-1 < \rho_i^r < 1$ . Strategies to eliminate inactive vertices [4] will not be discussed here.

The optimization parameters  $x$  may now be identified as a  $n$ -dimensional vector consisting of the variable nominal values, tolerances, tuning variables and all the appropriate slack variables  $\rho$ . A total of  $m$  constraint functions may be formed. in general,

$$n = k_o + k_\epsilon + k_t(i + n_v) \quad (22)$$

and

$$m = \left( \sum_{i=1}^{n_\omega} n_v(i) \right) + 2k_t n_v + \dots \quad (23)$$

where  $k_o$ ,  $k_\epsilon$  and  $k_t$  are the numbers of nominal parameter values, tolerated and tuned parameters, respectively;  $n_v < 2^k \epsilon$  is the number of distinct vertices chosen;  $n_\omega$  is the number of frequency points considered;  $n_v(i)$  is the number of vertices chosen at the  $i$ th frequency point and  $2k_t n_v$  is the number of slack variable bounds.

Any optimization algorithm may be used to minimize  $C(\chi)$  subject to  $g_i(\chi) \geq 0$ ,  $i = 1, 2, \dots, m$ .

#### EXAMPLE

Consider the constraints

$$\phi_2 - \phi_1 - 2 \geq 0 \quad (24)$$

$$-\phi_2^2 + 16\phi_1 \geq 0 \quad (25)$$

#### Tolerance Example

$$\text{Minimize } \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}$$

subject to

$$g_1 = \epsilon_1 \geq 0, \quad g_2 = \epsilon_2 \geq 0, \quad g_3 = \phi_1^0 \geq 0, \quad g_4 = \phi_2^0 \geq 0$$

$$g_5(i) = (\phi_2^0 + \epsilon_2 \mu_2(i)) - (\phi_1^0 + \epsilon_1 \mu_1(i)) - 2 \geq 0 \quad i = 1, 2, \dots \quad (26)$$

$$g_6(i) = -(\phi_2^0 + \epsilon_2 \mu_2(i))^2 + 16(\phi_1^0 + \epsilon_1 \mu_1(i)) \geq 0 \quad i = 1, 2, \dots \quad (27)$$

where  $-1 \leq \mu_1(i) \leq 1$  and  $-1 \leq \mu_2(i) \leq 1$ .

Optimality requires

$$\begin{bmatrix} -\frac{1}{\epsilon_1} \\ -\frac{1}{\epsilon_2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \sum_i u_5(i) \begin{bmatrix} -\mu_1(i) \\ \mu_2(i) \\ -1 \\ 1 \end{bmatrix} + \sum_i u_6(i) \begin{bmatrix} 16\mu_1(i) \\ -2\mu_2(i) (\phi_2^0 + \epsilon_2 \mu_2(i)) \\ 16 \\ -2(\phi_2^0 + \epsilon_2 \mu_2(i)) \end{bmatrix} \quad (28)$$

$$u_1 g_1 = \dots = u_4 g_4 = u_5(i) g_5(i) = u_6(i) g_6(i) = 0 \quad i = 1, 2, \dots \quad (29)$$

$$u_1, \dots, u_4, u_5(i), u_6(i) \geq 0 \quad i = 1, 2, \dots \quad (30)$$

Assume that  $\epsilon_1$ ,  $\epsilon_2$ ,  $\phi_1^0$  and  $\phi_2^0$  are not zero, therefore, set  $u_1, u_2, u_3$  and  $u_4$  to zero. Minimize  $g_5(i)$  of (26) and  $g_6(i)$  of (27) w.r.t.  $\mu(i)$ . This leads, respectively, to

$$(\phi_2^0 - \epsilon_2) - (\phi_1^0 + \epsilon_1) - 2 \geq 0$$

using  $\mu(i) = [1 \ -1]^T$  and

$$-(\phi_2^0 + \epsilon_2)^2 + 16(\phi_1^0 - \epsilon_1) \geq 0$$

using  $\mu(i) = [-1 \ 1]^T$ . The optimality conditions (28) - (30) are correspondingly reduced yielding the solution

$$\epsilon_1 = 0.5, \epsilon_2 = 0.5, \phi_1^0 = 3.5, \phi_2^0 = 7.5,$$

which have been verified by numerical optimization.

#### Tuning Example

Suppose  $\phi_1$  has a 10% tuning range and  $\phi_2$  is toleranced. Consider the problem of minimizing

$$\frac{1}{\epsilon_2'}$$

subject to

$$g_1 = t_1' \geq 0, g_2 = \epsilon_2' \geq 0, g_3 = \phi_1^0 \geq 0, g_4 = \phi_2^0 \geq 0$$

$$g_5 = .1 - \frac{t_1'}{\phi_1^0} \geq 0 \quad (31)$$

$$g_6(i) = (\phi_2^0 + \epsilon_2' \mu_2(i)) - (\phi_1^0 + t_1' \rho_1'(i)) - 2 \geq 0 \quad i = 1, 2, \dots \quad (32)$$

$$g_7(i) = -(\phi_2^0 + \epsilon_2' \mu_2(i))^2 + 16(\phi_1^0 + t_1' \rho_1'(i)) \geq 0 \quad i = 1, 2, \dots \quad (33)$$

$$g_8(i) = 1 - \rho_1'(i) \geq 0 \quad i = 1, 2, \dots \quad (34)$$

$$g_9(i) = 1 + \rho_1'(i) \geq 0 \quad i = 1, 2, \dots \quad (35)$$

The variables are  $t_1', \epsilon_2', \phi_1^0, \phi_2^0$  and  $\rho_1'(i)$ , and  $-1 \leq \mu_2(i) \leq 1$ .

Optimality requires

$$\begin{bmatrix} 0 \\ -\frac{1}{\epsilon_2'^2} \\ \epsilon_2' \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ 0 \end{bmatrix} + u_5 \begin{bmatrix} -\frac{1}{\phi_1^0} \\ \phi_1^0 \\ 0 \\ \frac{t_1'}{\phi_1^0} \\ \frac{t_1'}{\phi_1^0} \\ 0 \\ 0 \end{bmatrix} + \sum_i u_6(i) \begin{bmatrix} -\rho_1'(i) \\ \mu_2(i) \\ -1 \\ 1 \\ -t_1' \xi_i \end{bmatrix} + \sum_i u_7(i) \begin{bmatrix} 16\rho_1'(i) \\ -2(\phi_2^0 + \epsilon_2' \mu_2(i))\mu_2(i) \\ 16 \\ -2(\phi_2^0 + \epsilon_2' \mu_2(i)) \\ 16t_1' \xi_i \end{bmatrix}$$

$$+ \sum_i u_8(i) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\xi_i \end{bmatrix} + \sum_i u_9(i) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \xi_i \end{bmatrix} \quad (36)$$



$$u_1 g_1 = \dots = u_5 g_5 = u_6(i) g_6(i) = \dots = u_9(i) g_9(i) = 0 \quad i = 1, 2, \dots \quad (37)$$

$$u_1, \dots, u_5, u_6(i), \dots, u_9(i) \geq 0 \quad i = 1, 2, \dots \quad (38)$$

where  $e_i$  is the  $i$ th unit vector. Minimize  $g_6(i)$  of (32) and  $g_7(i)$  of (33) w.r.t.  $u_2(i)$ . We use  $u_2(i) = -1$  in (32) and  $u_2(i) = 1$  in (33) for this purpose. The corresponding  $\rho_1'(i) = -1$  and  $\rho_1'(i) = 1$ , respectively, are obtained by maximizing  $g_6(i)$  and  $g_7(i)$  w.r.t.  $\rho_1'(i)$ . This yields the solution  $t_1' = 0.5432$ ,  $\epsilon_2^1 = 1.444$ ,  $\phi_1^0 = 5.4321$ ,  $\phi_2^0 = 8.3333$ .

#### CONCLUSIONS

A theory of optimal worst case design embodying centering, tolerancing and tuning with circuit applications has been presented. The concept of a tunable constraint region that allows variable specifications as set by the customer has also been incorporated. This may find application, for example, in tunable filters. Components can be separated into effectively tuned and effectively toleranced parameters to simplify the solution, but possibly at the expense of optimality. The purely toleranced and purely tuned problems become special cases. Further simplification has been discussed in the light of one-dimensional convexity.

It may be added that as far as the authors are aware, this seems to be the most general formulation to date dealing with the centering, tolerancing and tuning problems at the design stage. Tuning uncertainties can also be taken care of in the formulation.

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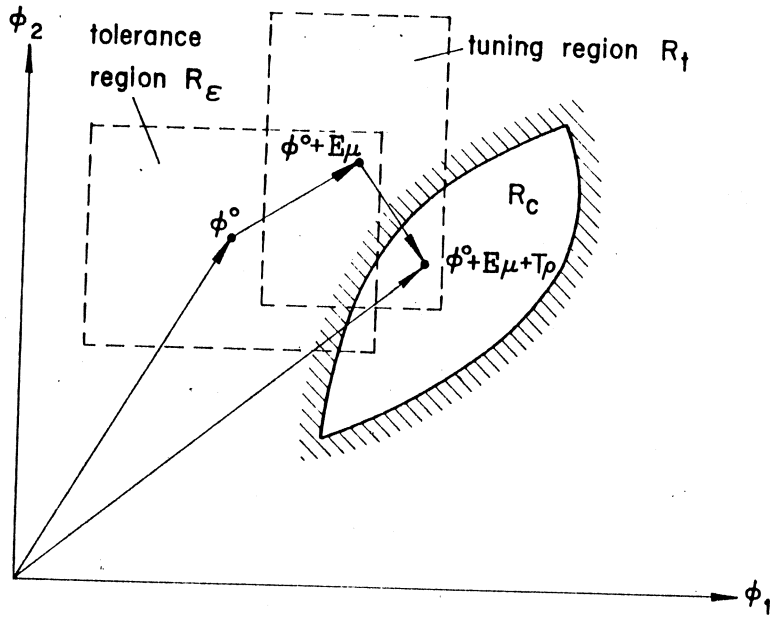


Fig. 1 An illustration of the regions  $R_e$ ,  $R_t$ , and  $R_c$ .

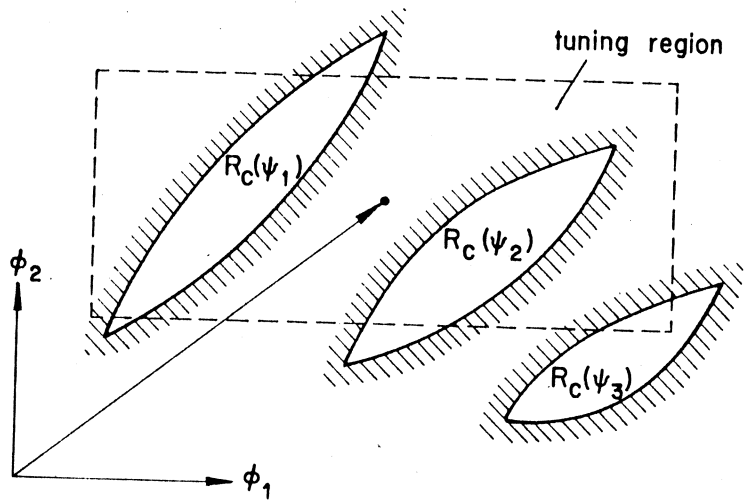


Fig. 2 An example of three different settings of the tunable constraint regions.

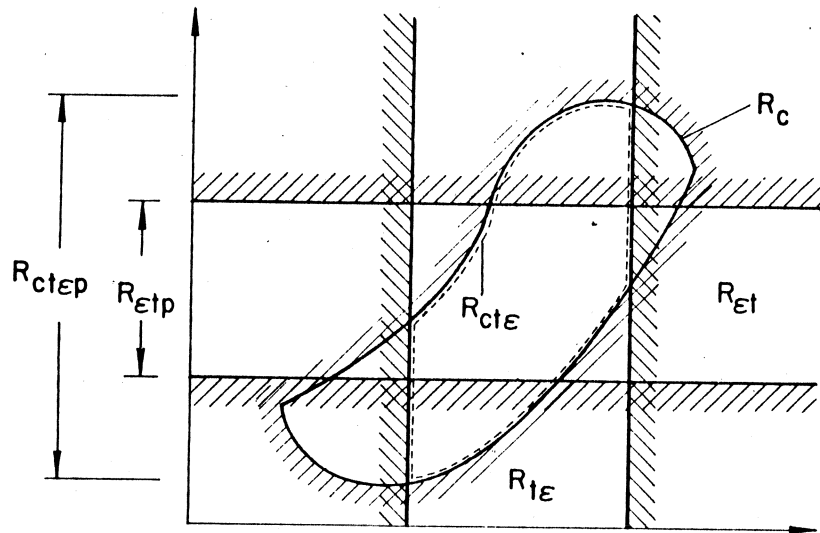


Fig. 3 A geometric interpretation of the reduced problem 1.

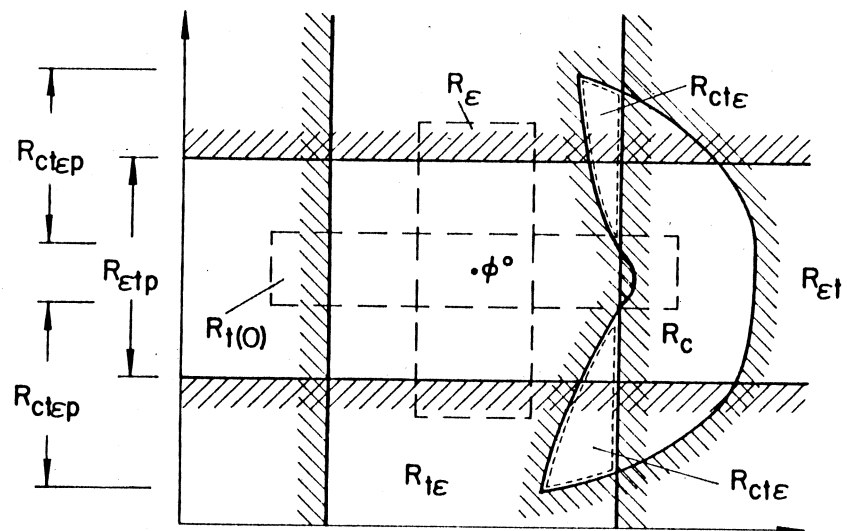


Fig. 4 An example of  $R_{\epsilon t p} \not\subseteq R_{ct\epsilon p}$ .